Jñānābha, Vol. 11, 1981

ON A CHAIN FOR THE *H*-FUNCTION TRANSFORM OF TWO VARIABLES

by

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(*Received : May 27, 1980; Revised : March 31, 1981*)

ABSTRACT

In this paper we study an integral transform whose kernel is the *H*-function of two variables and obtain a new chain for this transform interconnecting a number of *H*-function transforms of two variables having different parameters. Interesting chains for a number of other integral transforms of one and two variables follow as special cases of our main result. These special cases are of interest in themselves and are also believed to be new. Since the *H*-function of two variables includes a large number of special functions of practical value as its particular cases, the results obtained here serve as key formulae for obtaining other similar results.

1. INTRODUCTION

The *H*-function transform of two variables [8, p. 2], occurring in the present paper, will be defined and represented by the following integral equation :

$$(1.1)\quad \phi\,(p,q) = pq \int_0^\infty \int_0^\infty H_1\,[\,px, qy\,]\, f\,(x,y)\; dx\; dy$$

$$= pq \int_0^\infty \int_0^\infty H\begin{matrix} 0,\,0 & : m_2,\,n_2\;;\;m_3,\,n_3 \\ p_1,\,q_1 & : p_2,\,q_2\;;\;p_3,\,q_3 \end{matrix}\left[\begin{matrix} px \\ qy \end{matrix}\right|$$

$$\left.\begin{matrix}(a_j;\ \alpha_j,\ A_j)_{1,p_1}\ :\ (c_j,\ \gamma_j)_{1,\ p_2}\ ;\ (e_j,\ E_j)_{1,\ p_3}\\ (b_j;\ \beta_j,\ B_j)_{1,\ q_1}\ :\ (d_j,\ \delta_j)_{1,\ q_2}\ ;\ (f_j,\ F_j)_{1,\ q_3}\end{matrix}\right] f(x,y)\,dx\,dy$$

provided that the integral on the right-hand side of (1.1) is absolutely convergent.

The function $H_1\,[\,px,\,qy\,]$ appearing in (1.1) is a special case of the H-function of two variables, defined by Mittal and Gupta [9, p. 117] in terms of a double Mellin–Barnes type contour integral; thus, following the contracted notations introduced by Srivastava and Panda [12, p. 266, Eq. (1.5) *et seq.*], we have

$$(1.2)\quad H\,[\,x.\,y\,] = H\begin{matrix}0,\ n_1\ :\ m_2,\ n_2\ ;\ m_3,\ n_3\\ p_1,\ q_1\ :\ p_2,\ q_2\ ;\ p_3,\ q_3\end{matrix}\left[\begin{matrix}x\\ \\ y\end{matrix}\right|$$

$$\left.\begin{matrix}(a_j;\ \alpha_j,\ A_j)_{1,p_1}\ :\ (c_j,\ \gamma_j)_{1,\ p_2}\ ;\ (e_j,\ E_j)_{1,\ p_3}\\ (b_j;\ \beta_j,\ B_j)_{1,\ q_1}\ :\ (d_j,\ \delta_j)_{1,\ q_2}\ ;\ (f_j,\ F_j)_{1,\ q_3}\end{matrix}\right]$$

$$= -\frac{1}{4\pi^2}\int_{L_1}\int_{L_2}\phi(s,\,t)\,\theta_1\,(s)\,\theta_2\,(t)\,x^s\,y^t\,ds\,dt$$

where

$$\phi\,(s,\,t) = \frac{\prod_{j=1}^{n_1}\Gamma\,(1-a_j+\alpha_j\,s+A_j\,t)}{\prod_{j=1}^{q_1}\Gamma\,(1-b_j+\beta_j s+B_j t)\prod_{j=n_1+1}^{p_1}\Gamma(a_j-\alpha_j s-A_j t)},$$

$$\theta_1\,(s) = \frac{\prod_{j=1}^{m_2}\Gamma\,(d_j-\delta_j s)\prod_{j=1}^{n_2}\Gamma\,(1-c_j+\gamma_j s)}{\prod_{j=m_2+1}^{q_2}\Gamma\,(1-d_j+\delta_j s)\prod_{j=n_2+1}^{p_2}\Gamma\,(c_j-\gamma_j s)},$$

$$\theta_2(t) = \frac{\prod_{j=1}^{m_3} \Gamma(f_j - F_j t) \prod_{j=1}^{n_3} \Gamma(1 - e_j + E_j t)}{\prod_{j=m_3+1}^{q_3} \Gamma(1 - f_j + F_j t) \prod_{j=n_3+1}^{p_3} \Gamma(e_j - E_j t)},$$

The conditions for the function $H[x, y]$ to represent an analytic function and the conditions for the integral in (1.2) to converge are given by Mittal and Gupta [9, p. 119]. It is assumed that all the H–functions of two variables, occurring in the present paper, always satisfy conditions of convergence corresponding appropriately to the conditions (i) to (vi) of Mittal and Gupta [9, p. 119].

We mention below some of the special cases of the integral transform (1.1) which will be required in our subsequent discussion :

(I) If we take $p_1 = q_1 = 0$, $m_2 = m_3 = q_2 = q_3 = 2$, $n_2 = n_3 = 0$, $p_2 = p_3 = 1$, $c_1 = -k - w$, $d_1 = r - w$, $d_2 = -r - w$, $d_1 = \delta_1 = \delta_2 = 1$, $e_1 = -k_1 - w_1$, $f_1 = r_1 - w_1$, $f_2 = -r_1 - w_1$, $E_1 = F_1 = F_2 = 1$ in (1.1), and use the known results [2, p. 216, Eq. (6)] and [9, p. 120, Eq. (1.4)], we get the following transform studied earlier by Nigam [10] :

$$\text{(1.3)} \quad \phi(p, q) = w \left[f(x, y) ; \begin{matrix} w, & k, & r \\ w_1, & k_1, & r_1 \end{matrix} ; p, q \right]$$

$$= p\, q \int_0^\infty \int_0^\infty e^{-1/2\,(px+qy)} (px)^{-w-\frac{1}{2}} (qy)^{-w_1-\frac{1}{2}}$$

$$. W_{k+1/2,\, r}(px)\, W_{k_1+1/2,\, r_1}(qy)\, f(x, y)\, dx\, dy$$

II If we take $p_1 = q_1 = 0$, $n_2 = n_3 = p_2 = p_3 = 0$,

$m_2=m_3=q_2=q_3=2, d_1=1/4-\varepsilon/2$, $d_2=1/4+\varepsilon/2$, $f_1=1/4-\nu/2$, $f_2=1/4+\nu/2$, $\delta_1=\delta_2=F_1=F_2=\frac{1}{2}$ and replace p by $p/2$ and q by $q/2$ in (1.1) we get the following double Meijer–Bessel function transform by virtue of the results [2, p. 216, Eq. (4)] and [9, p. 120, Eq. (1.4)] :

(1.4) $$\phi(p,q) = K\,[\,f(x,y)\,;\,\varepsilon,\nu\,;\,p,q\,]$$

$$= 2\,p\,q \int_0^\infty \int_0^\infty (px)^{1/2}\,(qy)^{1/2}\,K_\varepsilon(px)\,K_\nu(qy)\,f(x,y)\,dx\,dy$$

III. If we take $p_1=q_1=0$, $m_2=m_3=q_2=q_3=1$, $n_2=n_3=p_2=p_3=0$, $d_1-f_1=0$, $\delta_1=F_1=1$ in (1.1) and use the known results [7, p. 61] and [9, p. 120, Eq. 1. 4], we obtain the following well known Laplace transform of two variables :

(1.5) $$\phi(p,q) = L\,[\,f(x,y)\,;\,p,q\,]$$

$$= p\,q \int_0^\infty \int_0^\infty e^{-px-qy}\,f(x,y)\,dx\,dy\,.$$

In what follows we use the abbreviation $h(x,y) \in C \begin{Bmatrix} c, & d \\ e, f; & \alpha, \beta \end{Bmatrix}$

to denote that

$$h(x,y) = \begin{cases} O\,(x^c\,y^d), & \max\{\,|x|\,,\,|y|\,\} \longrightarrow 0\,, \\ O\,(x^e\,y^f\,e^{-px-qy}), & \min\{\,|x|\,,\,|y|\,\} \longrightarrow \infty \end{cases}$$

where c, d, e, f, α and β are complex numbers.

The following symbols have also been used in the present paper :

(i) $H_1'\,[\,px, qy\,]$ stands for

$$H\begin{matrix} 0,\ 0 & : m'_2,\ n'_2\ ;\ m'_3,\ n'_3 \\ p'_1,\ q'_1 & : p'_2,\ q'_2\ ;\ p'_3,\ q'_3 \end{matrix}\left[\begin{matrix} px \\ qy \end{matrix}\right|$$

$$\left.\begin{matrix} (a_j'\ ;\ \alpha_j',\ A_j')_{1,p_1'} : (c_j',\ \gamma_j')_{1,p_2'}\ ;\ (e_j',\ E_j')_{1,p_3'} \\ (b_j'\ ;\ \beta_j',\ B_j')_{1,q_1'} : (d_j',\ \delta_j')_{1,q_2'}\ ;\ (f_j',\ F_j')_{1,q_3'} \end{matrix}\right]$$

(ii) $H_1^{(N)}[px, qy]$ stands for

$$H\begin{matrix} 0,\ 0 & : m_2'',\ n_2'';\ m_3'',\ n_3'' \\ p_1'',\ q_1'' & : p_2'',\ q_2'';\ p_3'',\ q_3'' \end{matrix}\left[\begin{matrix} px \\ qy \end{matrix}\right|$$

$$\left.\begin{matrix} (a_j^{(N)}\ ;\ \alpha_j^{(N)},\ A_j^{(N)})_{1,p_1''} : (c_j^{(N)},\ \gamma_j^{(N)})_{1,p_2''};(e_j^{(N)},\ E_j^{(N)})_{1,p_3''} \\ (b_j^{(N)}\ ;\ \beta_j^{(N)},\ B_j^{(N)})_{1,q_1''} : (d_j^{(N)},\ \delta_j^{(N)})_{1,q_2''};\ (f_j^{(N)},\ F_j^{(N)})_{1,q_3''} \end{matrix}\right]$$

for a positive integer N.

(iii) $(a \pm b, c) = (a + b, c), (a - b, c)$.

(iv) $\left\{ (a_j^{(M)},\ \alpha_j^{(M)})_{1,u} \right\}_{M=1}^{N}$ stands for

$$(a_j^{(1)},\ (\alpha_j^{(1)})_{1,u},\ (a_j^{(2)},\ \alpha_j^{(2)})_{1,u}, \ldots, (a_j^{(N)},\ \alpha_j^{(N)})_{1,u}.$$

The following theorem obtained recently by the author [11] will also be required in our investigation.

THEOREM. *IF*

$$(1.6)\ f(p, q) = pq \int_0^\infty \int_0^\infty x^{\rho'} y^{\sigma'} H_1'[px, qy]\ \phi(x^\lambda, y^\mu)dx\,dy,$$

$$(1.7)\quad p^{\xi} q^{\eta} \phi (p^{a}, q^{b}) = pq \int_0^{\infty} \int_0^{\infty} x^{\rho} y^{\sigma} H_1 [px, qy]\, h(x,y) dx\, dy,$$

and

$$(1.8)\quad \theta_1(p, q) = pq \int_0^{\infty} \int_0^{\infty} x^{u_1} y^{v_1} H_1^{(1)} [px, qy]\, f(x,y) dx\, dy,$$

then

$$(1.9)\quad \theta_1 (p, q) = ab\, p^{A-u_1} q^{B-v_1} \int_0^{\infty} \int_0^{\infty} s^{\rho} t^{\sigma} h(s, t)$$

$$H^{0,\,0 \;:\; n_2+m_2'+n_2'',\; m_2+n_2'+m_2'' \;;\; n_3+m_3'+n_3'',\; m_3+n_3'+m_3''}_{q_1+p_1'+q_1'',\; p_1+q_1'+p_1'' \;:\; q_2+p_2'+q_2'',\; p_2+q_2'+p_2'' \;;\; q_3+p_3'+q_3'',\; p_3+q_3'+p_3''} \left[\begin{matrix} s^{-a}\, p^{-\lambda} \\ t^{-b}\, q^{-\mu} \end{matrix} \middle| \begin{matrix} I^{(1)} : K^{(1)} \;;\; P^{(1)} \\ J^{(1)} : L^{(1)} \;;\; Q^{(1)} \end{matrix} \right] ds\, dt$$

where

$$A = \rho' + \frac{\gamma(1-\xi)}{a},\quad B = \sigma' + \frac{\mu(1-\eta)}{b},$$

$$I^{(1)} = (1-b_j \;;\; a\beta_j,\, bB_j)_{1,\, q_1},\ (a_j' + \{A+1\}\, \alpha_j' + \{B+1\}\, A_j' \;;\; \lambda\alpha_j',\ \mu A_j')_{1,\, p_1'},$$

$$(1-b_j^{(1)} - \{u_1 - A + 1\}\, \beta_j^{(1)} - \{v_1 - B + 1\}\, B_j^{(1)} ; \lambda\beta_j^{(1)}, \mu B_j^{(1)})_{1,\, q_1''}$$

$$J^{(1)} = (1-a_j;\ a\,\alpha_j,\, bA_j)_{1,\, p_1},(b_j' + \{A+1\}\beta_j' + \{B+1\}B_j^{(1)} ; \lambda\beta_j', \mu B_j')_{1,\, q_1'}$$

$$(1-a_j^{(1)} - \{u_1 - A + 1\}\alpha_j^{(1)} - \{v_1 - B + 1\}A_j^{(1)} ; \lambda\alpha_j^{(1)}, \mu A_j^{(1)})_{1,\, p_1''}$$

$$K^{(1)} = (1-d_j,\, a\delta_j)_{1,\, m_2},\ (c_j' + \{A+1\}\, \gamma_j',\ \lambda\gamma_j')_{1,\, n_2'},$$

$$(1-d_j^{(1)} - \{u_1-A+1\}\ \delta_j^{(1)}\ ,\ \lambda\delta_j^{(1)})_{1,\ q_2''}\ ,$$

$$(1-d_j,\ a\delta_j)_{m_2+1,\ q_2}\ ,\ (c_j'+\{A+1\}\ \gamma_j',\ \lambda\gamma_j')_{n_2'+1,\ p_2'}\ ,$$

$$L^{(1)} = (1-c_j,\ a\gamma_j)_{1,\ n_2}\ ,\ (d_j'+\{A+1\}\ \delta_j',\ \lambda\delta_j')_{1,\ m_2'}\ ,$$

$$(1-c_j^{(1)} - \{u_1-A+1\}\ \gamma_j^{(1)}\ ,\ \lambda\ \gamma_j^{(1)}\ )_{1,\ p_2''}\ ,$$

$$(1-c_i,\ a\ \gamma_j)_{n_2+1,\ p_2}\ ,\ (d_j'+\{A+1\}\ \delta_j',\ \lambda\delta_j')_{m_2'+1,\ q_2'}$$

$$P^{(1)} = (1-f_j,\ bF_j)_{1,\ m_3}\ ,\ (e_j'-\{B+1\}\ E_j',\ \mu E_j')_{1,\ n_3'}\ ;$$

$$(1-f_j^{(1)}-\{v_1-B+1\}\ F_j^{(1)},\ \mu F_j^{(1)})_{1,\ q_3''}\ ,$$

$$(1-f_j,\ bF_j)_{m_3+1,\ q_3}\ ,\ (e_j'+\{B+1\}\ E_j'\ ,\ \mu\ E_j')_{n_3'+1,\ p_3'}$$

and

$$Q^{(1)} = (1-e_j, bE_j)_{1,\ n_3}\ ,\ (f_j'+\{B+1\}\ F_j'\ ,\ \mu F_j')_{1,\ m_3'}\ ,$$

$$(1-e_j^{(1)}-\{v_1-B+1\}\ E_j^{(1)}\ ,\ \mu E_j^{(1)})_{1,\ p_3''}\ ,$$

$$(1-e_j,\ bE_j)_{n_3+1,\ p_3}\ ,\ (f_j'+\{B+1\}\ F_j'\ ,\ \mu F_j')_{m_3'+1,\ q_3'}$$

The details of the proof and the conditions of its validity of the above theorem can be found in the paper cited above [11]

2. MAIN THEOREM

If

$$(2.1)\quad f\,(\mathrm{p},\ q) = \mathrm{p}q \int_0^\infty \int_0^\infty x^{\rho'}\ y^{\sigma'}\ \mathrm{H}_1'\ [\ \mathrm{p}\ x,\ q\ y\ ]\ \phi\ (x^{\lambda}, y^{\mu})\, dx\ dy\ ,$$

$$(2.2)\quad p^{\xi}q^{\eta}\phi(p^{a}, q^{b}) = pq\int_0^\infty \int_0^\infty x^{\rho} y^{\sigma} H_1 [\, px, qy\,] h(x,y)\,dx\,dy\,,$$

$$(2.3)\quad \theta_1(p, q) = pq\int_0^\infty \int_0^\infty x^{u_1} y^{v_1} H_1^{(1)} [\, px, qy\,] f(x,y)\,dx\,dy\,,$$

$$(2.4)\quad \theta_2(p, q) = pq\int_0^\infty \int_0^\infty x^{u_2} y^{v_2} H_1^{(2)} [\, px, qy\,] \theta_1(1/x, 1/y)\,dx\,dy\,.$$

$$(2.5)\quad \theta_3(p, q) = pq\int_0^\infty \int_0^\infty x^{u_3} y^{v_3} H_1^{(3)} [\, px, qy\,] \theta_2(1/x, 1/y)\,dx\,dy\,,$$

and

$$(2.6)\quad \theta_N(p, q) = pq\int_0^\infty \int_0^\infty x^{u_N} y^{v_N} H_1^{(N)} [px, qy]\theta_{N-1}(1/x, 1/y)\,dx,\,dy,$$

then

$$(2.7)\quad (\theta_N (p, q) = ab\, p^{A-\sum_{i=1}^{N} u_i}\, q^{B-\sum_{i=1}^{N} v_i} \int_0^\infty \int_0^\infty s^{\rho} t^{\sigma} h(s, t)$$

$$H^{0,\,0\,:\,n_2+m_2'+Nn_2'',\, m_2+n_2'+Nm_2''\,;}_{q_1+p_1'+Nq_1'',\, p_1+q_1'+Np_1''\,:\,q_2+p_2'+Nq_2'',\, p_2+q_2'+Np_2''\,;}$$

$$\begin{matrix} n_3+m_3'+Nn_3'',\, m_3+n_3'+Nm_3'' \\ q_3+p_3'+Nq_3'',\, p_3+q_3'+Np_3'' \end{matrix} \left[\begin{matrix} s^{-a}\, p^{-\lambda} \\ t^{-b}\, q^{-\mu} \end{matrix} \,\middle|\, \begin{matrix} I^{(N)} \;:\; K^{(N)} \;;\; P^{(N)} \\ J^{(N)} \;:\; L^{(N)} \;;\; Q^{(N)} \end{matrix} \right] ds\,dt,$$

where

$$A = \rho' + \frac{\lambda(1-\xi)}{a},\; B = \sigma' + \frac{\mu(1-\eta)}{b}\,,$$

$$I^{(N)} = (1-b_j\;;\; a\beta_j\;,\; bB_j)_{1,\,q_1}\,,$$

$$(a_j'+\{A+1\}\,\alpha_j'+\{B+1\}\,A_j'\;;\;\lambda\alpha_j',\;\mu A_j')_{1,\,p_1'}\,,$$

$$\left\{ (1-b_j^{(M)} - \left\{\sum_{i=1}^{M} u_i - A + 1\right\}\beta_j^{(M)} - \left\{\sum_{i=1}^{M} v_i - B + 1\right\} B_j^{(M)} ;\right.$$

$$\left. \lambda\,\beta_j^{(M)},\ \mu B_j^{(M)})_{1,\,q''_1} \right\}_{M=1}^{N}$$

$$J^{(N)} = (1-a_j;\ a\alpha_j,\ bA_j)_{1,\,p_1},\ (b_j'+\{A+1\}\beta_j'+\{B+1\}B_j';\lambda\beta_j',\mu B_j')_{1,q_1'},$$

$$\left\{ (1-a_j^{(M)} - \{\sum_{i=1}^{M} u_i - A + 1\}\alpha_j^{(M)} - \{\sum_{i=1}^{M} v_i - B + 1\} A_j^{(M)} ;\right.$$

$$\left. \lambda\alpha_j^{(M)},\ \mu A_j^{(M)})_{1,\,p''_1} \right\}_{M=1}^{N}$$

$$K^{(N)} = (1-d_j,\ a\delta_j)_{1,\,m_2},\ (c_j'+\{A+1\}\ \gamma_j',\ \lambda\gamma_j')_{1,\,n_2'},$$

$$\left\{ (1-d_j^{(M)} - \{\sum_{i=1}^{M} u_i - A + 1\}\ \delta_j^{(M)},\ \lambda\ \delta_j^{(M)})_{1,\,q''_2} \right\}_{M=1}^{N},$$

$$(1-d_j,\ a\delta_j)_{m_2+1,\,q_2},\ (c_j'+\{A+1\}\gamma_j',\ \lambda\gamma_j')_{n_2'+1,\,p_2'}\ .$$

$$L^{(N)} = (1-c_j,\ a\gamma_j)_{1,n_2},\ (d_j'+\{A+1\}\delta_j',\ \lambda\delta_j')_{1,m_2'},$$

$$\left\{ (1-c_j^{(M)} - \{\sum_{i=1}^{M} u_i - A + 1\}\gamma_j^{(M)},\ \lambda\gamma_j^{(M)})_{1,\,p''_2} \right\}_{M=1}^{N},$$

$$(1-c_j,\ a\gamma_j)_{n_2+1,\,p_2},\ (d_j'+\{A+1\}\ \delta_j',\ \lambda\delta_j')_{m_2'+1,\,q_2'}$$

$$P^{(N)} = (1-f_j,\ bF_j)_{1,\ m_3},\ (e_j'+\{B+1\}E_j',\ \mu E_j')_{1,\ n_3'},$$

$$\left\{ (1-f_j^{(M)} - \{\sum_{i=1}^{M} v_i - B+1\}F_j^{(M)},\ \mu F_j^{(M)})_{1,\ q_3''} \right\}_{M=1}^{N},$$

$$(1-f_j,\ bF_j)_{m_3+1,\ q_3},\ (e_j'+\{B+1\}\ E_j',\ \mu E_j')_{n_3'+1,\ p_3'};$$

and

$$Q^{(N)} = (1-e_j,\ bE_j)_{1,\ n_3},\ (f_j'+\{B+1\}\ F_j',\ \mu F_j')_{1,\ m_3'},$$

$$\left\{ (1-e_j^{(M)} - \{\sum_{i=1}^{M} v_i - B+1\}\ E_j^{(M)},\ \mu E_j^{(M)})_{1,\ p_3''} \right\}_{M=1}^{N},$$

$$(1-e_j,\ bE_j)_{n_3+1,\ p_3},\ (f_j'+\{B+1\}\ F_j',\ \mu F_j')_{m_3'+1,\ q_3'}.$$

The above theorem holds when the integral transforms defined by (2.1), (2.3), ... , (2.6) of $|\ x^{\rho'} y^{\sigma'}\ \phi\ (x^\lambda, y^\mu)\ |$, $|\ x^{u_1} y^{v_1}\ f\ (x, y)\ |$, $|\ x^{u_j}\ ^{v_j}\ \theta_{j-1}\ (1/x)\ |$ ($j=2, \ldots, N$), respectively, exist, $h(x, y) \varepsilon\ C \begin{bmatrix} c,\ d \\ \cdots \\ e, f\ ;\alpha, \beta \end{bmatrix}$, $Re\ (\alpha) > 0$, $Re\ (\beta) > 0$, $\lambda > 0$, $\mu > 0$, $a > 0$, $b > 0$, and the following sets of conditions are satisfied for a positive integer N :

(i) $Re\ (R^{(N)}) < 0$, $Re\ (S^{(N)}) < 0$, $Re\ (T^{(N)}) > 0$, $Re\ (W^{(N)}) > 0$,

$$|\arg p| < \frac{T^{(N)}\pi}{2},\ |\arg q| < \frac{W^{(N)}\pi}{2},$$

where

$$R^{(N)} = \sum_{1}^{p''} (\alpha_j^{(N)}) + \sum_{1}^{p_2''} (\gamma_j^{(N)}) - \sum_{1}^{q_1''} (B_j^{(N)}) - \sum_{1}^{q_2''} (\delta_j^{(N)}),$$

$$S^{(N)} = \sum_{1}^{p_1''} (A_j^{(N)}) + \sum_{1}^{p_3''} (E_j^{(N)}) - \sum_{1}^{q_1''} (B_j^{(N)}) - \sum_{1}^{q_3''} (F_j^{(N)}),$$

$$T^{(N)} = -\sum_{1}^{p_1''} (\alpha_j^{(N)}) - \sum_{1}^{q_1''} (\beta_j^{(N)}) + \sum_{1}^{m_2''} (\delta_j^{(N)}) - \sum_{m_2''+1}^{q_2''} (\delta_j^{(N)})$$

$$+ \sum_{1}^{n_2''} (\gamma_j^{(N)}) - \sum_{n_2''+1}^{p_2''} (\gamma_j^{(N)}),$$

and

$$W^{(N)} = -\sum_{1}^{p_1''} (A_j^{(N)}) - \sum_{1}^{q_1''} (B_j^{(N)}) + \sum_{1}^{m_3''} (F_j^{(N)}) - \sum_{m_3''+1}^{q_3''} (F_j^{(N)})$$

$$+ \sum_{1}^{n_3''} (E_j^{(N)}) - \sum_{n_3''+1}^{p_3''} (E_j^{(N)}).$$

(ii) $Re\,(R^*) < 0$, $Re\,(S^*) < 0$, $Re\,(T^*) > 0$, $Re\,(W^*) > 0$,

where

$$R^* = a\sum_{1}^{q_1} (\beta_j) + \lambda \sum_{1}^{p_1'} (\alpha_j') + a\sum_{1}^{q_2} (\delta_j) + \lambda \sum_{1}^{p_2'} (\gamma_j')$$

$$- a\sum_{1}^{p_1} (\alpha_j) - \lambda \sum_{1}^{q_1'} (\beta_j') - a\sum_{1}^{p_2} (\gamma_j) - \lambda \sum_{1}^{q_2'} (\delta_j')$$

$$+ \lambda \sum_{l=1}^{N-1} \left[\sum_{1}^{q_1''} (\beta_j^{(l)}) + \sum_{1}^{q_2''} (\delta_j^{(l)}) - \sum_{1}^{p_1''} (\alpha_j^{(l)}) - \sum_{1}^{p_2''} (\gamma_j^{(l)}) \right],$$

$$S^* = b\sum_{1}^{q_1}(B_j)+\mu\sum_{1}^{p_1'}(A_j')+b\sum_{1}^{q_3}(F_j)+\mu\sum_{1}^{p_3'}(E_j')$$

$$-\ b\sum_{1}^{p_1}(A_j)-\mu\sum_{1}^{q_1'}(B_j')-b\sum_{1}^{p_3}(E_j)-\mu\sum_{1}^{q_3'}(F_j')$$

$$+\ \mu\sum_{l=1}^{N-1}\left\{\sum_{1}^{q_1''}(B_j^{(l)})+\sum_{1}^{q_3''}(F_j^{(l)})-\sum_{1}^{p_1''}(A_j^{(l)})-\sum_{1}^{p_3''}(E_j^{(l)})\right\},$$

$$T^* = -\ a\sum_{1}^{q_1}(\beta_j)-\sum_{1}^{p_1'}(\alpha_j')-a\sum_{1}^{p_1}(\alpha_j)-\lambda\sum_{1}^{q_1'}(\beta_j')$$

$$+\ a\sum_{1}^{n_2}(\gamma_j)-a\sum_{n_2+1}^{p_2}(\gamma_j)+\lambda\sum_{1}^{m_2'}(\delta_j')-\lambda\sum_{m_2'+1}^{q_2'}(\delta_j')$$

$$+\ a\sum_{1}^{m_2}(\delta_j)-a\sum_{m_2+1}^{q_2}(\delta_j)+\lambda\sum_{1}^{n_2'}(\gamma_j')-\lambda\sum_{n_2'+1}^{p_2'}(\gamma_j')$$

$$+\ \lambda\sum_{l=1}^{N-1}\left\{-\sum_{1}^{q_1''}(\beta_j^{(l)})-\sum_{1}^{p_1''}(\alpha_j^{(l)})+\sum_{1}^{n_2''}(\gamma_j^{(l)})\right.$$

$$\left.-\sum_{n_2''+1}^{p_2''}(\gamma_j^{(l)})+\sum_{1}^{m_2''}(\delta_j^{(l)})-\sum_{m_2''+1}^{q_2''}(\delta_j^{(l)})\right\},$$

and

$$W^* = -\ b\sum_{1}^{q_1}(B_j)-\mu\sum_{1}^{p_1'}(A_j')-b\sum_{1}^{p_1}(A_j)-\mu\sum_{1}^{q_1'}(B_j')$$

$$+\ b\sum_{1}^{n_3}(E_j)-b\sum_{n_3+1}^{p_3}(E_j)+\mu\sum_{1}^{m_3'}(F_j')-\mu\sum_{m_3'+1}^{q_3'}(F_j')$$

$$+\ b\sum_{1}^{m_3}(F_j)-b\sum_{m_3+1}^{q_3}(F_j)+\mu\sum_{1}^{n_3'}(E_j')-\mu\sum_{n_3'+1}^{p_3'}(E_j')$$

$$+ \mu \sum_{l=1}^{N-1} \left\{ -\sum_{1}^{q_1''} (B_j^{(l)}) - \sum_{1}^{p_1''} (A_j^{(l)}) + \sum_{1}^{n_3''} (E_j^{(l)}) \right.$$

$$\left. - \sum_{n_3''+1}^{p_3''} (E_j^{(l)}) + \sum_{1}^{m''} (F_j^{(l)}) - \sum_{m_3''+1}^{q_3''} (F_j^{(l)}) \right] .$$

$$Re \left\{ \sum_{r=1}^{N} u_r + \frac{\lambda}{a} \left(\frac{1-c_i}{\gamma_i} \right) + \frac{d_j'}{\delta_j'} + \frac{d_k^{(N)}}{\delta_k^{(N)}} \right.$$

$$\left. + \sum_{l=1}^{N-1} \left(\frac{1-c_h^{(l)}}{\gamma_h^{(l)}} - \left\{ \sum_{r=1}^{l} u_r - A + 1 \right\} \right) + 2 \right] > 0$$

$(i=1, \dots, n_2 \; ; \; j=1, \dots, m_2' \; ; \; k=1, \dots, m_2'' \; ; \; h=1, \dots, n_2'')$.

$$Re \left\{ \sum_{r=1}^{N} v_r + \frac{\mu}{q} \left(\frac{1-e_i}{E_i} \right) + \frac{f_j'}{F_j'} + \frac{f_k^{(N)}}{F_k^{(N)}} \right.$$

$$\left. + \sum_{l=1}^{N-1} \left(\frac{1-c_h^{(l)}}{E_h^{(l)}} - \left\{ \sum_{r=1}^{l} v_r - B + 1 \right\} \right) + 2 \right] > 0$$

$(i=1, \dots, n_3 \; ; \; j=1, \dots, m_3' \; ; \; k=1, \dots, m_3'' \; ; \; h=1, \dots, n_3'')$

$$Re \left\{ \sum_{r=1}^{N} u_r - \frac{\lambda}{a} \left(\frac{d_i}{\delta_i} \right) + \frac{c_j'-1}{\gamma_j'} + \frac{c_k^{(N)}-1}{\gamma_k^{(N)}} \right.$$

$$\left. - \sum_{l=1}^{N-1} \left(\frac{d_h^{(l)}}{\delta_h^{(l)}} + \left\{ \sum_{r=1}^{l} u_r - A + 1 \right\} \right) + 2 \right] < 0$$

$(i=1, \dots, m_2 \; ; \; j=1, \dots, n_2' \; ; \; k=1, \dots, n_2'' \; ; \; h=1, \dots, m_2'')$

$$Re\left[\sum_{r=1}^{N} v_r - \frac{\mu}{b}\left(\frac{f_i}{F_i}\right) + \frac{e_j'-1}{E_j'} + \frac{e_k^{(N)}-1}{E_k^{(N)}}\right.$$

$$\left. - \sum_{l=1}^{N-1}\left(\frac{f_h^{(l)}}{F_h^{(l)}} + \left\{\sum_{r=1}^{l} v_r - B+1\right\}\right)+2\right] < 0$$

$(i=1, \dots, m_3\ ;\ j=1, \dots, n_3'\ ;\ k=1, \dots, n_3''\ ;\ h=1, \dots, m_3'')$.

$$Re\left[\rho + c + \frac{d_i}{\delta_i} + a/\lambda\left(\frac{1-c_j'}{\gamma_j'}\right) - a/\lambda\,(A+1) + a/\lambda\left(\frac{d_k^{(N-1)}}{\delta_k^{(N-1)}}\right)\right.$$

$$\left. + a/\lambda\left(\sum_{r=1}^{N-1} u_r - A+1\right) + 1\right] > 0$$

$(i=1, \dots, m_2\ ;\ j=1, \dots, n_2'\ ;\ k=1, \dots, m_2'')$.

$$Re\left[\rho + d + \frac{f_i}{F_i} + b/\mu\left(\frac{1-e'}{E_j'}\right) - b/\mu(B+1) + b/\mu\left(\frac{f_k^{(N-1)}}{F_k^{(N-1)}}\right)\right.$$

$$\left. + b/\mu\left(\sum_{r=1}^{N-1} v_r - B+1\right) + 1\right] > 0$$

$(i=1, \dots, m_3\ ;\ j=1, \dots, n_3'\ ;\ k=1, \dots, m_3'')$.

$$Re\left[\rho + c + \frac{d_i}{\delta_i} + a/\lambda\left(\frac{1-c_j'}{\gamma_j'}\right) - a/\lambda(A+1) + a/\lambda\left(\frac{d_k^{(N)}}{\delta_k^{(N)}}\right)\right.$$

$$\left. + a/\lambda\left(\sum_{r=1}^{N} u_r - A+1\right) + 1\right] > 0$$

$(i=1, \ldots, m_2 ; j=1, \ldots, n_2' ; k=1, \ldots, m_2'')$

$$Re\left[\sigma + d + \frac{f_i}{F_i} + b/\mu\left(\frac{1-e_j'}{E_j'}\right) - b/\mu\,(B+1) + b/\mu\left(\frac{f_k^{(N)}}{F_k^{(N)}}\right) + b/\mu\left(\sum_{r=1}^{N} v_r - B + 1\right) + 1\right] > 0$$

$(i=1, \ldots, m_3 ; j=1, \ldots, n_3' ; k=1, \ldots, m_3'')$.

Proof :

Substituting the value of $\theta_1\,(1/x, 1/y)$ from (1.9) in (2.4), interchanging the order of the (s, t)-integrals and (x, y)-integrals (which is permissible since the (s, t)-integrals, (x, y)-integrals and the resulting integral are all absolutely convergent under the conditions stated with the main theorem) , we get

$$(2.8)\quad \theta_2(p, q) = abpq \int_0^\infty \int_0^\infty s^\rho t^\sigma h(s, t) \left\{ \int_0^\infty \int_0^\infty x^{u_1+u_2-A}\, y^{v_1+v_2-B} \right.$$

$$H_1^{(2)}[px, qy]\; H^{0,\,0}_{q_1+p_1'+q_1'',\, p_1+q_1'+p_1''}$$

$$: n_2+m_2'+n_2'' , m_2+n_2'+m_2'' ; n_3+m_3'+n_3'' , m_3+n_3'+m_3''$$

$$: q_2+p_2'+q_2'' , p_2+q_2'+p_2'' ; q_3+p_3'+q_3'' , p_3+q_3'+p_3''$$

$$\left. \left[\begin{array}{c|c} s^{-\alpha} x^\lambda & I^{(1)} : K^{(1)} ; P^{(1)} \\ t^{-\beta} y^\mu & J^{(1)} : L^{(1)} ; Q^{(1)} \end{array}\right] dx\, dy \right\} ds\, dt ,$$

where $I^{(1)}$, $J^{(1)}$, $K^{(1)}$, $L^{(1)}$, $P^{(1)}$ and $Q^{(1)}$ stand for the quantities mentioned above in (1.9) .

Now, evaluating the (x, y)-integrals involved in (2.8) with the help of a known result [4, p. 98, Eq. (2.2.1)], we obtain the following result :

$$(2.9)\quad \theta_2(p, q) = ab\, p^{A-u_1-u_2}\, q^{B-v_1-v_2} \int_0^\infty \int_0^\infty s^\rho t^\sigma h(s, t)$$

$$H^{0,\,0}_{q_1+p_1'+2q_1'',\; p_1+q_1'+2p_1''} {}^{:\; n_2+m_2'+2n_2'',\; m_2+n_2'+2m_2'';}_{:\; q_2+p_2'+2q_2'',\; p_2+q_2'+2p_2'';}$$

$$ {}^{n_3+m_3'+2n_3'',\; m_3+n_3'+2m_3''}_{q_3+p_3'+2q_3'',\; p_3+q_3'+2p_3''} \left[\begin{matrix} s^{-a}\, p^{-\lambda} \\ t^{-b}\, q^{-\mu} \end{matrix} \middle| \begin{matrix} I^{(2)} : K^{(2)} ; P^{(2)} \\ J^{(2)} : L^{(2)} ; Q^{(2)} \end{matrix}\right] ds\, dt$$

where $I^{(2)}$, $J^{(2)}$, $K^{(2)}$, $L^{(2)}$, $P^{(2)}$ and $Q^{(2)}$ stand for the same quantities as $I^{(N)}$, $J^{(N)}$, $K^{(N)}$, $L^{(N)}$, $P^{(N)}$ and $Q^{(N)}$, respectively, in (2.7) when $N=2$.

Repeating this process successively with the correspondences (2.5) , ... , we arrive at the required result stated above.

3. SPECIAL CASES OF THE MAIN THEOREM

If we reduce all the H–function transform of two variables in the theorem to Nigam's transform defined by (1.3), as indicated above, we get the following interesting chain :

COROLLARY 1. If

$$(3.1)\quad f(p, q) = W\left[x^{\rho'} y^{\sigma'} \phi(x^\lambda, y^\mu) : \begin{matrix} w', k', r' \\ w_1', k_1', r_1' \end{matrix} ; p, q \right]$$

$$(3.2)\quad p^\xi q^\eta \phi(p^a, q^b) = W\left[x^\rho y^\sigma h(x, y) ; \begin{matrix} w, k, r \\ w_1, k_1, r_1 \end{matrix} ; p, q \right]$$

$$(3.3)\quad \theta_1(p, q) = W\left[x^{u_1} y^{v_1} f(x,y);\ \begin{matrix} w^{(1)}, k^{(1)}, r^{(1)} \\ w^{(1)}, k_1^{(1)}, r_1^{(1)} \end{matrix};\ p, q\right]$$

$$(3.4)\quad \theta_2(p, q) = W\left[x^{u_2} y^{v_2} \theta_1(1/x, 1/y);\ \begin{matrix} w^{(2)}, k^{(2)}, r^{(2)} \\ w_1^{(2)}, k_1^{(2)}, r_1^{(2)} \end{matrix};\ p, q\right]$$

$$(3.5)\quad \theta_N(p, q) = W\left[x^{u_N} y^{v_N} \theta_{N-1}(1/x, 1/y);\ \begin{matrix} w^{(N)}, k^{(N)}, r^{(N)} \\ w_1^{(N)}, k_1^{(N)}, r_1^{(N)} \end{matrix};\ p, q\right]$$

then

$$(3.6)\quad \theta_N(p, q) = a\, b\, p^{A-\sum_{i=1}^{N} u_i}\, q^{B-\sum_{i=1}^{N} v_i} \int_0^\infty \int_0^\infty s^{\rho} t^{\sigma} h(s, t)$$

$$H^{2,\,2+2N}_{3+2N,\,3+N}\left[s^{-a} p^{-\lambda} \left| \begin{matrix} (1\pm r+w, a), \{(w^{(M)} \pm r^{(M)} - \sum_{i=1}^{N} u_i + A, \lambda)\}_{M=1}^{N}, \\ (A-k'-w'+1, \lambda) \\ (A\pm r'-w'+1, \lambda), \{(k^{(M)} + w^{(M)} - \sum_{i=1}^{M} u_i + A, \lambda)\}_{M=1}^{N}, \\ (k+w+1, a) \end{matrix} \right. \right]$$

$$H^{2,\,2+2N}_{3+2N,\,3+N}\left[t^{-b} q^{-\mu} \left| \begin{matrix} (1\pm r_1+w_1, b), \{(w_1^{(M)} \pm r_1^{(M)} - \sum_{i=1}^{M} v_i + B, \mu)\}_{M=1}^{N}, \\ (B\pm k_1'-w_1'+1, \mu) \\ (B\pm r_1'-w_1'+1, \mu), \{(k_1^{(M)} + w_1^{(M)} - \sum_{i=1}^{M} v_i + B, \mu)\}_{M=1}^{N}, \\ (k_1+w_1+1, b) \end{matrix} \right. \right] ds\, dt,$$

provided that the conditions easily obtainable from the main theorem are satisfied.

If we reduce all the Nigam's transforms of two variables occurring in Corollary 1 to the corresponding transforms involving one variable, we arrive at the chain involving Mainra transforms obtained earlier by Jain [5, p. 129].

If we reduce all H-function transforms of two variables involved in the theorem to the double Meijer-Bessel function transform defined by (1.4), as indicated above, we get the following interesting chain of double Meijer-Bessel function transforms :

COROLLARY 2. If

(3.7) $f(p, q) = K[\, x^{\rho'} y^{\sigma'} \phi(x^{\lambda}, y^{\mu})\,;\ \varepsilon', \nu'\,;\ p, q\,]$

(3.8) $p^{\xi} q^{\eta} \phi(p^{a}, q^{b}) = K[\, x^{\rho} y^{\sigma} h(x, y)\,;\ \varepsilon, \nu;\ p, q\,]$

(3.9) $\theta_1(p, q) = K\left[\, x^{u_1} y^{v_1} f(x, y);\ \varepsilon^{(1)}, \nu^{(1)};\ p, q\,\right]$

(3.10) $\theta_2(p, q) = K\left[\, x^{u_2} y^{v_2} \theta_1(1/x, 1/y)\,;\ \varepsilon^{(2)}, \nu^{(2)}\,;\ p, q\,\right]$

(3.11) $\theta_N(p, q) = K\left[\, x^{u^{(N)}} y^{v^{(N)}} \theta_{N-1}(1/x, 1/y);\ \varepsilon^{(N)}, \nu^{(N)};\ p, q\,\right]$

then

$$\theta_N(p, q) = a\, b\, p^{A - \sum_{i=1}^{N} u_i}\, q^{B - \sum_{i=1}^{N} v_i} \int_0^{\infty}\int_0^{\infty} s^{\rho} t^{\sigma} h(s, t)$$

$$. H^{2,\, 2+2N}_{2+2N,\, 2}\left[s^{-a} (p/2)^{-\lambda} \,\middle|\, \begin{matrix} \left(\frac{3}{4} \pm \frac{\varepsilon}{2}, \frac{1}{2}\right); \left\{\left(\frac{3}{4} \pm \frac{\varepsilon^{(M)}}{2} - \frac{1}{2}\left\{ \sum_{i=1}^{M} u_i - A + 1 \right\}, \frac{\lambda}{2}\right)\right\}_{M=1}^{N} \\ \left(\frac{1}{4} \pm \frac{\varepsilon'}{2} + \frac{1}{2}\{A+1\}, \frac{\lambda}{2}\right) \end{matrix} \right]$$

$$.H^{2,\,2+2N}_{2+2N,\,2}\left[t^{-b}(q/2)^{-\mu} \middle| \begin{matrix} \left(\frac{3}{4}\pm\frac{\nu}{2},\frac{1}{2}\right)\ \left\{\left(\frac{3}{4}\pm\frac{\nu^{(M)}}{2}-\frac{1}{2}\left\{\sum_{i=1}^{M} v_i-B+1\right\},\frac{\mu}{2}\right)\right\}_{M=1}^{N} \\ \left(\frac{1}{4}\pm\frac{\nu'}{2}+\frac{1}{2}\{B+1\},\frac{\mu}{2}\right) \end{matrix} \right] ds\, dt,$$

provided that the conditions easily obtainable from our main theorem are satisfied.

If we reduce all H-function transforms of two variables involved in the theorem to the Laplace transform of two variables defined by (1.5), as indicated above, we get the following interesting chain involving the well-known Laplace transform of two variables :

COROLLARY 3. If

(3.13) $\quad f(p,q) = L\left[x^{\rho'} y^{\sigma'} \phi\left(x^{\lambda}, y^{\mu}\right) ;\ p, q \right]$

(3.14) $\quad p^{\xi} q^{\eta} \phi\left(p^{a}, q^{b}\right) = L\left[x^{\rho} y^{\sigma} h(x,y) ;\ p, q \right]$

(3.15) $\quad \theta_1(p,q) = L\left[x^{u_1} y^{v_1} f(x,y) ;\ p, q \right]$

(3.16) $\quad \theta_2(p,q) = L\left[x^{u_2} y^{v_2} \theta_1(1/x, 1/y) ;\ p, q \right]$

$\quad \cdots \qquad \cdots \qquad \cdots$

(3.17) $\quad \theta_N(p,q) = L\left[x^{u_N} y^{v_N} \theta_{N-1}(1/x, 1/y) ;\ p, q \right]$

then

(3.18) $$\theta_N(p\ q) = ab\, p^{A-\sum_{i=1}^{N} u_i}\, q^{B-\sum_{i=1}^{N}} \int_0^{\infty}\int_0^{\infty} s^{\rho} t^{\sigma} h(s,t)$$

$$H^{1,N}_{N+1,1}\left[s^{-\alpha} p^{-\lambda} \middle| \begin{matrix} (1, a), \left\{ (A- \sum_{i=1}^{M} u_i , \lambda) \right\}_{M=1}^{N} \\ (A+1, \lambda) \end{matrix} \right]$$

$$H^{1,N}_{N+1,1}\left[t^{-b} q^{-\mu} \middle| \begin{matrix} (1, b), \left\{ (B- \sum_{i=1}^{M} v_i , \mu) \right\}_{M=1}^{N} \\ (B+1, \mu) \end{matrix} \right] ds\, dt,$$

provided that the conditions easily obtainable from the main theorem are satisfied.

Similarly, by suitably specialising the parameters of the *H*–function transforms of two variables, occurring in the main theorem, to the corresponding Bhise transform [1, p. 57] involving one variable, we get the chain involving *G*–function transforms obtained earlier by Gupta and Mittal [3, p. 106] .

ACKNOWLEDGEMENTS

The author is extremely grateful to Dr. K. C. Gupta for his kind help and guidance in the preparation of this paper. He is also highly grateful to Professor H. M. Srivastava for his very valuable suggestions for the improvement of this paper.

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