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A COMPLEX INVERSION FORMULA AND A TAUBERIAN THEOREM FOR THE GENERALIZED WHITTAKER TRANSFORM

by

SUSHIL KUMAR SINHA
Qtrs. No. H/45, Hinoo, Ranchi-834002
Bihar, India

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ABSTRACT

The author establishes a complex inversion formula expressing the α-th sum of the determining function of the integral

$$(*) \qquad \int_0^\infty (pt)^{\sigma-1/2}\, e^{-qpt/2}\, W_{k,m}(\rho pt)\, dA(t)$$

in terms of an integral involving the function represented by (*), when (*) is summable (C,α). He also gives a Tauberian theorem for this integral transform which was introduced earlier by H.M. Srivastava [3].

1. INTRODUCTION

Over a decade ago, Srivastava [3] gave a generalization of the Laplace-Stieltjes integral

$$(1.1) \qquad f(p) = \int_0^\infty e^{-pt}\, dA(t)$$

in the from :

$$(1.2) \qquad f(p) = \int_0^\infty (pt)^{\sigma-1/2}\, e^{-qpt/2}\, W_{k,m}(\rho pt)\, dA(t),$$

where $W_{k,m}$ is the Whittaker function [2], and $A(t)$ is a function of bounded variation in any finite interval $0 \leqslant t \leqslant T$. When $\sigma=m$ and $\rho=q=1$, the transform (1.2) reduces to the Varma transform [5], *viz*

$$(1.3)\quad f(p) = \int_0^\infty (pt)^{m-1/2} e^{-pt/2} W_{k,m}(pt)\, dA(t).$$

When $k+m=\frac{1}{2}$, (1.3) reduces to (1.1). { for a number of other known special cases of the Srivastava transform (1.2), see [3, p.385] . }

It is also well known that if $A(x) = \sum_{\lambda_n \leqslant x} a_n$,

where $0 < \lambda_0 < \lambda_1 < \ldots$ and $\lambda_n \to \infty$, (1.1) reduces to Dirichlet's series

$$(1.4)\quad f(p) = \Sigma a_n\, e^{-\lambda_n p}$$

The purpose of the present paper is to obtain a complex inversion formula expressing the α-th sum of the determining function of the integral (1.2) in term of an integral involving functions represented by (1.2), when (1.2) is summable (C, α). We also deduce a Tauberian theorem for (1.2) on the lines of the proof given for a Tauberian theorem of (1.4) by Cnandrasekharan and Minakshisundaram [4].

2. NOTATIONS

We introduce the following notations :

(i) $\phi_{k,m}(pt) = (pt)^{\sigma-1/2} e^{-q pt/2} W_{k,m}(\rho pt)$.

(ii) $[\alpha]$ denotes the integral part of α. If $\alpha > 1$, we denote by h the greatest integer less then α ; if $0 < \alpha \leqslant 1$, then $h=0$, and if $\alpha = 0$, then $h = -1$.

(iii) In (1.2), we shall take k, m, σ, q and ρ to be real.

(iv) ξ and η will stand for the real and imaginary parts of complex number p, $p = \xi + i\eta$.

(v) $A_\alpha(x)$ denotes α-th Cesàro sum of $A(x)$.

(vi) $\bar{A}(x) = \int_0^x t\, dA(t)$

(vii) $\bar{A}_\alpha(x) = \frac{1}{\Gamma(\alpha+1)} \int_0^\infty (x-t)^\alpha t\, dA(t), \ (\alpha \geqslant 1)$

$$= -\frac{1}{\Gamma(\alpha+1)} \int_0^x A(t)\ (d/dt)\ [(x-t)^\alpha t]\ dt,\ (\alpha > 1).$$

Thus, if $\alpha \geqslant 1$, we have

$$\bar{A}_\alpha(x) = \frac{1}{\Gamma(\alpha+1)} \int_0^x (x-t)^\alpha\ [x-(x-t)]\ dA(t)$$

$$= x\, A_\alpha(x) - (\alpha+1)\, A_{\alpha+1}(x).$$

3. A COMPLEX INVERSION FORMULA

Theorem 3.1 *The α-th Cesàro sum $A_\alpha(t)$ of the determining function $A(t)$ of the integral (1.2) is given by the formula*

$$A_\alpha(t) = \frac{1}{2\pi i} \lim_{d \to \infty} \int_{c-id}^{c+id} \frac{\Gamma(m-k-z+3/2)}{\Gamma(2m-z-\alpha)\ \Gamma(1-z)} \cdot t^{-z-\alpha-1}\ \psi(z+\alpha+1)\ dz,$$

where

$$\psi(z) = \int_0^\infty p^{-z} f(p)\ dp,$$

provided that :

(i) the integral (1.2) is summable (C, α) for $\xi \geqslant \xi_0 > 0$ and $c > \xi_0$,

(ii) $\int_0^\infty x^{-z-\alpha-1} f(x) dx$ convrges absolutely, $(z=c+id, -\infty < d < \infty)$

(iii) $\int_0^\infty x^{c-1} A_\alpha(x) dx$ converges absolutely, and

(iv) $\mathrm{Re}\,(2m-z-\alpha) > 0$, $\mathrm{Re}\,(1-z) > 0$ and $m-(\alpha+1)/2 > 0$.

Proof : From Condition (i), we have for $\xi > \xi_0$.

$$(3.1)\quad f(p) = p^{\alpha+1} \int_0^\infty \phi_{k+(\alpha+1)/2,\ m-(\alpha+1)/2}(pt)\, A_\alpha(t)dt,$$

where, we may assume that p is real. Now multiplying both sides of (3.1) by $p^{-z-\alpha-1}$ and integrating with respect to p from 0 to ∞, we have

$$(3.2)\quad \int_0^\infty p^{-z-\alpha-1} f(p)dp$$

$$= \int_0^\infty p^{-z}dp \int_0^\infty \phi_{k+(\alpha+1)/2,\ m-(\alpha+1)/2}(pt)\, A_\alpha(t)dt$$

$$= \int_0^\infty A_\alpha(t)dt \int_0^\infty p^{-z}\, \phi_{k+(\alpha+1)/2,\ m-(\alpha+1)/2}(pt)dp$$

$$= \frac{\Gamma(2m-z-\alpha)\ \Gamma(1-z)}{\Gamma(m-k-z+3/2)} \int_0^\infty t^{z-1} A_\alpha(t)dt,$$

where we have evaluated the p–integral by means of Goldstein's formula [6].

On applying Mellin's complex inversion formula [1, p. 46], we have

$$A_\alpha(t) = \frac{\Gamma(m-k-z+3/2)}{\Gamma(2m-z-\alpha)\ \Gamma(1-z)} \frac{1}{2\pi i} \lim_{d\to\infty} \int_{c-id}^{c+id} t^{-z}\psi(z+\alpha+1)dz$$

Since $A_\alpha(t)$ is absolutely continuous, provided that the integrals

$$\int_0^\infty x^{c-1} A_\alpha(x)dx \quad \text{and} \int_0^\infty x^{-z-\alpha-1} f(x)dx,$$

converges absolutely in order to justify the change in the order of integration, we first note that the integral

$$\int_0^\infty \phi_{k+(\alpha+1)/2,\ m-(\alpha+1)/2}(pt)\ A_\alpha(t)dt$$

and

$$\int_0^\infty p^{-z} \phi_{k+(\alpha+1)/2\,,\, m-(\alpha+1)/2}(pt)dp$$

exists, the former due to the condition (i) and latter due to the first two inequalities of the condition (iv).

Also

$$\int_0^\infty |A_\alpha(t)| dt \int_0^\infty |p^{-z}| \quad \phi_{k+(\alpha+1)/2\,,\, m-(\alpha+1)/2}(pt) \,|\, dp$$

$$= \int_0^\infty |A_\alpha(t)| \; dt \left| \int_0^\infty p^{-c} \phi_{k+(\alpha+1)/2, m-(\alpha+1)/2}(pt)dp \right|$$

$$= \frac{\Gamma(2m-c-\alpha) \quad \Gamma(1-c)}{\Gamma(m-k-c+3/2)} \int_0^\infty t^{c-1} \,|\, A_\alpha(t) \,|\, dt.$$

The right-hand side exists by virtue of Conditions (iii) and (iv). Hence the change of the order of integration is justified by Fubini's theorem. Thus Theorem 3.1 is completely established.

4. A TAUBERIAN THEOREM

Theorem 4.1 *Let the integral (1.2) with $m > 0$ converge to the function $f(\xi)$, for $p = \xi$, where ξ is positive and real, and $f(0+)$ exist. Then a necessary and sufficient condition that $\lim_{x \to \infty} A(x)$ should exist is that*

(4.1) $\quad \bar{A}(x) = \int_0^x t dA(t) = O(x), \; (x \to \infty).$

Proof : The necessary condition follows from the identity

(4.2) $\quad x^{-1} \bar{A}(x) = A(x) - x^{-1} A_1(x).$

For, if $\lim_{x \to \infty} A(x)$ exists, then $A(x)$ and $x^{-1} A_1(x)$ eonverge to the same limit and their difference tends to zero.

For the sufficiency part, we get

$$C_1(x) = x^{-1} A_1(x) = \int_0^x t^{-2} \bar{A}(t)dt,$$

and

$$f_1(\xi) = \xi \int_0^\infty C_1(t)\, \phi_{k+1/2,m-1/2}(\xi t)dt$$

$$= \xi \int_\xi^\infty u^{-2} f(u)du,$$

where

$$f(u) = \int_0^\infty \phi_{k,m}(ut)\, dA(t) = u \int_0^\infty \phi_{k+1/2,m-1/2}(ut)\, A(t)dt$$

$$= u^2 \int_0^\infty \phi_{k+1,\, m-1}(ut)\, A_1(t)dt.$$

Hence

$$\xi \int_\xi^\infty u^{-2} f(u)du = \xi \int_\xi^\infty du \int_0^\infty \phi_{k+1,m-1}(ut)\, A_1(t)dt$$

$$= \xi \int_0^\infty A_1(t)\, dt \int_\xi^\infty \phi_{k+1,m-1}(ut)\, du, \text{ by Fubini's theorem,}$$

$$= \xi \int_0^\infty t^{-1} A_1(t)\, \phi_{k+1/2,m-1/2}(\xi t)\, dt$$

$$= \xi \int_0^\infty C_1(t)\, \phi_{k+1/2,m-1/2}(\xi t)\, dt.$$

Now since $f(0+)$ exists, it follows that $f_1(0+)$ also exists and $f_1(0+) = f(0+)$.

Further

$$f_1(\xi) = \xi \int_0^\infty C_1(t)\, \phi_{k+1/2,m-1/2}(\xi t)\, dt$$

$$= \xi \int_0^\infty \phi_{k+1/2,m-1/2}(\xi t)\, dt \int^t u^{-2} \bar{A}(u)\, du$$

$$= \xi \int_0^\infty u^{-2} \bar{A}(u)\, du \int_u^\infty \phi_{k+1/2, m-1/2} (\xi t)\, dt$$

$$= \int_0^\infty \phi_{k,m} (\xi u)\, u^{-2} \bar{A} (u)\, du.$$

Now

$$[\Gamma (2m) / \Gamma (m-k+\tfrac{1}{2}]\, C_1 (x) - f_1 (\xi)$$

$$= [\Gamma (2m) / \Gamma (m-k+\tfrac{1}{2})] \int_0^x t^{-2} \bar{A}(t)\, dt - \int_0^\infty t^{-2} \phi_{k,m} (\xi t)\, \bar{A} (t)\, dt$$

$$= \int_0^x [\{\Gamma(2m) / \Gamma(m-k+\tfrac{1}{2})\} - \phi_{k,m} (\xi t)]\, t^{-2} \bar{A} (t)\, dt$$

$$- \int_x^\infty t^{-2} \phi_{k,m} (\xi t)\, \bar{A} (t)\, dt$$

(4.3) $= I_1 - I_2$ (say).

On using the inequality given by Singh ([7], Lemma 5.1), viz

$$[\Gamma (2m) / \Gamma (m-k+\tfrac{1}{2}] - \phi_{k,m} (x)$$

$$\leqslant x [\Gamma (2m-1) / \Gamma (m-k-\tfrac{1}{2})], \ (m \geqslant 0,\ x > 0),$$

We have

$$| I_1 | \leqslant \int_0^x | \{\Gamma (2m) / \Gamma (m-k+\tfrac{1}{2})\} - \phi_{k,m} (\xi t) |\, t^{-2} \bar{A}(t) |\, dt$$

$$\leqslant [\Gamma (2m-1) / \Gamma (m-k-\tfrac{1}{2})]\, \xi \int_0^x t^{-1} | \bar{A} (t) |\, dt$$

$$= O [\xi \int_0^x | \bar{A} (t) |\, t^{-1} dt] = o (\xi x)$$

Also

$$| I_2 | = \int_x^\infty | \phi_{k,m} (\xi t) |\, t^{-2} \bar{A}(t)\, dt$$

$$= \xi \int_0^{\infty} u^{-2}\,\bar{A}(u)\,du \int_u^{\infty} \phi_{k+1/2,m-1/2}(\xi t)\,dt$$

$$= \int_0^{\infty} \phi_{k,m}(\xi u)\,u^{-2}\,\bar{A}(u)\,du.$$

Now

$$[\,\Gamma(2m) / \Gamma(m-k+\tfrac{1}{2}\,]\,C_1(x) - f_1(\xi)$$

$$= [\,\Gamma(2m) / \Gamma(m-k+\tfrac{1}{2})\,] \int_0^{x} t^{-2}\,\bar{A}(t)\,dt - \int_0^{\infty} t^{-2}\,\phi_{k,m}(\xi t)\,\bar{A}(t)\,dt$$

$$= \int_0^{x} [\,\{\Gamma(2m) / \Gamma(m-k+\tfrac{1}{2})\} - \phi_{k,m}(\xi t)\,]\,t^{-2}\,\bar{A}(t)\,dt$$

$$- \int_x^{\infty} t^{-2}\,\phi_{k,m}(\xi t)\,\bar{A}(t)\,dt$$

(4.3) $= I_1 - I_2$ (say).

On using the inequality given by Singh ([7], Lemma 5.1), viz

$$[\,\Gamma(2m) / \Gamma(m-k+\tfrac{1}{2}\,] - \phi_{k,m}(x)$$

$$\leqslant x\,[\,\Gamma(2m-1) / \Gamma(m-k-\tfrac{1}{2})\,],\ (m \geqslant 0,\ x > 0),$$

We have

$$|\,I_1\,| \leqslant \int_0^{x} |\,\{\Gamma(2m) / \Gamma(m-k+\tfrac{1}{2})\} - \phi_{k,m}(\xi t)\,|\,t^{-2}\,\bar{A}(t)\,|\,dt$$

$$\leqslant [\,\Gamma(2m-1) / \Gamma(m-k-\tfrac{1}{2})\,]\,\xi \int_0^{x} t^{-1}\,|\,\bar{A}(t)\,|\,dt$$

$$= O\,[\,\xi \int_0^{x} |\,\bar{A}(t)\,|\,t^{-1}\,dt\,] = o(\xi x)$$

Also

$$|\,I_2\,| = \int_x^{\infty} |\,\phi_{k,m}(\xi t)\,|\,t^{-2}\,\bar{A}(t)\,dt$$

$$= \int_x^\infty O\left[\rho^k (\xi t)^{k+\sigma-1/2} e^{-(1/2)\xi t(\eta+\rho)} \right] t^{-2} \, | \bar{A}(t) | \, dt, \ (t \to \infty).$$

$$= \int_x^\infty O\left[\rho^k (\xi t)^{k+\sigma-3/2} e^{-(1/2)\xi t(\eta+\rho)} \right] dt, \ (t \to \infty).$$

$$= O\left[\rho^k (\xi x)^{k+\sigma-3/2} e^{-(1/2)\xi x(\eta+\rho)} \right], \ (x \to \infty),$$

$$= O(1), \ (x \to \infty).$$

Finally, let us choose $\xi x = 1$. This adjustment is possible because ξ has to tend to zero to give our result. This gives

(4.4) $|I_1| = o(1), \ (x \to \infty);$

and

(4.5) $|I_2| = o(1), \ (x \to \infty).$

On applying (4.4) and (4.5) in (4.3), we get

$$[\Gamma(2m) / \Gamma(m-k+\tfrac{1}{2})] \, C_1(x) - f_1(\xi) = o(1), \ (x \to \infty, \ \xi x = 1).$$

This result together with (4.1), when applied to (4.2), shows that $\lim_{x \to \infty} A(x)$ exists and, in particular,

$$\lim_{x \to \infty} A(x) = [\Gamma(m-k+\tfrac{1}{2}) / \Gamma(2m)] \, f_1(0+)$$

$$= [\Gamma(m-k+\tfrac{1}{2}) / \Gamma(2m)] \, f(0+).$$

Thus Theorem 4.1 is completely established.

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