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SOME INTEGRALS INVOLVING THE MULTIVARIABLE H- FUNCTION*

by

R. S. GARG

Department of Mathematics, College of Arts & Science, Banasthali Vidyapith - 304022, Rajasthan, India

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ABSTRACT

In this paper we evaluate one single and two multiple integrals involving the products of hypergeometric functions, Jacobi polynomials, and the multivariable H- function, which was introduced and studied in a series of papers by *H. M. Srivastava* and *R. Panda.* The results are believed to be new and are quite general in character. On specialization of the parameters, a number of interesting integrals can be deduced as particular cases of our main results. Some special cases are also discussed briefly.

1. INTRODUCTION

The H- function of several complex variables (or the multivariable H- function) was introduced and studied earlier in a series of papers by Srivastava and Panda (see, e. g., [6], [7] and [8]). The parameters of this function will be displayed here in the following contracted notation which is due essentially to Srivastava and Panda ([6], p. 130, Eq. (1.3)*et seq.*) :

$$(1.1)\quad H\left[x_1,\ldots, x_r \right] \equiv H^{0,\, n\ :\ m_1,\, n_1;\ldots;\, m_r,\, n_r}_{p,\, q\ :\ p_1,\, q_1;\ldots;\, p_r,\, q_r}\left[\begin{matrix} x_1 \\ \vdots \\ x_r \end{matrix}\right]$$

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$$\left. \begin{matrix} (a_j ; a'_j , \ldots , a^{(r)}_j)_{1,p} : (c'_j , \varepsilon'_j)_{1,p_1} ; \ldots ; (c^{(r)}_j , \varepsilon^{(r)}_j)_{1,p_r} \\ (b_j ; \beta'_j , \ldots , \beta^{(r)}_j)_{1,q} : (d'_j , \delta'_j)_{1,q_1} ; \ldots ; (d^{(r)}_j , \delta^{(r)}_j)_{1,q_r} \end{matrix} \right]$$

$$= \frac{1}{(2\pi w)^r} \int_{L_1} \ldots \int_{L_r} \phi \, (s_1, \ldots , s_r) \prod_{i=1}^{r} \left\{ \theta_i \, (s_i) \, (x_i)^{s_i} \, ds_i \right\},$$

where $w = \sqrt{-1}$,

(1.2) $$\phi \, (s_1, \ldots , s_r) = \prod_{j=1}^{n} \left(1-a_j + \sum_{i=1}^{r} \alpha^{(i)}_j s_i \right)$$

$$\cdot \left[\prod_{j=1}^{q} \Gamma \left(1-b_j + \sum_{i=1}^{r} \beta^{(i)}_j s_i \right) \prod_{j=n+1}^{p} \Gamma \left(a_j - \sum_{i=1}^{r} \alpha^{(i)}_j s_i \right) \right]^{-1},$$

(1.3) $$\theta_i \, (s_i) = \prod_{j=1}^{m_i} \Gamma \left(d^{(i)}_j - \delta^{(1)}_j s_i \right) \prod_{j=1}^{n_i} \Gamma \left(1-c^{(i)}_j + \varepsilon^{(i)}_j s_i \right)$$

$$\cdot \left[\prod_{j=m_i+1}^{q_i} \Gamma \left(1-d^{(i)}_j + \delta^{(i)}_j s_i \right) \prod_{j=n_i+1}^{p_i} \Gamma \left(c^{(i)}_j - \varepsilon^{(i)}_j s_i \right) \right]^{-1},$$

$$\forall \; i \in \{1, \ldots ,r\} .$$

Throughout this paper, the symbol $(a_j ; \alpha'_j , \ldots ,\alpha^{(r)}_j)_{1,p}$ abbreviates the p–member array $(a_1 ; \alpha_1' , \ldots , \alpha^{(r)}_1) ,\ldots ,$ $(a_p ; \alpha_p' , \ldots , \alpha^{(r)}_p)$, and $(c_j, \varepsilon_j)_{1,p}$ the p–member array $(c_1, \varepsilon_1),\ldots, (c_p, \varepsilon_p)$, $p \geqslant 0$, the array being empty if $p=0$.

The multiple integral (1.1) converges absolutely, if

(1.4) $U_i > 0$ and $| \arg x_i | < \frac{1}{2} U_i \pi$,

where

(1.5) $$U_i = - \sum_{j=n+1}^{p} \alpha^{(i)}_j - \sum_{j=1}^{q} \beta^{(i)}_j + \sum_{j=1}^{n_1} \varepsilon^{(i)}_j - \sum_{j=n_i+1}^{p_i} \varepsilon^{(i)}_j +$$

$$\sum_{j=1}^{m_i} \delta_j^{(i)} - \sum_{j=m_i+1}^{q_i} \delta_j^{(i)}; \qquad \forall\, i \in \{1, \ldots, r\}.$$

Also, we have ([6] , p. 131, Eq. (1.9)) :

$$(1.6) \quad H\left[x_1, \ldots, x_r\right] = \begin{cases} O(|x_1|^{v_1} \ldots |x_r|^{v_r}), \max\{|x_1|, \ldots, |x_r|\} \to 0, \\ O(|x_1|^{w_1} \ldots |x_r|^{w_r}), n \equiv 0, \min\{|x_1|, \ldots, |x_r|\} \to \infty, \end{cases}$$

where

$$(1.7) \quad v_i = \min_{1 \leqslant j \leqslant m_i} \left[Re\,(d_j^{(i)} / \delta_j^{(i)}) \right],$$

$$w_i = \max_{1 \leqslant j \leqslant n_i} \left[Re \left\{ c_j^{(i)} - 1)/ \varepsilon_j^{(i)} \right\} \right], \quad \forall\, i \in \{1, \ldots, r\}.$$

2. INTEGRAL FORMULAS

The following single and multiple integrals have been evaluated :

First integral :

$$(2.1) \quad \int_0^t (t-x)^{\rho-1} x^{\sigma-1} (1+Bx)^{-\rho-\sigma} {}_2F_1\left[a, b; c; \frac{zx}{1+Bx}\right]$$

$$.H\left[y_1 \left(\frac{t-x}{1+Bx}\right)^{k_1} \left(\frac{x}{1+Bx}\right)^{h_1}, \ldots, y_r \left(\frac{t-x}{1+Bx}\right)^{k_r} \left(\frac{x}{1+Bx}\right)^{h_r} \right] dx = \frac{t^{\rho+\sigma-1}}{(1+Bt)^{\sigma}} \sum_{u=0}^{\infty} \frac{(a)_u \, (b)_u}{(c)_u \; u\,!} \left(\frac{zt}{1+Bt}\right)^u$$

$$.G_{\rho,\sigma,u}\left[y_1 t^{h_1+k_1} (1+Bt)^{-h_1}, \ldots, y_r t^{h_r+k_r} (1+Bt)^{-h_r} \right]$$

where

$$(2.2) \quad G_{\rho,\sigma,w}\left[y_1, \ldots, y_r \right] \equiv H^{0,\, n+2\,:\, m_1, n_1; \ldots; m_r, n_r}_{p+2,\, q+1\,:\, p_1, q_1; \ldots; p_r, q_r}$$

$$\left[\begin{array}{c} y_1 \\ \vdots \\ y_r \end{array}\middle| \begin{array}{l} (1-\rho\,;k_1,\ldots,k_r),\,(1-\sigma-u;h_1,\ldots,h_r),\,(a_j\,;\alpha'_j,\ldots,\alpha^{(r)}_j)_{1,p} : \\ (1-\rho-\sigma-u\,;\,h_1+k_1,\ldots,h_r+k_r)\,,\,(b_j\,;\beta'_j,\ldots,\beta^{(r)}_j)_{1,q} : \end{array}\right.$$

$$\left.\begin{array}{l} (c'_j,\varepsilon'_j)_{1,p_1}\,;\ldots;\,(c^{(r)}_j,\varepsilon^{(r)}_j)_{1,p_r} \\ (d'_j,\delta'_j)_{1,q_1}\,;\ldots;\,(d^{(r)}_j,\delta^{(r)}_j)_{1,q_r} \end{array}\right] .$$

The integral (2.1) converges under the following (sufficient) conditions :

(i) $|z|<1$ or $|z|=1$ and Re $(c-a-b)>0$

(ii) Re $(\sigma+\sum_{i=1}^{r} h_i v_i)>0$, Re $(\rho+\sum_{i=1}^{r} k_i v_i)>0$ $B>0$,

$h_i,\ k_i,\ U_i>0$ and $|\arg y_i|<\frac{1}{2}U_i\pi$, $\forall\ i\in\{1,\ldots,r\}$.

where U_i and v_i are defined by (1.5) and (1.7), respectively.

(iii) The series occurring on the right-hand side of (2.1) is assumed to be absolutely convergent.

Second integral :

$$(2.3)\quad \int_0^{t_1}\!\!\cdots\int_0^{t_r}\prod_{i=1}^{r}\left\{(t_i-x_i)^{\rho_i-1}(x_i)^{\sigma_i-1}(1+B_i x_i)^{-\rho_i-\sigma_i}\right.$$

$${}_2F_1\left[\lambda_i,\mu_i;\nu_i;\frac{z_i x_i}{1+B_i x_i}\right].H\left[y_1\left(\frac{t_1-x_1}{1+B_1x_1}\right)^{k_1}\left(\frac{x_1}{1+B_1x_1}\right)^{h_1},\ldots,\right.$$

$$\left.y_r\left(\frac{t_r-x_r}{1+B_rx_r}\right)^{k_r}\left(\frac{x_r}{1+B_rx_r}\right)^{h_r}\right]dx_1\ldots dx_r$$

$$=\prod_{i=1}^{r}\left\{(t_i)^{\rho_i+\sigma_i-1}(1+B_it_i)^{-\sigma_i}\sum_{u_i=0}^{\infty}\frac{(\lambda_i)_{u_i}(\mu_i)_{u_i}}{(\nu_i)_{u_i}\ u_i\,!}\right. .$$

$$\left(\frac{z_i t_i}{1+B_i t_i}\right)^{u_i}\Bigg\} . K_{\rho_r,\,\sigma_r,\,u_r}\Big[\ y_1 t_1^{\ h_1+k_1}\,(1+B_1 t_1)^{-h_1}, \ldots,$$

$$y_r\, t_r^{\ h_r+k_r}\ (1+B_r\, t_r)^{-h_r}\Big],$$

where

$$(2.4)\quad K_{\rho_r,\,\sigma_r,\,u_r}\Big[\ y_1, \ldots, y_r\Big] \equiv H^{0,n\,:\,m_1,\,n_1+2;\,\ldots\,;\,m_r,\,n_r+2}_{p,q\,:\,p_1+2,\,q_1+1;\ldots;\,p_r+2,q_r+1}$$

$$\left[\begin{matrix} y_1 \\ \vdots \\ y_r \end{matrix}\ \middle|\ \begin{matrix} (a_j;\alpha'_j,\ldots,\alpha^{(r)}_j)_{1,p}:(1-\rho_1,k_1),\,(1-\sigma_1-u_1,h_1)\,(c'_j,\varepsilon'_j)_{1,p_1} \\ (b_j;\beta'_j,\ldots,\beta^{(r)}_j)_{1,q}:(d'_j,\delta'_j)_{1,q_1},\,(1-\rho_1-\sigma_1-u_1,\,h_1+k_1) \end{matrix}\right.$$

$$\left.\begin{matrix} ;\ldots;\,(1-\rho_r,k_r),\,(1-\sigma_r-u_r,h_r),\,(c^{(r)}_j,\varepsilon^{(r)}_j)_{1,p_r} \\ ;\ldots;\,(d^{(r)}_j,\delta^{(r)}_j)_{1,q_r},\,(1-\rho_r-\sigma_r-u_r,\,h_r+k_r) \end{matrix}\right].$$

The integral (2.4) converges under the following (sufficient) conditions :

(i) $|z_i| \leqslant 1$, and $\mathrm{Re}\,(\upsilon_i-\lambda_i-\mu_i) > 0,\ \forall\ i \in \{1, \ldots, r\}$

(ii) $\mathrm{Re}\,(\rho_i+h_i v_i) > 0$, $\mathrm{Re}\,(\sigma_i+h_i v_i) > 0$, $B > 0$.
$h_i, k_i, U_i > 0$ and $|\arg y_i| < \frac{1}{2} U_i \pi,\ \forall\ i \in \{1, \ldots, r\}$

where U_i and v_i are defined by (1.5) and (1.7), respectively.

(iii) The series occurring on the right-hand side of (2.4) is assumed to be absolutely convergent.

Third integral :

$$(2.5)\quad \int_0^1 \cdots \int_0^1 \prod_{i=1}^{r} \left\{(1-x_i)^{\upsilon_i}\, x_i^{\,k_i}\, P^{(\mu_i,\,\upsilon_i)}_{u_i}(1-2x_i)\,P^{(\rho_i,\,\sigma_i)}_{g_i}(1-2x_i)\right\}$$

$$H\left[\, y_1 x_1^{h_1}\,, \ldots,\, y_r x_r^{h_r} \right] dx_1 \ldots dx_r$$

$$= \prod_{i=1}^{r} \left\{ \sum_{N_i=0}^{g_i} \Gamma(\upsilon_i+u_i+1)\Gamma(\rho_i+g_i+1)\,(-g_i)_{N_i}(\rho_i+\sigma_i+g_i+1)_{N_i}(-1)^{u_i} \right.$$

$$\left. \cdot \left[u_i!\, g_i!\, N_i!\, \Gamma\,(\rho_i+1+N_i) \right]^{-1} \right\} L_{k_r,\, \upsilon_r, N_r}\left[\, y_1, \ldots, y_r \right],$$

where

$$\text{(2.6)} \quad L_{k_r,\, \upsilon_r, N_r}\left[\, y_1, \ldots, y_r \right] \equiv H^{\,0,n:m_1,n_1+2;\ldots;m_r,n_r+2}_{\,p,q:p_1+2,q_1+2;\ldots:p_r+2,q_r+2}$$

$$\left[\begin{matrix} y_1 \\ \vdots \\ y_r \end{matrix} \left| \begin{matrix} (a_j\,;\alpha'_j\,, \ldots, \alpha^{(r)}_j)_{1,p} : (-k_1-N_1,h_1),(\mu_1-k_1-N_1,h_1),\,(c'_j,\varepsilon'_j)_{1,p_1} \\ (b_j\,;\beta'_j\,, \ldots, \beta^{(r)}_j)_{1,q} : (d'_j,\delta'_j)_{1,q_1},(-\upsilon_1-k_1-u_1-N_1-1,h_1),(\mu_1-k_1+ \end{matrix}\right.\right.$$

$$\left.\begin{matrix} :\ldots;(-k_r-N_r,h_r),(\mu_r-k_r-N_r,h_r),(c^{(r)}_j,\,\varepsilon^{(r)}_j)_{1,p_r} \\ u_1-N_1,h_1):\ldots;\,(d^{(r)}_j,\,\delta^{(r)}_j)_{1,q_r},(-\upsilon_r-k_r-u_r-N_r-1,h_r),(\mu_r-k_r+u_r-N_r,h_r) \end{matrix}\right]$$

provided that $Re\,(k_i+h_i v_i+1) > 0$, $Re\,(\upsilon_i) > -1$,

$|\arg y_i| < \frac{1}{2}\, U_i\, \pi\,,\ U_i > 0,\quad \forall\ i \in \{\, 1, \ldots, r\,\}$,

U_i and v_i are defined by (1.5) and (1.7), respectively.

Derivation of the integral formulae (2.1), (2.3) and (2.5)

To derive integral (2.1), we first write the H-function (occurring in the integrand) in terms of the Mellin–Barnes contour integral with the help of (1.1) and change the order of integrations therein. We, then apply a known result ([1], p. 10, Eq. (11)) in order to evaluate the x-integral and interpret the resulting contour integral as a multivariable H-function, with the help of (1.1) ; the integral formula follows at once.

In a similar manner, by applying multiple analogues of the formulae ([1], p. 10, Eq. (11)) and ([2], p. 288, Eq. (20)) instead of ([1], p. 10, Eq. (11)). we can derive easily the second and third integrals.

3. SPECIAL CASES

Our integral formulae can suitably be specialized to a number of known or new integrals involving a large families of special functions (or product of several such functions).

For example, if we put $B=0$, $a=-m$, $b=\alpha+\beta+m+1$, $c=\alpha+1$, $z\to z/2$ in (2.1) and use a known result ([5], p. 254, Eq. (1)) we get the following interesting result :

$$(3.1)\quad \int_0^t (t-x)^{\rho-1} x^{\sigma-1} P_m^{(\alpha,\beta)}(1-zx)\, H\left[y_1 (t-x)^{k_1} x^{h_1}, \ldots, y_r (t-x)^{k_r} x^{h_r} \right] dx$$

$$= t^{\rho+\sigma-1} \sum_{u=0}^{m} \frac{\Gamma(1+\alpha+m)\,(-m)_u\,(\alpha+\beta+m+1)_u}{m\,!\;u\,!\;\Gamma(\alpha+u+1)} \left(\frac{zt}{2}\right)^u$$

$$G_{\rho,\sigma,u}\left[y_1\, t^{h_1+k_1}, \ldots, y_r\, t^{h_r+k_r} \right],$$

where $G_{\rho,\sigma,u}\left[y_1, \ldots, y_r \right]$ is given by (2.2).

The integral formula (3.1) is valid under the following (sufficient) conditions :

(i) $\text{Re}\,(\sigma+\sum_{i=1}^{r} h_i v_i) > 0$, $\text{Re}\,(\rho+\sum_{i=1}^{r} k_i v_i) > 0$,

(ii) $h_i, k_i, U_i > 0$ and $|\arg y_i| < \frac{1}{2} U_i \pi$, $\forall\ i \in \{1, \ldots, r\}$,

U_i and v_i are defined by (1.5) and (1.7), respectively.

If we take $t=1$, $z=2$, and $r=2$ in (3.1), we shall fairly easily get the result given recently by Prasad and Singh ([4], p. 126, Eq. (2.1)). Again, if we put $t=1$, $z=2$, $\alpha=\beta$, $k_i=0$ $(i=1, \ldots r)$ in (3.1), we shall obtain another known integral due to Srivastava and Panda ([6], p. 131, Eq. (2.2)). The integral formula of Srivastava and Panda [*loc. cit.*] contains many known integrals as its particular cases.

On the other hand, if we put $a=\alpha+m+1$, $b=-\beta-m$, $c=1+\alpha$. $z \to z/2$ in (2.1) and using the known results ([1], p. 105, Eq. (1) and (2)) and ([5], p. 254, Eq. (1)) therein, it would reduce to the following integral, which is believed to be new :

$$(3.2) \quad \int_0^t (t-x)^{\rho-1} x^{\sigma-1} (1-\tfrac{1}{2}zx)^{\beta} P_m^{(\alpha,\beta)}(1-zx) \, H\left[y_1 (t-x)^{k_1} x^{h_1}, \ldots, y_r (t-x)^{k_r} x^{h_r} \right] dx$$

$$= t^{\rho+\sigma-1} \sum_{u=0}^{\infty} \frac{(1+\alpha+u)_m (-\beta-m)_u}{m! \, u!} \left(\tfrac{1}{2} z t \right)^u G_{\rho,\sigma,u}\left[y_1 t^{h_1+k_1}, \ldots, y_r t^{h_r+k_r} \right],$$

provided that the conditions mentioned with the integral (3.1) are satisfied and the series occurring on the right-hand side of (3.2) is absolutely convergent. Also, $G_{\rho,\sigma,u}[y_1, \ldots, y_r]$ is defined by (2.2).

Also, our integral formula (2.3) would reduce to Mishra's integral ([3], p. 173, Eq. (2.1)), if we put $r=1$, $\lambda_1=-m$, $\mu_1=\alpha+\beta+m+1$, $\upsilon_1=1+\alpha$, $B \to 0$ in it.

We conclude by remarking that a number of interesting variations of our integral formulas can be obtained when one or more k_i, h_i $(i=1, \ldots, r)$ tend to zero. The details are reasonably straightforward, and we may very well leave them as an exercise to the interested reader.

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