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ON THE RAPIDITY OF ALMOST CONVERGENCE BY POSITIVE LINEAR OPERATORS

by

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ABSTRACT

The aim of the present note is to sharpen a result of R. N. Mohapatra [2] on quantitative results for almost convergence by positive linear operators.

1. Introduction

Let C [K] be the linear space of continuous real-valued functions on the set K. A sequence of positive linear operators $\{L_n\}_{n=1}^{\infty}$ on C [K] is almost convergent to g ε C [K] uniformly in K, provided

$$t_p^k(f)(x) = \frac{1}{p}\sum_{n=k+1}^{k+p} L_n(f)(x), \qquad p=1,2,\ldots;\ k=1,2\ldots,$$

converges to g (x) when $p \to \infty$ uniformly in k and uniformly in K.

Recently, Mohapatra [2] gave quantitative estimates for the rapidity on almost convergence of a sequence of positive linear operators to a given continuous function on a closed and bounded

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interval. The purpose of this note is to sharpen his result.

We define, following Mohapatra [2], the norm in the sequence space as

$$\| t_p(f) \| = \sup_{k} \sup_{x \varepsilon k} | t_p^k(f)(x) |$$

by setting $t_p(f) = \left\{ t_p^k(f) \right\}$ for each p.

Now $\left\{ L_n(f) \right\}$ is almost convergent to g ε C [K] uniformly in K,

if and only if $\| t_p(f) - g \| \to 0$ as $p \to \infty$.

We prove the following

Theorem. Let $\left\{ L_n \right\}_{n=1}^{\infty}$ be a sequence of positive linear operators on C [a, b], and let r be a fixed positive integer. Suppose that $\left\{ t_p^k(1) \right\}(x)$ is uniformly bounded for x ε [a, b] for each positive integer p, uniformly in k = 1, 2, ... Also let $w(f'; .)$ be the modulus of continuity of f ε C^1 [a, b]. Then, for p = 1, 2,...,

$$(1.1) \quad \| t_p(f) - f \| \leqslant \| f \| \cdot \| t_p(1) - 1 \| + \| f' \| \cdot \| t_p(1) \|^{\frac{1}{2}} \cdot \mu_p^1$$

$$+ \; w(f'; \mu_p^1) \left\{ \mu_p^1 \left\| t_p(1) \right\|^{\frac{1}{2}} + \frac{(\mu_p^r)^2}{2r \cdot (\mu_p^1)^{2r-1}} \right\}$$

where $\mu_p^s = \left\| t_p^k (t-x)^{2s}(x) \right\|^{\frac{1}{2}}$ *for* $s = 1$ *or* r, $\| \cdot \|$ norm being sup norm over [a, b] .

Furthermore, if in addition

$$\left\{ t_p^k (1) \right\} (x) = 1 \text{ and } \left\{ t_p^k (t) \right\} (x) = x,$$

then

$$(1.2) \quad \| t_p (f) - f \| \leqslant w (f' ; \mu_p^1) \left\{ \mu_p^1 + \frac{(\mu_p^r)^2}{2\, r .\, (\mu_p^1)^{2r-1}} \right\}.$$

Remark. For $r=1$, Our theorem is sharper than a result of Mohapatra [2, p. 243, Theorem 4].

2. Proof of the Theorem

We know that

$$(2.1) \qquad f(t) - f(x) = (t-x) f'(x) + \int_x^t \left\{ f'(\eta) - f'(x) \right\} d\eta .$$

Also, for $\delta > 0$,

$$(2.2) \quad | f'(\eta) - f'(x) | \leqslant w (f' ; \delta) \left\{ 1 + \frac{|\eta - x|}{\delta} \right\}$$

$$(2.3) \qquad \leqslant w (f' ; \delta) \left\{ 1 + \frac{|\eta - x|^{2r-1}}{\delta^{2r-1}} \right\} .$$

Using (2.3) in (2.1), we obtain

$$(2.4) \qquad \left| t_p^k (f) (x) - f(x) \quad t_p^k (1) (x) \right|$$

$$\leqslant \left| f'(x). \quad t_p^k (t - x) (x) \right|$$

$$+ \left| t_p^k \left[\int_x^t \left\{ f'(\eta) - f'(x) \right\} d\eta \right] (x) \right|$$

$$\leqslant \left| f'(x) \, . \; t_p^k \left(|t-x| \right) (x) + w \left(f' \, ; \, \delta \right) \, t_p^k \, . \right.$$

$$\left[\left| \int_x^t \left\{ 1 + \frac{| \eta - x |^{2r-1}}{\delta^{2r-1}} \right\} d\eta \right| \right] (x$$

$$\leqslant \left| f' \; x) \right| \left(t_p^k \, (t-x)^2 \, (x) \, . \; t_p^k \, (1) \, (x) \right)^{\frac{1}{2}} +$$

$$w \left(f' \, ; \, \delta \right) \, t_p^k \left\{ \left| t - x \right| + \frac{(t-x)^{2r}}{2r . \delta^{2r-1}} \right\} (x)$$

$$\leqslant \left| f'(x) \right| \left(t_p^k \, (t-x)^2 (x) \, . \; t_p^k \, (1) \, (x) \right)^{\frac{1}{2}} +$$

$$w \left(f' \, ; \, \delta \right. \left\{ \left(t_p^k \, (t-x)^2 (x) . \; t_p^k \, (1) \, (x) \right)^{\frac{1}{2}} + \frac{t_p^k \, (t-x)^{2r} \, (x)}{2 \, r . \delta^{2r-1}} \right\}$$

$$(2.5) \quad \leqslant \| f' \| \, . \, \mu_p^1 \, . \; \| t_p \, (1) \|^{\frac{1}{2}} + w \left(f' \, ; \, \delta \right) \left\{ \mu_p^1 \, . \; \| t_p \, (1) \|^{\frac{1}{2}} + \frac{(\mu_p^r)^2}{2 \, r . \delta^{2r-1}} \right\} .$$

Choosing $\delta = \mu_p^1$, the above reduces to

$$(2.6)\quad \left| t_p^k(f)(x) - f(x)\,.\, t_p^k(1)(x) \right|$$

$$\leqslant \|f'\|\,.\,\mu_p^1\,.\,\|t_p(1)\|^{\frac{1}{2}} + w(f';\mu_p^1)\left\{ \mu_p^1\,.\,\|t_p(1)\|^{\frac{1}{2}} + \frac{(\mu_p^r)^2}{2r.\,(\mu_p^1)^{2r-1}} \right\}$$

Clearly

$$(2.7)\quad \left| -f(x) + f(x)\; t_p^k(1)(x) \right| \leqslant \left\| f \right\|\,.\, \left\| t_p(1) - 1 \right\|.$$

On adding (2.6) and (2.7) we get (1.1).

In case $\mu_p^1 = 0$, we use (2.2) in the above proof and find, for every $\delta > 0$, that

$$t_p^k(f)(x) = f(x)\; t_p^k(1)(x).$$

So

$$\left| t_p^k(f)(x) - f(x) \right| = \left| f(x)\, t_p^k(1)(x) - f(x) \right|$$

$$\leqslant \|f\|\,.\,\|t_p(1) - 1\|.$$

Again, if $t_p^k(1)(x) = 1$ and $t_p^k(t)(x) = x$, then $t_p^k(t-x)(x) = 0$, and from (2.4) we thus get (1.2).

3. Concluding Remarks

If, in addition to the hypotheses of our theorem, $f \in C^2[a,b]$, then we can get a result better than the corresponding result of Mohapatra [2,p.244, Theorem 5].

Further results on the extended Chebyshev system analogous to the results of Censor [1] can be established.

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