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ON THE GEOMETRIC MEANS OF AN ENTIRE FUNCTION OF SLOW GROWTH

by

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1. Introduction . Throughout this paper we shall assume f (z) to be a non–constant entire function of order zero, other than polynomial. For an entire function of this nature, the logarithmic order ρ^* and lower logarithmic order λ^* are given as [5] :

$$(1.1)\quad \lim_{r\to\infty} \frac{\sup}{\inf} \frac{\log\log M(r)}{\log\log r} = \frac{\rho^*}{\lambda^*},\ (1 \leqslant \lambda^* \leqslant \rho^* \leqslant \infty);$$

where $M(r) = \max_{|z|=r} |f(z)|$.

The geometric mean of $|f(z)|$, for $|z| = r$, has been defined as [*4, p. 144*]:

$$(1.2)\qquad G(r) = \exp\left[\frac{1}{2\pi}\int_0^{2\pi} \log\left|f(re^{i\theta})\right| d\theta\right].$$

Further let

$$(1.3)\qquad g_\delta(r) = \exp\left[\frac{\delta+1}{r^{\delta+1}}\int_0^r x^\delta \log G(x)\, dx\right],$$

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$$(1.4) \qquad \overset{*}{g}_{\delta}(r) = \exp\left[\frac{\delta+1}{(\log r)^{\delta+1}} \int_1^r \log G(x) (\log x)^{\delta} x^{-1} dx \right].$$

Then the following results are known [*3, p. 22*] :

$$(1.5) \quad \lim_{r \to \infty} \begin{matrix} sup \\ inf \end{matrix} \frac{\log \log \Phi(r)}{\log \log r} = \begin{matrix} \rho^* \\ \lambda^* \end{matrix},$$

where $\Phi(r)$ may be replaced by $G(r)$, $g_{\delta}(r)$ and $\overset{*}{g}_{\delta}(r)$.

Also, we have [*2, p. 98*] :

$$(1.6) \quad \lim_{r \to \infty} \begin{matrix} sup \\ inf \end{matrix} \frac{\log n(r)}{\log \log r} = \begin{matrix} \rho^*-1 \\ \lambda^*-1 \end{matrix},$$

where n (r) represents the number of zeros of f (z) in $|z| \leqslant r$. Using Jensen's formula [*1, p.* 2] in (1.2), we have

$$(1.7) \qquad \log G(r) = \log |f(0)| + \int_0^r \frac{n(x)}{x} dx.$$

Our aim in this note is to prove some results on the means of an entire function. This paper has been divided into two parts. Part A contains the results on the growth of $[\Phi(r)]^{n(r)}$, $[G(r)/g_{\delta}(r)]$ and $[G(r)/\overset{*}{g}_{\delta}(r)]$. Part B contains the growth relations of $[G(r)/\overset{*}{g}_{\delta}(r)]$.

PART A

2. Theorem 1.

$$(2.1) \quad \lim_{r \to \infty} \begin{matrix} sup \\ inf \end{matrix} \frac{\log \log [\Phi(r)]^{n(r)}}{\log \log r} = \begin{matrix} 2\rho^*-1 \\ 2\lambda^*-1 \end{matrix}.$$

Proof. From (1.7), we have

$$\log G(r) = \log G(r_0) + \int_{r_0}^{r} \frac{n(x)}{x}\, dx$$

$$\leqslant n(r)\,(\log r)\,(1+o(1)).$$

Therefore,

$$\log [G(r)]^{n(r)} \leqslant (n(r))^2 (\log r)\,(1+o(1)).$$

Proceeding to limits and using (1.6), we get

$$(2.2) \qquad \lim_{r\to\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log \log [G(r)]^{n(r)}}{\log \log r} \leqslant \begin{matrix} 2\rho^*-1 \\ 2\lambda^*-1 \end{matrix},$$

Again,

$$\log G(r^2) > \int_{r}^{r^2} \frac{n(x)}{x}\, dx \geqslant n(r) \log r,$$

which gives

$$(2.3) \qquad \lim_{r\to\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log \log [G(r)]^{n(r)}}{\log \log r} \geqslant \begin{matrix} 2\rho^*-1 \\ 2\lambda^*-1 \end{matrix}.$$

Combining (2.2) and (2.3), we get

$$(2.4) \qquad \lim_{r\to\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log \log [G(r)]^{n(r)}}{\log \log r} = \begin{matrix} 2\rho^*-1 \\ 2\lambda^*-1 \end{matrix}.$$

Since log G(r) is an increasing fuuction of r, therefore

$$(2.5) \qquad \log g_{\delta}(r) = \frac{\delta+1}{r^{\delta+1}} \int_{0}^{r} x^{\delta} \log G(x)\, dx \leqslant \log G(r).$$

Further,

$$(2.6) \quad \log g_\delta(R) \geqslant \frac{\delta+1}{R^{\delta+1}} \int_r^R x^\delta \log G(x)\, dx, \quad (R > r)$$

$$\geqslant \log G(r) \left[\frac{R^{\delta+1} - r^{\delta+1}}{R^{\delta+1}} \right].$$

From (2.5) and (2.6), we have

$$(2.7) \quad \log g_\delta(r) \leqslant \log G(r) \leqslant \left[\frac{R^{\delta+1}}{R^{\delta+1} - r^{\delta+1}} \right] \log g_\delta \; (R).$$

Similarly, we can show that

$$(2.8) \quad \log \overset{*}{g_\delta}(r) \leqslant \log G(r)$$

$$\leqslant \left[\frac{(\log R)^{\delta+1}}{(\log R)^{\delta+1} - (\log r)^{\delta+1}} \right] \log \overset{*}{g_\delta}(R), \; (R > r > 1).$$

On substituting $R = 2r$ in (2.7) and $R = r^2$ in (2.8), and proceeding to limits, we get

$$(2.9) \quad \lim_{r \to \infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log \log [g_\delta(r)]^{n(r)}}{\log \log r} = \begin{matrix} 2\rho^*-1 \\ 2\lambda^*-1 \end{matrix},$$

and

$$(2.10) \quad \lim_{r \to \infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log \log [\overset{*}{g_\delta}(r)]^{n(r)}}{\log \log r} = \begin{matrix} 2\rho^*-1 \\ 2\lambda^*-1 \end{matrix},$$

respectively.

Hence theorem 1 follows in view of (2.4), (2.9) and (2.10).

Theorem 2.

$$(2.11)\qquad \lim_{r\to\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log\log [G(r) / g_\delta(r)]}{\log\log r} = \begin{matrix} \rho^*-1 \\ \lambda^*-1 \end{matrix},$$

and

$$(2.12)\qquad \lim_{r\to\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log\log [G(r) / \overset{*}{g}_\delta(r)]}{\log\log r} = \begin{matrix} \rho^* \\ \lambda^* \end{matrix}.$$

Proof. Combining (1.2) and (1.3) and using (1.7), we obtain

$$(2.13)\qquad \log \left[\frac{G(r)}{g_\delta(r)} \right] = \frac{1}{r^{\delta+1}} \int_0^r n(x)\, x^\delta\, dx.$$

Similarly, from (1.2), (1.4) and (1.7), we obtain

$$(2.14)\qquad \log \left[\frac{G(r)}{\overset{*}{g}_\delta(r)} \right] = \frac{1}{(\log r)^{\delta+1}} \int_1^r n(x) \log x)^{\delta+1}\, x^{-1}\, dx.$$

It can be easily seen that

$$(2.15)\qquad \frac{n(r)}{\delta+1} \left[\frac{R^{\delta+1}-r^{\delta+1}}{R^{\delta+1}} \right] \leqslant \log \left[\frac{G(R)}{g_\delta(R)} \right] \leqslant \frac{n(R)}{\delta+1}, \quad (R > r),$$

and

$$(2.16)\qquad \frac{n(r)}{\delta+2} \left[\frac{(\log R)^{\delta+2}-(\log r)^{\delta+2}}{(\log R)^{\delta+1}} \right] \leqslant \log \left[\frac{G(R)}{g^*_\delta(R)} \right] \leqslant \frac{n(R)\log R}{\delta+2}, \quad (R > r > 1).$$

Putting $R = 2r$ in (2.15) and $R = r^2$ in (2.16), and proceeding to limits, (2.11) and (2.12) follows in view of (1.6).

PART B

3. **Theorem** 3.

$$(3.1) \qquad \frac{\beta}{\delta+\rho^*+1} \leqslant \liminf_{r \to \infty} \frac{\log [G(r) / g_\delta^*(r)]}{(\log r)^{\rho^*}} \leqslant$$

$$\leqslant \limsup_{r \to \infty} \frac{\log [G(r) / g_\delta^*(r)]}{(\log r)^{\rho^*}} \leqslant \frac{\alpha}{\delta+\rho^*+1},$$

where the quantities α and β are given by

$$(3.2) \qquad \lim_{r \to \infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{n(r)}{(\log r)^{\rho^*-1}} = \begin{matrix} \alpha \\ \beta \end{matrix}.$$

Proof. From (2.14), we have

$$(3.3) \qquad \log \left[\frac{G(r)}{g_\delta^*(r)} \right] = \frac{1}{(\log r)^{\delta+1}} \int_1^r n(x) (\log x)^{\delta+1} x^{-1} \, dx$$

From (3.2), we have, for any $\varepsilon > 0$ and $r > r_0$,

$$(3.4) \qquad (\log r)^{\rho^*-1} (\beta-\varepsilon) < n(r) < (\log r)^{\rho^*-1} (\alpha+\varepsilon).$$

Using left- hand inequality in (3.3), we get

$$\log \left[\frac{G(r)}{g_\delta^*(r)} \right] > \frac{\beta-\varepsilon}{(\log r)^{\delta+1}} \int_{r_0}^r (\log x)^{\delta+\rho^*} x^{-1} \, dx$$

$$= \frac{(\beta-\varepsilon)(\log r)^{\rho^*}}{\delta+\rho^*+1} (1-o(1)).$$

Proceeding to limit, we get

$$\liminf_{r \to \infty} \frac{\log [G(r) / \overset{*}{g}_{\delta}(r)]}{(\log r)^{\rho^*}} \geqslant \frac{\beta}{\delta+\rho^*+1} .$$

Similarly, using right hand inequality of (3.4) in (3.3), we get

$$\limsup_{r \to \infty} \frac{\log [G(r) / g^*_{\delta}(r)]}{(\log r)^{\rho^*}} \leqslant \frac{\alpha}{\delta+\rho^*+1} .$$

This completes the Proof of theorem 3.

Theorem 4. If

(3.5) $$\liminf_{r \to \infty} \frac{n(r)}{(\log r)^{p} \log \log r} > 1 ,$$

then

(3.6) $$\liminf_{r \to \infty} \frac{\log [G(r) / g^*_{\delta}(r)]}{(\log r)^{p+1} \log \log r} \geqslant \frac{1}{p+\delta+2} ,$$

and if

(3.7) $$\limsup_{r \to \infty} \frac{n(r)}{(\log r)^{p} \log \log r} < 1 ,$$

then

(3.8) $$\limsup_{r \to \infty} \frac{\log [G(r) / g^*_{\delta}(r)]}{(\log r)^{p+1} \log \log r} \leqslant \frac{1}{p+\delta+2} ,$$

where p $\geqslant$ 0.

Proof. From (3.5), we have, for any $\varepsilon > 0$ and $r > r_0$,

$$n(r) > (1-\varepsilon)(\log r)^p \log \log r.$$

Using this inequality in (2.14), we get

$$\log \left[\frac{G(r)}{\overset{*}{g}_\delta(r)} \right] > \frac{1-\varepsilon}{(\log r)^{\delta+1}} \int_{r_0}^{r} (\log x)^{p+\delta+1} (\log \log x)\, x^{-1}\, dx$$

$$= \frac{(1-\varepsilon)(\log r)^{p+1} \log \log r}{p+\delta+2} \; (1-o(1)).$$

Proceecding to limit, we get

$$\liminf_{r \to \infty} \frac{\log [G(r)/\overset{*}{g}_\delta(r)]}{(\log r)^{p+1} \log \log r} \geqslant \frac{1}{P+\delta+2}.$$

Again, from (3.7), we have, for any $\varepsilon > 0$ and $r > r_0$,

$$n(r) < (1+\varepsilon)(\log r)^p \log \log r,$$

which together with (2.14) gives

$$\log \left[\frac{G(r)}{\overset{*}{g}_\delta(r)} \right]$$

$$< 0(1) + \frac{1+\varepsilon}{(\log r)^{\delta+1}} \int_{r_0}^{r} (\log x)^{p+\delta+1} (\log \log x)\, x^{-1}\, dx$$

from which (3.8) follows.

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