Jñānābha, Vol. 11, 1981

MULTIPLE INTEGRALS INVOLVING THE *H*-FUNCTION OF TWO VARIABLES

by

NAVRATNA RATHIE

Department of Mathematics M. R. Engineering College, Jaipur-302004, India

(*Received : January 10, 1980; Revised : October 7, 1980*)

ABSTRACT

In this paper we establish two formulas involving multiple integrals. The first multiple integral involves the product of Fox's *H*-function and the *H*-function of two variables, while the second involves the product of two *H*-functions of two variables. The integrals evaluated in this paper are believed to be new. Since the *H*-function of two variables includes a large number of special functions of one and two variables as its particular cases, the results established here serve as key formulas giving us a large number of new and interesting results by specialising the parameters involved. Some interesting particular cases have also been given.

1. INTRODUCTION

The *H*-function of two variables, occurring in this paper, was defined by Mittal and Gupta [4]. The parameters of this function are displayed in the following contracted notation which is due essentially to Srivastava and Panda [7, p. 266, Eq. (1.5)] ;

$$(1.1)\quad H\,[x,y] = H^{0,\, n_1\,:\, m_2,\, n_2;\, m_3,\, n_3}_{p_1,\, q_1:\, p_2,\, q_2\,:\, p_3,\, q_3}\left[\begin{matrix} x \\ y \end{matrix}\left|\begin{matrix} (a_j;\alpha_j,A_j)_{1,p_1} & :(c_j,\gamma_j)_{1,p_2} & ; & (e_j,\,E_j)_{1,p_3} \\ (b_j;\beta_j,B_j)_{1,q_1} & :(d_j,\delta_j)_{1,q_2} & : & (f_j,F_j)_{1,q_3} \end{matrix}\right.\right]$$

$$= -\frac{1}{4\pi^2} \int_{L_1} \int_{L_2} \phi(s, t)\, \theta_1(s)\, \theta_2(t)\, x^s y^t\, ds\, dt,$$

where

$$\phi(s, t) = \prod_{j=1}^{n_1} \Gamma(1-a_j+\alpha_j s+A_j t) \left[\prod_{j=n_1+1}^{p_1} \Gamma(a_j-\alpha_j s-A_j t) \prod_{j=1}^{q_1} \Gamma(1-b_j+\beta_j s+B_j t) \right]^{-1},$$

$$\theta_1(s) = \prod_{j=1}^{m_2} \Gamma(d_j-\delta_j s) \prod_{j=1}^{n_2} \Gamma(1-c_j+\gamma_j s) \left[\prod_{j=m_2+1}^{q_2} \Gamma(1-d_j+\delta_j s) \prod_{j=n_2+1}^{p_2} \Gamma(c_j-\gamma_j s) \right]^{-1}$$

$$\theta_2(t) = \prod_{j=1}^{m_3} \Gamma(f_j-F_j t) \prod_{j=1}^{n_3} \Gamma(1-e_j+E_j t) \left[\prod_{j=m_3+1}^{q_3} \Gamma(1-f_j+F_j t) \prod_{j=n_3+1}^{p_3} \Gamma(e_j-E_j t) \right]^{-1},$$

x and y are not equal to zero, and an empty product is interpreted as unity.

The conditions for the function $H[x, y]$ to be an analytic function and conditions for the integral in (1.1) to converge are fully discussed by Mittal and Gupta [4, p. 119]. It is assumed throughout this paper that the conditions corresponding appropriately to Conditions (i) to (vi) in their paper [4, p. 119] are always satisfied by the H-function of two variables.

We use the notation :

$$H_{p_1, q_1 : \ldots ; \ldots}^{0, n_1 : \ldots ; \ldots} \left[\begin{matrix} x \\ y \end{matrix} \middle| \begin{matrix} (a_j : \alpha_j, A_j)_{1, p_1} : \ldots ; \ldots \\ (b_j : \beta_j, B_j)_{1, q_1} ; \ldots ; \ldots \end{matrix} \right]$$

to denote the fact that the parameters shown as ... are the same as those of $H\,[x, y]$ defined by (1.1).

Also the following symbols have been used in the present paper ;

(i) $H_1^{(S)}\,[\,w_S\, x_S,\ \varepsilon_S\, y_S\,]$ stands for

$$H^{0,\,0\ :\ m_2^{(S)},\, n_2^{(S)};\ m_3^{(S)},\, n_3^{(S)}}_{p_1^{(S)}, q_1^{(S)}:\ p_2^{(S)},\, q_2^{(S)};\ p_3^{(S)},\, q_3^{(S)}}\left[\begin{matrix} w_S x_S \\ \varepsilon_S\, y_S \end{matrix}\ \middle|\ \begin{matrix} (a_j^{(S)};\alpha_j^{(S)},\, A_j^{(S)})_{1,p_1^{(S)}} :(c_j^{(S)},\, \gamma_j^{(S)})_{1,p_2^{(S)}} ;\ (e_j^{(S)},\, E_j^{(S)})_{1,\, p_3^{(S)}} \\ (b_j^{(S)};\, \beta_j^{(S)}, B_j^{(S)})_{1,q_1^{(S)}} :(d_j^{(S)},\, \delta_j^{(S)})_{1,\, q_2^{(S)}} ;(f_j^{(S)},\, F_j^{(S)})_{1,\, q_3^{(S)}} \end{matrix}\right]$$

If we take $n_2^{(S)}=n_3^{(S)}=0$ in $H_1^{(S)}\,[\,w_S\, x_S,\ \varepsilon_S y_S\,]$ and the rest of the parameters are unchanged, we shall denote it by

$$H_1^{\{s\}}\,[\,w_S\, x_S\ \ \varepsilon_S\, y_S\,].$$

(ii) $H_1'\,[\,x, y$ stands for

$$H^{0,\ 0\ :\ m_2',\, n_2';\ m_3',\, n_3'}_{p_1',\, q_1'\ :\ p_2',\, q_2';\ p_3',\, q_3'}\left[\begin{matrix} x \\ y \end{matrix}\ \middle|\ \begin{matrix} (a_j',\, \alpha_j',\, A_j')_{1,p_1'} :(c_j',\, \gamma_j')_{1,p_2'} ; \\ (b_j';\, \beta_j',\, B_j')_{1,q_1'} :(d_j',\, \delta_j')_{1,q_2'} : \end{matrix}\ \begin{matrix} (e_j',\, E_j')_{1,\, p_3'} \\ (f_j',\, F_j')_{1,q_3'} \end{matrix}\right]$$

(iii) $(a \pm b,\, c)$ stands for $(a+b,\, c)$, $(a-b,\, c)$.

2. MAIN INTEGRALS

$$(2.1)\quad \int_0^\infty \int_0^\infty \cdots \int_0^\infty \int_0^\infty x_1^{\rho_1-1}\, y_1^{\sigma_1-1} \ldots x_S^{\rho_S-1}\, y_S^{\sigma_S-1}\ H_1^{(1)}[w_1 x_1, \varepsilon_1 y_1] \ldots$$

$$H_1^{(s)}\,[w_S\, x_S, \varepsilon_S\, y_S]\, H^{m,\, n}_{p,\, q}\left[A\left(x_1^{u_1}\ y_1^{v_1}\ \ldots\ x_S^{u_S}\ y_S^{v_S}\right) \middle|\ \begin{matrix} (g_j,\, G_j)_{1,p} \\ (h_j,\, H_j)_{1,q} \end{matrix}\right]$$

$$dx_1\ dy_1\ \ldots\ dx_S\ dy_S$$

$$= \left[w_1^{-\rho_1} \varepsilon_1^{-\sigma_1} \dots w_S^{-\rho_S} \varepsilon_S^{-\sigma_S} \right] H^{m+a,n+b}_{p+c,q+d} \left[A(w_1^{-u_1} \varepsilon_1^{-v_1} \dots w_S^{-u_S} \varepsilon_S^{-v_S}) \middle| \begin{matrix} B \\ C \end{matrix} \right]$$

where

$$H^{m,n}_{p,q}\left[x \middle| \begin{matrix} (g_j, G_j)_{1,p} \\ (h_j, H_j)_{1,q} \end{matrix} \right]$$

is the well-known (Fox's) H-function,

$$a = \sum_{r=1}^{S} (n_2^{(r)} + n_3^{(r)}),\ b = \sum_{r=1}^{S} (m_2^{(r)} + m_3^{(r)}),$$

$$c = \sum_{r=1}^{S} (q_1^{(r)} + q_2^{(r)} + q_3^{(r)}),\ d = \sum_{r=1}^{S} (p_1^{(r)} + p_2^{(r)} + p_3^{(r)}),$$

$B = (g_j, G_j)_{1,n}$, $(1 - d_j^{(r)} - \rho_r\, \delta_j^{(r)},\ u_r\, \delta_j^{(r)})_{1,m_2^{(r)}}$, $r = 1, \dots, S$,

$(1 - f_j^{(r)} - \sigma_r\, F_j^{(r)},\ v_r\, F_j^{(r)})_{1,q_3^{(r)}}$, $r = 1 \dots, S$,

$(g_j, G_j)_{n+1,p}$, $(1 - d_j^{(r)} - \rho_r\, \delta_j^{(r)},\ u_r\, \delta_j^{(r)})_{m_2^{(r)}+1, q_2^{(r)}}$, $r = 1, \dots, S$,

$(1 - b_j^{(r)} - \rho_r\, \beta_j^{(r)} - \sigma_r\, B_j^{(r)},\ u_r\, \beta_j^{(r)} + v_r\, B_j^{(r)})_{1,q_1^{(r)}}$, $r = 1, \dots, S$

and

$C = (h_j, H_j)_{1,m}$, $(1 - c_j^{(r)} - \rho_r\, \gamma_j^{(r)},\ u_r\, \gamma_j^{(r)})_{1,n_2^{(r)}}$, $r = 1, \dots, S$,

$(1 - e_j^{(r)} - \sigma_r E_j^{(r)}\ v_r\, E_j^{(r)})_{1,p_3^{(r)}}$, $r = 1, \dots, S$, $(h_j, H_j)_{m+1,q}$,

$(1 - c_j^{(r)} - \rho_r\, \gamma_j^{(r)},\ u_r\, \gamma_j^{(r)})_{n_2^{(r)}+1, p_2^{(r)}}$, $r = 1, \dots, S$

$(1 - a_j^{(r)} - \rho_r\, \alpha_j^{(r)} - \sigma_r\, A_j^{(r)},\ u_r\, \alpha_j^{(r)} + v_r\, A_j^{(r)})_{1,p_1^{(r)}}$, $r = 1, \dots, S$.

The result (2.1) is valid under the following sets of conditions for $r=1, \ldots, S$:

(i) $u_r > 0$, $v_r > 0$,

(ii) $Re\ (\rho_r + u_r (\frac{h_i}{H_i}) + \frac{d_j^{(r)}}{\delta_j^{(r)}}) > 0\ (i=1,\ldots,m;\ j=1,\ldots,m_2^{(r)})$,

(iii) $Re\ (\sigma_r + v_r\ (\frac{h_i}{H_i}) + \frac{f_j^{(r)}}{F_j^{(r)}}) > 0\ (i=1,\ldots,m;\ j=1,\ldots,m_3^{(r)})$,

(iv) $Re\ (\rho_r + u_r\ (\frac{g_i - 1}{G_i}) + \frac{c_j^{(r)} - 1}{\gamma_j^{(r)}}) < 0\ (i=1,\ldots,n;\ j=1,\ldots,n_2^{(r)})$.

(v) $Re\ (\sigma_r + v_r (\frac{g_i - 1}{G_i}) + \frac{e_j^{(r)} - 1}{E_j^{(r)}}) < 0\ (i=1,\ldots,n;\ j=1,\ldots,n_3^{(r)})$.

Remark A result similar to (2.1) was given by Rekha Panda [6, p. 158, Eq. (1.8)] who gave her results in terms of the H–function of several complex variables, due to Srivastava and Panda [7, p. 271, Eq. (4.1) **et seq.**]

(2.2) $$\int_0^\infty \int_0^\infty \cdots \int_0^\infty \int_0^\infty x_1^{\rho_1 - 1}\, y_1^{\sigma_1 - 1} \cdots x_S^{\rho_S - 1}\, y_S^{\sigma_S - 1}\, H_1^{\{1\}}[g_1 x_1, h_1 y_1] \cdots$$

$$H_1^{\{S\}}[g_S x_S, h_S y_S]\, H_1' \Big[M(x_1^{u_1}\, y_1^{v_1} \ldots x_S^{u_S}\, y_S^{v_S}),$$

$$N(x_1^{w_1}\, y_1^{\varepsilon_1} \ldots x_S^{w_S}\, y_S^{\varepsilon_S}) \Big]\, dx_1\, dy_1 \quad dx_S\, dy_S$$

$$= \Big[g_1^{-\rho_1}\, h_1^{-\sigma_1} \ldots g_S^{-\rho_S}\, h_S^{-\sigma_S} \Big]$$

$$H^{0,\ U\ :\ldots;\ldots}_{p_1'+V,\ q_1'+W\ :\ldots;\ldots} \left[\begin{matrix} M(g_1^{-u_1}\, h_1^{-v_1} \ldots g_S^{-u_S}\, h_S^{-v_S}) \\ N(g_1^{-w_1}\, h_1^{-\varepsilon_1} \ldots g_S^{-w_S}\, h_S^{-\varepsilon_S}) \end{matrix} \middle| \begin{matrix} I:\ldots;\ldots \\ J:\ldots;\ldots \end{matrix} \right],$$

where

$$U = \sum_{r=1}^{S} (m_2^{(r)}+m_3^{(r)}),\; V=\sum_{r=1}^{S} (q_1^{(r)}+q_2^{(r)}+q_3^{(r)}),$$

$$W = (p_1^{(r)}+p_2^{(r)}+p_3^{(r)}) ,$$

$$I=(1-d_j^{(r)}-\rho_r\, \delta_j^{(r)};\, u_r\, \delta_j^{(r)},\, w_r\, \delta_j^{(r)})_{1,\, m_2^{(r)}},\; r=1,\ldots, S,$$

$$(1-f_j^{(r)}-\sigma_r\, F_j^{(r)};\, v_r\, F_j^{(r)},\, \varepsilon_r\, F_j^{(r)})_{1,\, q_3^{(r)}},\; r=1,\ldots, S,$$

$$1-d_j^{(r)}-\rho_r\, \delta_j^{(r)};\, u_r\, \delta_j^{(r)},\, w_r\, \delta_j^{(r)})_{m_2^{(r)}+1,q_2^{(r)}},\; r=1,\ldots, S,$$

$$(a_j'\, ;\, \alpha_j'\, ,\, A_j')_{1,p_1'},\; (1-b_j^{(r)}-\rho_r\, \beta_j^{(r)}-\sigma_r\, B_j^{(r)};\, u_r\, \beta_j^{(r)}+v_r\, B_j^{(r)},$$

$$w_r\beta_j^{(r)}+\varepsilon_r\, B_j^{(r)})_{1,q_1^{(r)}}\, ,\; r=1,\ldots, S$$

and

$$J=(b_j';\, \beta_j')_{1,q_1'}\, ,\; (-a_j^{(r)}-\rho_r\, \alpha_j^{(r)}-\sigma_r\, A_j^{(r)};\, u_r\, \alpha_j^{(r)}+v_r\, A_j^{(r)},$$

$$w_r\, \alpha_j^{(r)}+\varepsilon_r\, A_j^{(r)})_{1\, p_1^{(r)}},\; r=1,\ldots, S,\; (1-c_j^{(r)}-\rho_r\, \gamma_j^{(r)}\, ;$$

$$u_r\, \gamma_j^{(r)};\, w_r\, \gamma_j^{(r)})_{1,p_2^{(r)}},\; r=1,\ldots, S.\; (1-e_j^{(r)}-\sigma_r\, E_j^{(r)}\, ;$$

$$v_r\, E_j^{(r)}\, ,\, \varepsilon_r\, E_j^{(r)})_{1,p_3^{(r)}},\; r=1,\ldots, S.$$

The above result (2.2) is valid under the following sets of conditions for $r=1 , \ldots , S$:

(i) $u_r, v_r, w_r, \varepsilon_r > 0$,

(ii) $Re\ (\rho_r+\dfrac{d_i^{(r)}}{\delta_i^{(r)}} + u_r\ (\dfrac{d_j'}{\delta_j'})+w_r\dfrac{f_k'}{F_k'}) > 0$

$$(i=1,\ldots, m_2^{(r)}\; ;\; j=1, \ldots , m_2'\; ;\; k=1, \ldots , m_3') ,$$

(iii) $Re\ (\sigma_r + \dfrac{f_i^{(r)}}{F_i^{(r)}} + v_r\ (\dfrac{d_j'}{\delta_j'}) + \varepsilon_r\ \dfrac{f_k'}{F_k'}) > 0$

$(i=1, \ldots, m_3^{(i)} ; j=1, \ldots, m_2' ; k=1, \ldots, m_3')$.

Proofs : From the left–hand side of (2.1), we have

$$(2.3) \quad \int_0^\infty \int_0^\infty x_S^{\rho_S-1} y_S^{\sigma_S-1} H_1^{(S)}[w_S x_S, \varepsilon_S y_S] \int_0^\infty \int_0^\infty x_{S-1}^{\rho_{S-1}-1} y_{S-1}^{\sigma_{S-1}-1}$$

$$H_1^{(S-1)}[w_{S-1} x_{S-1}, \varepsilon_{S-1} y_{S-1}] \ldots \int_0^\infty \int_0^\infty x_1^{\rho_1-1} y_1^{\sigma_1-1} H_1^{(1)}[w_1 x_1, \varepsilon_1 y_1]$$

$$H_{p,q}^{m,n}\left[A\left(x_1^{u_1} y_1^{v_1} \ldots x_S^{u_S} y_S^{v_S} \right) \middle| \begin{matrix} (g_j, G_j)_{1,p} \\ (h_j, H_j)_{1,q} \end{matrix} \right] dx_S\, dy_S \ldots dx_1 dy_1$$

Now integrating the last integral with respect to $x_1 y_1$ with the help of the result given in [3, p. 165, Eq. (4.2.3)] and substituting the result thus obtained in (2.3) and then integrating with respect to $x_2 y_2$ and so on. Repeating this procedure s times, we arrive at the desired result (2.1).

The result (2.2) can be established similarly by using the known result [3, p. 168, Eq. (4. 2. 5)].

3. Particular Cases :

I. **If we replace** u_1 by $-u_1$, u_2 by $-u_2$ and so on in (2.1), we get the following interesting multiple integral which is also believed to be new :

$$(3.1) \quad \int_0^\infty \int_0^\infty \ldots \int_0^\infty \int_0^\infty x_1^{\rho_1-1} y_1^{\sigma_1-1} \ldots x_S^{\rho_S-1} y_S^{\sigma_S-1} H_1^{(1)}[w_1 x_1, \varepsilon_1 y_1] \ldots$$

$$H_1^{(S)}[w_S x_S, \varepsilon_S y_S] H_{p,q}^{m,n}\left[A\left(x_1^{-u_1} y_1^{v_1} \ldots x_S^{-u_S} y_S^{v_S} \right) \middle| \begin{matrix} (g_j, G_j)_{1,p} \\ (h_j, H_j)_{1,q} \end{matrix} \right]$$

$$dx_1\, dy_1 \ldots dx_S\, dy_S$$

$$= \left\{ \begin{matrix} -\rho_1 & -\sigma_1 & & -\rho_S & -\sigma_S \\ w_1 & \varepsilon_1 & \ldots & w_S & \varepsilon_S \end{matrix} \right\} H^{m+e,n+f}_{p+g,q+h} \left[A \begin{pmatrix} u_1 & -v_1 & & u_S & -v_S \\ w_1 & \varepsilon_1 & \ldots & w_S & \varepsilon_S \end{pmatrix} \middle| \begin{matrix} E \\ F \end{matrix} \right]$$

where

$$e = \sum_{r=1}^{S} (m_2^{(r)} + n_3^{(r)}), f = \sum_{r=1}^{S} (n_2^{(r)} + m_3^{(r)}),$$

$$g = \sum_{r=1}^{S} (p_1^{(r)} + p_2^{(r)} + q_3^{(r)}), h = \sum_{r=1}^{S} (q_1^{(r)} + q_2^{(r)} + p_3^{(r)}),$$

$$E = (g_j, G_j)_{1,n}, (c_j^{(r)} + \rho_r \gamma_j^{(r)}, u_r \gamma_j^{(r)})_{1,n_2^{(r)}}, r = 1, \ldots, S,$$

$$(1 - f_j^{(r)} - \sigma_r F_j^{(r)}, v_r F_j^{(r)})_{1,q_3^{(r)}}, r = 1 \ldots, S,$$

$$(g_j, G_j)_{n+1,p}, (c_j^{(r)} + \rho_r \gamma_j^{(r)}, u_r \gamma_j^{(r)})_{n_2^{(r)}+1, p_2^{(r)}}, r = 1, \ldots, S,$$

$$(a_j^{(r)} + \rho_r \alpha_j^{(r)} + \sigma_r A_j^{(r)}, u_r \alpha_j^{(r)} - v_r A_j^{(r)})_{1,p_1^{(r)}}, r = 1, \ldots, S$$

and

$$F = (h_j, H_j)_{1,m}, (d_j^{(r)} + \rho_r \delta_j^{(r)}, u_r \delta_j^{(r)})_{1,m_2^{(r)}}, r = 1, \ldots, S,$$

$$(1 - e_j^{(r)} - \sigma_r E_j^{(r)}, v_r E_j^{(r)})_{1,p_3^{(r)}}, r = 1, \ldots, S, (h_j, H_j)_{m+1,q},$$

$$(d_j^{(r)} + \rho_r \delta_j^{(r)}, u_r \delta_j^{(r)})_{m_2^{(r)}+1, q_2^{(r)}}, r = 1, \ldots, S$$

$$(b_j^{(r)} + \rho_r \beta_j^{(r)} + \sigma_r B_j^{(r)}, u_r \beta_j^{(r)} - v_r B_j^{(r)})_{1,q_1^{(r)}}, r = 1, \ldots, S.$$

The result (3.1) is valid under the following sets of conditions for $r = 1, \ldots, S$:

(i) $u_r > 0$, $v_r > 0$,

(ii) $Re\,(\rho_r + u_r(\frac{g_i - 1}{G_i}) + \frac{d_j^{(r)}}{\delta_j^{(r)}}) > 0$ $(i = 1, \ldots, n;\ j = 1, \ldots, m_2^{(r)})$,

(iii) $Re\,(\sigma_r + v_r(\frac{h_i}{H_i}) + \frac{f_j^{(r)}}{F_j^{(r)}}) > 0$ $(i = 1, \ldots, m;\ j = 1, \ldots, m_3^{(r)})$,

(iv) $Re\,(\rho_r - u_r(\frac{h_i}{H_i}) + \frac{c_j^{(r)} - 1}{\gamma_j^{(r)}}) < 0$ $(i = 1, \ldots, m;\ j = 1, \ldots, n_2^{(r)})$.

(v) $Re\,(\sigma_r + v_r(\frac{g_i - 1}{G_i}) + \frac{e_j^{(r)} - 1}{E_j^{(r)}}) < 0$ $(i = 1, \ldots, n;\ j = 1, \ldots, n_3^{(r)})$.

By specialising the parameters of the functions occurring in the integrands, our main integrals yield as special cases many new multiple integrals involving the products of various other elementary special functions and different classes of orthogonal polynomials occurring frequently in mathematical physics and other branches of applied mathematics having similar structures. Some of such integrals are given below :

II. If we take $p_1^{(r)}=q_1^{(r)}=n_2^{(r)}=n_3^{(r)}=p_2^{(r)}=p_3^{(r)}=0$,

$m_2^{(r)}=m_3^{(r)}=q_2^{(r)}=q_3^{(r)}=2$, $d_1^{(r)}=\mu_r/2$, $d_2^{(r)}=-\mu_r/2$,

$f_1^{(r)}=\nu_r/2$, $f_2^{(r)}=-\nu_r/2$, $\delta_1^{(r)}=\delta_2^{(r)}=F_1^{(r)}=F_2^{(r)}=w_r=\varepsilon_r=1$,

replace x_r by x_r^2 and y_r by y_r^2 for $r=1,\ldots,S$ in (2.1) and then using the known result [1, , p. 216] we get the following multiple integral :

$$\int_0^\infty \int_0^\infty \cdots \int_0^\infty \int_0^\infty x_1^{\rho_1-1}\, y_1^{\sigma_1-1} \ldots x_S^{\rho_S-1}\, y_S^{\sigma_S-1}\, K_{\mu_1}(x_1) K_{\nu_1}(y_1) \ldots$$

$$K_{\mu_S}(x_S)\, K_{\nu_S}(y_S)\, H^{m,n}_{p,q}\left[A(x_1^{2u_1}\, y_1^{2v_1} \ldots x_S^{2u_S}\, y_S^{2v_S}) \middle| \begin{matrix} (g_j, G_j)_{1,p} \\ (h_j, H_j)_{1,q} \end{matrix} \right]$$

$$dx_1\, dy_1 \ldots dx_S\, dy_S$$

$$= \left[\prod_{r=1}^{S} 2^{2(\rho_r+\sigma_r-2)} \right] H^{m,n+4}_{p+4S,q} \left[A(\prod_{r=1}^{S} 2^{2(u_r+v_r)}) \middle| \begin{matrix} B' \\ C' \end{matrix} \right],$$

where

$B'=(g_j, G_j)_{1,p}$ ' $(1\pm\mu_r/2-\rho_r, u_r)$, $r=1, \ldots, S$,

$(1\pm\nu_r/2-\sigma_r, v_r)$ $r=1,\ldots S$,

$C' = (h_j, H_j)_{1,q}$.

The conditions of validity of (3.2) are :

$$u_r > 0,\ v_r > 0,\ Re\,(2\rho_r \pm \mu_r + 2u_r(\frac{h_j}{H_j})\,) > 0\,,$$

$$Re\,(2\sigma_r \pm \nu_r + 2v_r\,(\frac{h_j}{H_j})\,) > 0\ ;\ \text{for all } j=1,\ldots,m;\ r=1,\ldots,S.$$

III. If we take $p_1^{(r)} = q_1^{(r)} = 0$, $m_2^{(r)} = m_3^{(r)} = q_2^{(r)} = q_3^{(r)} = 2,$

$p_2^{(r)} = p_3^{(r)} = 1,\ c_1^{(r)} = 1 - \mu_r,\ d_1^{(r)} = \nu_r + \frac{1}{2},\ d_2^{(r)} = -\nu_r + \frac{1}{2},$

$_1^{(r)} = 1 - \lambda_r.\ f_1^{(r)} = \xi_r + \frac{1}{2},\ f_2^{(r)} = -\xi_r + \frac{1}{2},\ \gamma_1^{(r)} = \delta_1^{(r)} = \delta_2^{(r)}$

$= E_1^{(r)} = F_1^{(r)} = F_2^{(r)} = 1$ in (2.2), we get with the help of the known result [1, p. 216], the following interesting multiple integral :

$$(3.3)\quad \int_0^\infty \int_0^\infty \cdots \int_0^\infty \int_0^\infty x_1^{\rho_1 - 1}\, y_1^{\sigma_1 - 1} \ldots x_S^{\rho_S - 1}\, y_S^{\sigma_S - 1}\, e^{-\frac{1}{2}(g_1 x_1 + h_1 y_1)} \ldots$$

$$e^{-\frac{1}{2}(g_S x_S + h_S y_S)}\, W_{\mu_1, \nu_1}(g_1 x_1)\, W_{\lambda_1, \xi_1}(h_1 y_1) \ldots W_{\mu_S, \nu_S}(g_S x_S)$$

$$W_{\lambda_S, \xi_S}(h_S y_S)\, H_1'\left[M(x_1^{u_1}\, y_1^{v_1} \ldots x_S^{u_S}\, y_S^{v_S}),\ N(x_1^{w_1}\, y_1^{\varepsilon_1} \ldots x_S^{w_S}\, y_S^{\varepsilon_S}) \right]$$
$$dx_1\, dy_1 \ldots dx_S\, dy_S$$

$$= \left[g_1^{-\rho_1}\, h_1^{-\sigma_1} \ldots g_S^{-\rho_S}\, h_S^{-\sigma_S} \right] H^{0,\ 4S\ :\ \ldots\ ;\ \ldots}_{p_1' + 4S,\ q_1' + 2S\ :\ \ldots\ ;\ \ldots}$$

$$\left[\begin{array}{l|l} M(g_1^{-u_1}\, h_1^{-v_1} \ldots g_S^{-u_S}\, h_S^{-v_S}) & I\ :\ \ldots\ ;\ \ldots \\ N(g_1^{-w_1}\, h_1^{-\varepsilon_1} \ldots g_S^{-w_S}\, h_S^{-\varepsilon_S}) & J'\ :\ \ldots\ ;\ \ldots \end{array} \right]$$

where

$I' = (\frac{1}{2} \pm \nu_r - \rho_r;\ u_r,\ w_r),\ r = 1,\ldots,S,\ (\frac{1}{2} \pm \xi_r - \sigma_r;\ v_r,\ \varepsilon_r),$

$r = 1,\ldots,S\ \ (a_j'\ ;\ \alpha_j',\ A_j)_{1, p_1'}\,,$

$$J' = (b_j'' ; \beta_j', B_j')_{1,q_1'}, (\mu_r - \rho_r ; u_r, w_r), r = 1,\ldots, S,$$

$$(\lambda_r = \sigma_r ; v_r, \varepsilon_r), r = 1,\ldots, S,$$

The conditions of validity of (3.3) are

$$u_r, v_r, w_r, \varepsilon_r > 0, \; Re\,(\rho_r \pm \nu_r + u_r(\frac{d_i'}{\delta_i'}) + w_r(\frac{f_j'}{F_j'}) + \tfrac{1}{2}) > 0,$$

$$Re\,(\sigma_r \pm \xi_r + v_r(\frac{d_i'}{\delta_i'}) + \varepsilon_r(\frac{f_j'}{F_j'}) + \tfrac{1}{2}) > 0 \text{ for all values of}$$

$$i = 1,\ldots, m_2' ; j = 1, \ldots, m_3' ; r = 1,\ldots, S.$$

IV. If we take $S = 1$ in (2.1), (2,2) and (3.1), we arrive at the results recenlty obtained by Gupta [3, p. 165, Eq. (4. 2. 3) ; p. 166, Eq. (4. 2. 4) and p. 168, Eq. (4. 2. 5)].

Also, by suitably specialising the parameters in (2.1) and (2.2), we obtain the known results due to Olkha [5], Vasishta and Goyal [8], and Goyal [2].

ACKNOWLEDGEMENTS

I am thankful to Dr. K. C. Gupta for his help and guidance in the preparation of this paper. I am also highly thankful to Professor H. M. Srivastava for his useful and valuable suggestions for the improvement of this paper which led to a better presentation.

REFERENCES

[1] A. Erdélyi, *et al.*. *Higher transcendental functions*, Vol . I, McGraw-Hill, New York, 1953.

[2] S.P. Goyal, On double integrals involving the *H*-functions, *Indian J. Math.* **19** (1977), 119–123.

[3] K. C. Gupta, *A study of integral transforms and Mellin–Barnes type of contour integrals*, D. Sc. thesis, Univ. of Rajasthan, India, 1978.

[4] P. K. Mittal and K. C. Gupta, An integral involving generalized function of two variables, *Proc. Indian Acad. Sci. Sect. A* **75** (1972), 117–123.

[5] G. S. Olkha, On multiple integrals for Meijer's G-function, *Indian J. Pure Appl. Math.* **3** (1972), 366–369.

[6] Rakha Panda, On a multiple integral involving the *H*–function of several variables, *Indian J. Math.* **19** (1977), 157–162.

[7] H. M. Srivastava and R. Panda, Some bilateral generating functions for a class of generalized hypergeometric polynomials, *J Reine Angew. Math* **283/-84** (1976), 265-274.

[8] S. K. Vasishta and S. P. Goyal, Multiple integrals involving the *H*-function of two variables, *Proc. Indian Acad. Sci Sect. A* **83** (1976), 41–49.