Jñānābha, Vol. 11, 1981

SOME GENERAL INTEGRAL RELATIONS FOR THE MULTIVARIABLE *H*-FUNCTION

by

R. S. GARG

Department of Mathematics B. V. College of Arts & Science

Banasthali Vidyapith-304022, Rajasthan, India

(*Received : May 12, 1981; Revised : August 27,1981*)

The aim of the present paper is to establish two multiple integral relations for Fox's H-function and the H-functions of several complex variables (or multivariable H-function) which was introduced and studied in a series of papers by H. M. Srivastava and R. Panda (see, for example, [7] and [8]) . The integral relations are quite general in character, not only because of the general nature of the H-functions involved, but also due to the presence of the functions f and g. { See also a recent paper on this subject by H. M. Srivastava, S. P. Goyal and R. K. Agrawal [9] } Thus, by appropriately reducing the H-functions in terms of simpler special functions, and by suitable choosing f or g (or both f and g) , one can easily obtain a considerably large number of (known or new) double and triple integral relations and integrals. A new finite integral formula involving the multivariable H-function has also been evaluated during the course of our study.

1. Introduction and Notations

The multivariable H-function occurring in this paper is a particular case of the general H-function of several variables, introduced by Srivastava and Panda ([7] and [8]); the parameters of this function will be displayed in the following contracted notation ([8] , p. 130, Eq. (1.3)) :

$$(1.1)\quad H_1\,[z_1,\ldots,z_s] = H\begin{matrix}0,\,0\,:\,m_1,\,n_1\,:\,m_s,\,n_s\\ p,\,q\,:\,p_1,\,q_1\,;\,p_s,\,q_s\end{matrix}\left[\begin{matrix}z_1\\ \vdots\\ z_s\end{matrix}\right.$$

$$\left|\begin{matrix}(a_j;\,\alpha_j',\ldots,\alpha_j^{(s)})_{1,p}\,:\,(c_j',\,\varepsilon_j')_{1,p_1}\,;\,\ldots\,;\,(c_j^{(s)},\,\varepsilon_j^{(s)})_{1,p_s}\\ (b_j;\,\beta_j',\ldots,\beta_j^{(s)})_{1,q}\,:\,(d_j',\,\delta_j')_{1,q_1}\,;\,\ldots\,;\,(d_j^{(s)},\,\delta_j^{(s)})_{1,q_s}\end{matrix}\right]$$

$$= (\tfrac{1}{2}\pi i)^s \int_{L_1}\cdots\int_{L_s} \phi(\xi_1,\ldots,\xi_s)\prod_{k=1}^{s}\{\,\theta_k(\xi_k)\,(z_k)^{\xi_k}\,d\xi_k\,\},$$

$$(i=\sqrt{-1})$$

where $\phi(\xi_1,\ldots,\xi_s)$, $\theta_k(\xi_k)$ and the various notations used here have been reproduced from [7] and [8] in our earlier work [1] .

The multiple integral (1.1) converges absolutely if,

$$(1.2)\quad U_k > 0 \text{ and } |\arg z_k| < (\tfrac{1}{2})\,U_k\pi,\ (k=1,\ldots,s),$$

where

$$(1.3)\quad U_k = -\sum_{j=1}^{p}\alpha_j^{(k)} - \sum_{j=1}^{q}\beta_j^{(k)} + \sum_{j=1}^{n_k}\varepsilon_j^{(k)} - \sum_{j=n_k+1}^{p_k}\varepsilon_j^{(k)} + \sum_{j=1}^{m_k}\delta_j^{(k)}$$

$$-\sum_{j=m_k+1}^{q_k}\delta_j^{(k)}\quad (k=1,\ldots,s)$$

Also, from the known asymptotic expansions given by Srivastava and Panda ([8], p. 131, Eq. (1.9)), we have

$$(1.4)\quad H_1[\,z_1,\ldots,z_s\,] = o\,(\,|z_1|^{u_1}\ldots|z_s|^{u_s}\,),\ \max\{\,|z_1|,\ldots,|z_s|\,\}\to 0,$$

where

$$(1.5)\quad u_k = \min_{1\leqslant j\leqslant m_k}\,[\,Re\,\{\,d_j^{(k)}/\delta_j^{(k)}\,\}\,],$$

for $k=1,\ldots,s$.

We shall use the following notation :

$$H\left[\begin{matrix} z_1 \\ \vdots \\ z_s \end{matrix} \middle| \begin{matrix} (\xi_j;\ \mu_j',\ldots,\ \mu_j^{(s)})_{1,u} \\ (\eta_j;\ \nu_j',\ldots,\ \nu_j^{(s)})_{1,v} \end{matrix}\right]$$

for the multivariable H–function :

$$(1.6)\quad H^{0,\ u\ :\ m_1,\ n_1\ ;\ \ldots\ ;\ m_s,\ n_s}_{p+u\ q+v:p_1,q_1\ ;\ \ldots\ ;\ p_s,\ q_s}\left[\begin{matrix} z_1 \\ \vdots \\ z_s \end{matrix} \middle| \begin{matrix} (\xi_j;\ \mu_j',\ \ldots,\ \mu_j^{(s)})_{1,u}, \\ (\eta_j;\ \nu_j',\ \ldots,\ \nu_j^{(s)})_{1,v}, \end{matrix}\right.$$

$$\left.\begin{matrix} (a_j;\ \alpha_j',\ldots,\ \alpha_j^{(s)})_{1,p}\ :\ (c_j',\ \varepsilon_j')_{1,p_1}\ ;\ \ldots\ ;\ (c_j^{(s)},\ \varepsilon_j^{(s)})_{1,p_s} \\ (b_j;\ \beta_j'\ \ldots,\ \beta_j^{(s)})_{1,q}\ :\ (d_j',\ \delta_j')_{1,q_1}\ ;\ \ldots\ ;\ (d_j^{(s)},\ \delta_j^{(s)})_{1,q_s} \end{matrix}\right].$$

The elementary integral relations contained in the following lemma will be required in the sequel :

Lemma (Srivastava, Goyal and Agrawal [9] , pp. 264–265). *Let the functions $f(x)$ and $g(x)$ be integrable over the semi–infinite interval $(0, \infty)$, and define*

$$(1.7)\quad F(r) = \int_0^{\pi/2} h(r, \theta)\ d\theta,$$

where $h(r,\theta)$ is an integrable function of two vnriables in the rectangular region :

$$0 \leq r \leq \theta,\ 0 \leq \theta \leq \pi/2.$$

Then

$$(1.8)\quad \int_0^\infty \int_0^\infty f(x^2+y^2)\ h\left((x^2+y^2)^{1/2},\ \tan^{-1}(y/x)\right)\ dx\ dy$$

$$= \tfrac{1}{2}\int_0^\infty f(t)\ f(\sqrt{t})\ dt,$$

and

$$(1.9)\quad \int_0^\infty \int_0^\infty \int_0^\infty (x^2+y^2)^{-\frac{1}{2}} f(x^2+y^2+z^2)\, g\{\tan^{-1}(x^2+y^2)^{\frac{1}{2}}/z\}$$

$$.\, h\{(x^2+y^2+z^2)^{\frac{1}{2}},\ \tan^{-1}(y/x)\}\, dx\, dy\, dz$$

$$=\int_0^\infty \int_0^\infty f(u^2+v^2)\, g(\tan^{-1}(v/u))\, F((u^2+v^2)^{\frac{1}{2}})\, du\, dv,$$

provided that the various integrals involved are absolutely convergent.

The above lemma has been recently established by Srivastava, Goyal and Agrawal [9], where details of proof are given.

2. A Useful Integral

$$(2.1)\quad \int_0^{\pi/2} e^{i(\alpha+\beta)\theta}(\sin\theta)^{\alpha-1}(\cos\theta)^{\beta-1}\, H^{1,N}_{p,Q+1}\left[ae^{i(\sigma+\rho)\theta}\sin^\sigma\theta\cos^\rho\theta \left|\begin{matrix}(g_j,G_j)_{1,p}\\(0,1),(h_j,H_j)_{1,Q}\end{matrix}\right.\right] H\left[z_1 e^{i\theta(\sigma_1+\rho_1)}\sin^{\sigma_1}\theta\cos^{\rho_1}\theta,\ \ldots,\ z_s e^{i\theta(\sigma_s+\rho_s)}\sin^{\sigma_s}\theta\cos^{\rho_s}\theta\right] d\theta$$

$$=\sum_{w=0}^{\infty}\Phi(w)\, e^{i\pi(\alpha+\sigma w)/2}\, H\left\{\begin{matrix} z_1\, e^{i\pi\sigma_1/2}\\ \vdots \\ z_s\, e^{i\pi\sigma_s/2}\end{matrix}\left|\begin{matrix}(1-\alpha-\sigma w;\ \sigma_1,\ldots,\sigma_s),\\ (1-\beta-\rho w;\ \rho_1,\ldots,\rho_s)\\ (1-\alpha-\beta-[\sigma+\rho]w;\ \sigma_1+\rho_1,\ldots,\sigma_s+\rho_s)\end{matrix}\right.\right\},$$

where

$$(2.2)\quad \Phi(w)=(-a)^w \prod_{j=1}^{N}\Gamma(1-g_j+G_jw)\left[\prod_{j=1}^{Q}\Gamma(1-h_j+H_jw)\prod_{j=N+1}^{P}\Gamma(g_j-G_jw)\, w!\right]^{-1}.$$

The (sufficient) conditions of validity of (2.1) are

(i) $$\min_{1 \leq j \leq s} \{ \sigma_j, \rho_j, \rho, \sigma, Re(\alpha), Re(\beta) \} > 0,$$

$$Re(\alpha) + \sum_{j=1}^{s} \sigma_j u_j > 0, \; Re(\beta) + \sum_{j=1}^{s} \rho_j u_j > 0 \; \{ u_j \text{ is given by } (1.5) \}$$

(ii) $U_j > 0$, $|\arg z_j| < \frac{1}{2} U_j \pi$ { U_j is given by (1.3) }, ($j = 1, \ldots, s$).

(iii) $A > 0$, $|\arg a| < \frac{1}{2} A \pi$, where

$$(2.3) \quad A = \sum_{j=1}^{N} G_j - \sum_{j=N+1}^{p} G_j - \sum_{j=1}^{Q} H_j + 1,$$

and

(iv) The series occurring on the right-hand side of (2.1) is absolutely convergent.

Proof. Writing

$$(2.4) \quad \triangle = \int_0^{\pi/2} e^{i(\alpha+\beta)\theta} (\sin\theta)^{\alpha-1} (\cos\theta)^{\beta-1} H_{p,Q+1}^{1,N} \left[a e^{i(\sigma+\rho)\theta} \sin^{\sigma}\theta \cos^{\rho}\theta \,\middle|\, \begin{matrix} \cdots \\ \cdots \end{matrix} \right]$$

using contour representation for the H-function involved in the right-hand side of (2.4), changing the order of integration, and applying MacRobert's result ([3], p. 450, Eq. (4)), we easily get

$$(2.5) \quad \triangle = e^{i\pi\alpha/2} H_{p+2,Q+2}^{1,N+2} \left[a e^{i\pi\sigma/2} \,\middle|\, \begin{matrix} (1-\alpha, \sigma), (1-\beta, \rho), (g_j, G_j)_{1,p} \\ (0, 1), (h_j, H_j)_{1,Q}, (1-\alpha-\beta, \sigma+\rho) \end{matrix} \right],$$

provided $A > 0$, $|\arg a| < \frac{1}{2} A \pi$, and $\min \{ \sigma, \rho, Re(\alpha), Re(\beta) \} > 0$.

Next, in order to establish (**2**.1), we use the Mellin-Barnes contour integral (1.1) for the multivariable H-function involved in the left-hand side of (2.1), change the order of integration and use (2.5) for the evaluation of the inner θ-integral so obtained. Thus we arrive at the right-hand side of (2.1) by virtue of (1.1).

Taking $N=P$, $G_j=H_k=1$ ($j=1, \ldots, p$; $k=1, \ldots, Q$), $\rho=1$, $\sigma \to 0$ in (2.1) and using a known result ([4], p. 151), we shal easily arrive at a known result given by the author ([2], Eq. (2.1)). The second result (2.2) of the same paper [2] is also contained in (2.1) as a particular case. The results established earlier by Vasishta and Goyal ([10], p. 13, Eq. (1.1)), Rathie ([5], p. 237, Eq. (2.3)), and others can also be deduced as special cases of our integral (2.1).

3. Integral Relations

The following integral relations are established in this paper

First Integral Relation

$$(3.1) \quad \int_0^\infty \int_0^\infty f(x^2+y^2)\, e^{i(\alpha+\beta)\tan^{-1}(y/x)}\, x^{\beta-1}\, y^{\alpha-1}\, (x^2+y^2)^{1-(\alpha+\beta)/2}$$

$$\cdot H^{1,N}_{P,Q+1}\left[B(x,y)\middle|\begin{matrix}\ldots\\ \ldots\end{matrix}\right] H_1\left[Y_1(x,y), \ldots, Y_s(x,y)\right] dx\, dy$$

$$= \tfrac{1}{2} \sum_{w=0}^{\infty} \Phi(w)\, e^{i\pi(\alpha+\sigma w)/2} \int_0^\infty f(t)\, H\left[\begin{matrix} z_1\, e^{i\pi\sigma_1/2}\, t^{\lambda_1} \\ \vdots \\ z_s\, e^{i\pi\sigma_s/2}\, t^{\lambda_s}\end{matrix}\middle|\begin{matrix}(1-\alpha-\sigma w; \sigma_1,\ldots,\sigma_s), (1-\beta-\rho w; \rho_1,\ldots,\rho_s) \\ (1-\alpha-\beta-[\sigma+\rho] w; \sigma_1+\rho_1,\ldots,\sigma_s+\rho_s)\end{matrix}\right] dt.$$

Second Integral Relation

$$(3.2) \quad \int_0^\infty \int_0^\infty \int_0^\infty (x^2+y^2)^{(1-\alpha-\beta)/2}\, x^{\beta-1}\, y^{\alpha-1}\, e^{i(\alpha+\beta)\tan^{-1}(y/x)}$$

$$.H^{1,N}_{P,Q+1}\left[B(x,y)\left|\begin{matrix}\cdots\\ \cdots\end{matrix}\right.\right] f(x^2+y^2+z^2)\, g\{\tan^{-1}((x^2+y^2)^{\frac{1}{2}}/z)\}$$

$$.H_1\left[Z_1(x,y,z),\ \ldots,\ Z_s(x,y,z)\right] dx\, dy\, dz$$

$$=\sum_{w=0}^{\infty} e^{i\pi(\alpha+\sigma w)/2}\,\Phi(w)\int_0^{\infty}\int_0^{\infty} f(u^2+v^2)\, g\{\tan^{-1}(v/u)\}$$

$$.\mathrm{H}\left[\begin{matrix} z_1\, e^{i\pi\sigma_1/2}(u^2+v^2)^{\lambda_1}\\ \vdots\\ z_s\, e^{i\pi\sigma_s/2}(u^2+v^2)^{\lambda_s}\end{matrix}\left|\begin{matrix}(1-\alpha-\sigma w;\sigma_1,\ldots,\sigma_s),(1-\beta-\rho w;\rho_1,\ldots,\rho_s)\\ (1-\alpha-\beta-[\sigma+\rho]w;\ \sigma_1+\rho_1,\ldots,\sigma_s+\rho_s)\end{matrix}\right.\right] du\, dv$$

where, for convenience, let

(3.3) $B(x,y) \equiv a\, x^{\rho} y^{\sigma} (x^2+y^2)^{-(\sigma+\rho)/2}\, e^{i(\sigma+\rho)\tan^{-1}(y/x)}$,

(3.4) $Y_k(x,y) \equiv z_k\, x^{\rho_k} y^{\sigma_k} (x^2+y^2)^{\lambda_k-(\sigma_k+\rho_k)/2}\, e^{i(\sigma_k+\rho_k)\tan^{-1}(y/x)}$,

(3.5) $Z_k(x,y\ z) \equiv z_k\, x^{\rho_k} y^{\sigma_k} (x^2+y^2)^{-(\sigma_k+\rho_k)/2}(x^2+y^2+z^2)^{\lambda_k}$

$. e^{i(\sigma_k+\rho_k)\tan^{-1}(y/x)}$, $(k = 1, \ldots, s)$

Φ (w) is given by (2.2) and the *remaining parameters of*

$H^{1,N}_{P,Q+1}$ *in the above equation are the same as those of the H-function occurring in (2.1).*

For the validity of (3.1) and (3.2) we assume that the functions f and g ase so constrained that the various integrals involved exist and various conditions mentioned with (2.1) hold.

Proofs of (3.1) and (3.2). we begin by setting in (1.6)

$$(3.6)\ h(r,\theta)=e^{i(\alpha+\beta)\theta}\,(\sin\theta)^{\alpha-1}\,(\cos\theta)^{\beta-1}$$

$$H^{1,\ N}_{P,Q+1}\left[ae^{i(\sigma+\rho)\theta}\sin^{\sigma}\theta\cos^{\rho}\theta\left|\begin{matrix}\dots\\ \dots\end{matrix}\right.\right]$$

$$H_1\Big[z_1 r^{2\lambda_1} e^{i(\sigma_1+\rho_1)\theta}\sin^{\sigma_1}\theta\cos^{\rho_1}\theta,\dots, z_s r^{2\lambda_s} e^{i(\sigma_s+\rho_s)\theta}$$

$$\sin^{\sigma_s}\theta\ \cos^{\rho_s}\theta\Big]$$

and evaluating the resulting integral by means of (2.1), we thus obtain

$$(3.7)\ F(r)=\sum_{w=0}^{\infty}\Phi(w)\,e^{i\pi(\alpha+\sigma w)/2}\,H\left[\begin{matrix}z_1\,e^{i\pi\sigma_1/2}\,r^{2\lambda_1}\\ \dots\\ z_s\,e^{i\pi\sigma_s/2}\,r^{2\lambda_s}\end{matrix}\right|$$

$$\left.\begin{matrix}(1-\alpha-\sigma w;\ \sigma_1,\dots,\sigma_s),(1-\beta-\rho w;\ \rho_1,\dots\ \rho_s)\\ \\ (1-\alpha-\beta-[\sigma+\rho]\,w;\ \sigma_1+\rho_1,\dots,\sigma_s+\rho_s)\end{matrix}\right],$$

substitution from (3.6) and (3.7) into the integral relations (1.8) and (1.9) give us the desired integral relations (3.1) and (3.2).

4. Some Interesting Deductions

The integral relations (3.1) and (3.2) are quite general in character, from which one can easily obtain a large number of double and triple integral relations and multiple integrals of interest to mathematical analysts and applied mathematicians. For example if we put $N=P=2$, $Q=1$, $G_1=G_2=H_1=H_2=1$ in (3.1) and use a known result ([4], p. 11, Eq. (1.7.7)) therein, we get the following integral relation involving Gaussian hypergeometric function and the multivariable H-function :

$$(4.1)\quad \int_0^\infty \int_0^\infty f(x^2+y^2)\, e^{i(\alpha+\beta)\tan^{-1}(y/x)} x^{\beta-1} y^{\alpha-1} (x^2+y^2)^{1-(\alpha+\beta)/2}$$

$$\cdot {}_2F_1[\lambda, \delta; \nu; B(x,y)]\, H_1[\Upsilon_1(x,y), \ldots, \Upsilon_s(x,y)]\, dx\, dy$$

$$= \tfrac{1}{2} \sum_{w=0}^{\infty} \frac{(\lambda)_w (\delta)_w}{(\nu)_w\, w!} e^{i\pi(\alpha+\sigma w)/2} \int_0^\infty f(t) H \left[\begin{matrix} z_1 e^{i\pi\sigma_1/2}\, t^{\lambda_1} \\ \vdots \\ z_s\, e^{i\pi\sigma_s/2}\, t^{\lambda_s} \end{matrix} \right| \begin{matrix} (1-\alpha-\sigma w; \sigma_1,\ldots,\sigma_s), (1-\beta-\rho w; \rho_1,\ldots,\rho_s) \\ \\ (1-\alpha-\beta-[\sigma+\rho] w; \sigma_1+\rho_1,\ldots,\sigma_s+\rho_s) \end{matrix} \left. \vphantom{\begin{matrix} a\\ \vdots \\ b \end{matrix}} \right] dt,$$

where $B(x,y)$ and $\Upsilon_k(x,y)$ $(k=1,\ldots,s)$ are defined by (3.3) and (3.4) respectively.

Taking $a=\rho=1$, $\nu=\beta$ in (4.1), letting all $\rho_j, \sigma \to 0$ $(j=1,\ldots,s)$ and using a known result ([6], p. 28, Eq. (1.7.6)), we arrive at a recent result due to Srivastava, Goyal and Agrawal ([9], Eq. (2.4)). Again, letting $\lambda \to 0$ in (4.1), we shall arrive at another result (2.9) due to them [9] in a slightly different form.

Also making substitutions in (3.2) similar to those mentioned above, we shall get the results (2.5) and (2.10) of Srivastava, Goyal and Agrawal [9].

Next, if in (3.2), we let

$$g(t) = e^{i(\alpha+\beta)t} (\sin t)^{\alpha-1} (\cos t)^{\beta-1} H^{1,N'}_{P',Q'+1}$$

$$\left[z' e^{i(\sigma'+\rho')t} \sin^{\sigma'} t \cos^{\rho'} t \,\middle|\, \begin{matrix} (e_j, E_j)_{1,P'} \\ (o,1),(f_j,F_j)_{1,Q'} \end{matrix} \right]$$

and simplify the right-hand side of the resulting equation by means of the integral relation (3.1), we shall abtain under suitable conditions of validity which are easily obtainable from (3.2) :–

$$(4.2)\quad \int_0^\infty\int_0^\infty\int_0^\infty (xz)^{\beta-1}\, y^{\alpha-1}\, (x^2+y^2)^{-\beta/2}\, (x^2+y^2+z^2)^{1-(\alpha+\beta)/2}$$

$$f\,(x^2+y^2+x^2)\; e^{i[\,(\alpha+\beta)\,\{tan^{-1}(\,y/x)+tan^{-1}(\,(x^2+y^2)^{1/2}/z)\}]}$$

$$.H^{1\;,\;N}_{P,Q+1}\left[B(x,y)\middle|\begin{matrix}\cdots\\ \cdots\end{matrix}\right].H^{1\;,\;N'}_{P',Q'+1}\Bigg[z'\; e^{i(\sigma'+\rho')tan^{-1}(\,(x^2+y^2)^{\frac{1}{2}}/z)}$$

$$z^{\rho'}\;(x^2+y^2)^{\sigma'/2}\,(x^2+y^2+z^2)^{-(\sigma'+\rho')/2}\left|\begin{matrix}(e_j,E_j)_{1,\,p'}\\ (0,1),(f_j,F_j)_{1,Q'}\end{matrix}\right]$$

$$H_1\Big[\,z_1(x,\,y,\,z),\;\ldots\;,\,z_s\,(x\;\,y,\,z)\;\Big]\;dx\;dy\,dz$$

$$=\;\tfrac{1}{2}\;\sum_{w=0}^{\infty}\sum_{W=0}^{\infty} e^{i\pi(2\alpha+\sigma w+\sigma' W)/2}\,\Phi(w)\Phi'(W)\frac{\Gamma(\alpha+\sigma' W)\Gamma(\beta+\rho' W)}{\Gamma(\alpha+\beta+[\sigma'+\rho']W)}\int_0^\infty f(t)$$

$$.\mathrm{H}\left[\begin{matrix}z_1\;\; e^{\,i\,\pi\,\sigma_1/2}\;\; t^{\lambda_1}\\ \vdots\\ z_s\;\; e^{\,i\,\pi\,\sigma_s/2}\;\; t^{\lambda_s}\end{matrix}\left|\begin{matrix}(1-\alpha-\sigma w;\sigma_1,\ldots,\sigma_s),(1-\beta-\rho w;\rho_1,\ldots,\rho_s)\\ (1-\alpha-\beta-[\sigma+\rho]w;\;\sigma_1+\rho_1,\ldots,\;\sigma_s+\rho_s)\end{matrix}\right.\right]dt,$$

where

$$\Phi'(W)=(-z')^W\prod_{j=1}^{N'}\Gamma(1-e_j+E_jW)\left[\;\prod_{j=1}^{Q'}\Gamma(1-f_j+F_jW)\prod_{j=N'+1}^{P'}\Gamma(e_j-E_jW)W!\right]^{-1}$$

Next, suppose that

$$f\,(t)\;=\,i^{m-1}\,H^{M,0}_{\mu,\;\nu}\left[\,\xi\,t\;\middle|\begin{matrix}(a_j',\;A_j)_{1,\;\mu}\\ (b_j',\;B_j)_{1,\nu}\end{matrix}\right]$$

and using a known result ([9], Eq. (3.6)), we get the following interesting multiple integral :

(4.3) $$\int_0^\infty \int_0^\infty (x^2+y^2)^{m-(\alpha+\beta)/2}\, x^{\beta-1}\, y^{\alpha-1}\, e^{i(\alpha+\beta)\tan^{-1}(y/x)}$$

$$.H^{M,0}_{\mu,\nu}\left[\zeta(x^2+y^2)\middle|\begin{matrix}(a_j',A_j)_{1,\mu}\\(b_j',B_j)_{1,\nu}\end{matrix}\right] H^{I,N}_{P,Q+1}\left[B(x,y)\middle|\begin{matrix}\cdots\\\cdots\end{matrix}\right]$$

$$.H_1\left[Y_1(x,y),\ldots,Y_s(x,y)\right]\, dx\, dy$$

$$=\tfrac{1}{2}\zeta^{-m}\sum_{w=0}^{\infty} e^{i\pi(\alpha+\sigma w)/2}\,\Phi(w) H\left[\begin{matrix}z_1\, e^{i\pi\sigma_1/2}\,\zeta^{-\lambda_1}\\ \vdots \\ z_s\, e^{i\pi\sigma_s/2}\,\zeta^{-\lambda_s}\end{matrix}\right|$$

$$\left.\begin{matrix}(1-a-\sigma w;\sigma_1,\ldots,\sigma_s),(1-\beta-\rho u;\rho_1,\ldots,\rho_s),(1-b_j'-mB_j;\lambda_1 B_j,\ldots,\lambda_s B_j)_{1,\nu}\\(1-\alpha-\beta-[\sigma+\rho]w;\sigma_1+\rho_1,\ldots,\sigma_s+\rho_s),(1-a_j'-mA_j;\lambda_1 A_j,\ldots,\lambda_s A_j)_{1,\mu}\end{matrix}\right]$$

which is valid under the following conditions :

(i) $\min_{1\leq j\leq s}\ \{\sigma_j, \rho_j, \lambda_j, \rho, \sigma, Re\,(\alpha), Re\,(\beta), Re\,(m)\} > 0,$

(ii) $Re\,(\alpha)+\sum_{j=1}^{s}\sigma_j u_j > 0,\ Re\,(\beta)+\sum_{j=1}^{s}\rho_j u_j > 0,$

$Re(m)+\min_{1\leq j\leq M}[Re(b_j'/B_j)]+\sum_{j=1}^{s}\lambda_j u_j>0$ $\{u_j$ is given by (1.5) $\}$.

(iii) $U_j > 0$, $|\arg z_j| < \frac{1}{2} U_j \pi$, $\{j=1,\ldots,s\}$

(iv) $A > 0$, $|\arg a| < \frac{1}{2} A\pi$ and

(v) $\Omega \equiv \sum_{j=1}^{M} B_j-\sum_{j=m+1}^{\nu} B_j-\sum_{j=1}^{\mu} A_j > 0$, $|\arg\zeta| < \frac{1}{2}\Omega\pi$,

{ U_j, u_j and A are given by (1.3), (1.5) and (2.3) respectively }.

In a similar manner, the equation (4.2) can be used to obtain another multiple integral analogous to (4.3).

Acknowledgements

The author is grateful to Dr. S. P. Goyal (Banasthali Vidyapith) for his generous help and valuable quidance during the preparation of the present paper. He is also thankful to Professor H. M. Srivastava of the University of Victoria, Canada, for substantial improvement which led to this presentation.

REFERENCES

[1] R. S. Garg, Some multiple integrals involving the multivariable *H*-function, *Indian J. Math.* **24** (1980) (to appear).

[2] R. S. Garg, Certain integral formulae for the multivariable *H*-function (communicated for publication).

[3] T. M. MacRobert, Beta-function formulae and integrals involving *E*-functions, *Math. Ann.* **142** (1961), 450-452.

[4] A. M. Mathai and R. K. Saxena, *The H-function with Applications in Statistics and Other Disciplines*, Wiley Eastern, New Delhi, 1978.

[5] A. K. Rathie, An integral involving *H*-function II, *Vijnana Parishad Anusandhan Patrika* **22** (1979), 235-238.

[6] L. J. Slater, *Generalized Hypergeometric Functions*, Cambridge Univ. Press, 1966.

[7] H.M. Srivastava and R. Panda, Some bilateral generating functions for a class of generalized hypergeometric polynomials, *J. Reine Angew. Math.* **283/284** (1976), 265-274.

[8] H. M. Srivastava and R. Panda, Expansion theorems for the *H*-function of several complex variables, *J. Reine Angew. Math.* **288** (1976) 129-145.

[9] H. M. Srivastava, S. P. Goyal and R. K. Agrawal, Some multiple integral relations for the *H*-function of several variables, *Bull. Inst. Math. Acad. Sinica* **9** (1981), 261-277.

[10] S. K. Vasishta and S. P. Goyal, Some finite integrals involving the *H*-function of two variables, *Univ. Studies Math.* **3** (1975-76) 13-22.