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ON A TWO-DIMENSIONAL H-FUNCTION TRANSFCRM

by

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The object of this paper is to establish a theorem for the two-dimensional H-function transform This theorem is then used to evaluate a general double infinite integral involving the product of Laguerre polynomials, and H-functions of one and two variables. Several other integrals have also been indicated briefly.

1. Introduction

The two-dimensional H-function transform is defined and represented by the following integral equation (Mittal and Goyal [9, p. 2] ; see also Joshi and Prajapat [7, p. 131])

$$(1.1)\quad H\{f(x,y) : s, t\} = st\int_0^\infty\int_0^\infty H_1[sx, ty]\, f(x,y)\, dx\, dy$$

provided that the double integral occurring on the right-hand side of (1.1) is absolutely convergent. { For a systematic study of two general classes of multidimensional (and multivariable) H-function transformations, see a series of recent papers by Srivastava and Panda [15, parts I and II]. }

Here

$$(1.2)\quad H_1[x,y] = H^{0,\,0\;:\;m_2,\,n_2\;;\;m_3,\,n_3}_{p_1,\,q_1\;:\;p_2,\,q_2\;;\;p_3,\,q_3}\left[\begin{matrix} x \\ y \end{matrix}\,\middle|\,\begin{matrix} (a_j\,;\,\alpha_j\,;\,A_j)_{1,\,p_1} : \\ (b_j\,;\,\beta_j\,;\,B_j)_{1,\,q_1} : \end{matrix}\right.$$

$$\left.\begin{matrix}(c_j, C_j)_{1, p_2} ; (e_j, E_j)_{1, p_3} \\ (d_j, \delta_j)_{1, q_2} ; (f_j, F_j)_{1, q_3}\end{matrix}\right]$$

$$= (-1/4\pi^2) \int_{L_1} \int_{L_2} \phi(\xi, \eta)\, \theta_1(\xi) \theta_2(\eta) x^{\xi} y^{\eta}\, d\xi\, d\eta$$

is the well–known H–function of two variables (see, for example, Goyal [3] and Mittal and Gupta [10] where ϕ (ξ, η), θ_1 (ξ) and θ_2 (η) and the conditions on the parameters are given in detail). The notation used here for this function is essentially the same as the notation intorduced by Srivastava and Panda [14, p. 266, Eq. (1.5)] who also gave analogous notations for their H–function of several complex variables (see also Gupta and Goyal [6, p. 26, Eq. (1.1)]).

2 MAIN THEOREM

Let

(i) $Re\,(a) > 0$, $Re\,(b) > 0$, $U > 0$, $V > 0$, $|\arg s| < \frac{1}{2}U\pi$, $|\arg t| < \frac{1}{2}V\pi$,

where

$$(2.1)\ U = -\sum_{j=1}^{p_1} a_j - \sum_{j=1}^{q_1} \beta_j + \sum_{j=1}^{m_2} \delta_j - \sum_{j=m_2+1}^{q_2} \delta_j + \sum_{j=1}^{n_2} \varepsilon_j - \sum_{j=n_2+1}^{p_2} \varepsilon_j$$

and

$$(2.2)\ V = -\sum_{j=1}^{p_1} A_j - \sum_{j=1}^{q_1} B_j + \sum_{j=1}^{m_3} F_j - \sum_{j=m_3+1}^{q_3} F_j + \sum_{j=1}^{n_3} E_j - \sum_{j=n_3+1}^{p_3} E_j$$

(ii) $f(x, y)$ is a continuous function for $x > 0$ and $y > 0$,

(iii) the double H–function transform defined by (1.1) of

$$\left| e^{-ax-by} L_r^{(\nu)} (ax+by) \right|$$

exists.

Then

$$(2.3)\quad H\{ e^{-ax-by} L_r^{(\nu)} (ax+by) f(x, y) ; s, t \}$$

$$= \sum_{k=0}^{r} \frac{(r-k)!\,(\nu+2k)\,\Gamma(\nu+k)\,(-ab)^k}{k!\,\Gamma(\nu+k+r+1)}$$

$$.H\{(xy)^k e^{-ax-by} L_{r-k}^{(\nu+2k)}(ax) L_{r-k}^{(\nu+2k)}(by)\, f(x,y); s, t\}$$

where $L_r^{(\nu)}(x)$ is the well-known Laguerre polynomial.

Proof. From (1.1) and a known formula due to Burchnall and Chaundy [1, p. 127]

$$(2.4) \quad L_r^{(\nu)}(x+y) = \sum_{k=0}^{r} (r-k)!\,(\nu+2k)\,\Gamma(\nu+k)(-xy)^k L_{r-k}^{(\nu+2k)}(x) L_{r-k}^{(\nu+2k)}(y)$$

we have

$$(2.5) \quad H\{e^{-ax-by} L_r^{(\nu)}(ax+by) f(x,y); s, t\}$$

$$= st \int_0^\infty \int_0^\infty e^{-ax-by} H_1[sx, ty] f(x,y).$$

$$\left[\sum_{k=0}^{r} \frac{(r-k)!\,(\nu+2k)\,\Gamma(\nu+k)}{k!\,\Gamma(\nu+k+r+1)} (-a\,b\,x\,y)^k \right.$$

$$\left. .L_{r-k}^{(\nu+2k)}(ax)\, L_{r-k}^{(\nu+2k)}(by) \right] dx\, dy.$$

Now, on interchanging the order of integration and summation in (2.5) (which is justified as the series involved is finite), and using (1.1) again, we easily arrive at the main result (2.3).

3. SPECIAL CASES

The main theorem established here involves a two- dimensional H-function transform which is quite general in nature. Thus with the help of interconnections between this transform and various classes of known integral transforms recorded by Mittal and Goyal [9, pp. 4-5], one can easily obtain a number of (known or new) theorems. We give below a few interesting special cases of the main theorem.

(i) If, in the main theorem, we take $p_1=q_1=0$ all α' s, β' s, ε' s, δ' s, A' s, B' s, E's and F' s equal to unity and apply a known result [10, p. 120, Eq. (1.4)] therein, we get the following corollary.

Corollary 1 (Goyal [5])

$$(3.1)\quad S\left[e^{-ax-by}L_r^{(\nu)}(ax+by)\,f(x,y);\begin{matrix} m,n \\ : \\ p,q\end{matrix}\ \begin{matrix}\alpha,\beta \\ ; \\ \eta,\delta\end{matrix}\ \begin{matrix}(a_p) \\ : \\ (b_q)\end{matrix}\ \begin{matrix}(c_\eta) \\ \\ (d_\delta)\end{matrix}\ ;s,t\right]$$

$$=\sum_{k=0}^{r}\frac{(r-k)\,!\,(\nu+2k)\,\Gamma(\nu+k)\,(-ab)^k}{k\,!\,\Gamma(\nu+k+r+1)}S\Bigg[(xy)^k\,e^{-ax-by}$$

$$.L_{r-k}^{(\nu+2k)}(ax)L_{r-k}^{(\nu+2k)}(by)\,f(xy);\begin{matrix} m,n \\ : \\ p,q\end{matrix}\ \begin{matrix}\alpha,\beta \\ ; \\ \eta,\delta\end{matrix}\ \begin{matrix}(a_p) \\ : \\ (b_q)\end{matrix}\ \begin{matrix}(c_\eta) \\ \\ (d_\delta)\end{matrix}\ ;s,t\Bigg]$$

where $S\,[f\,(x,y)\,;s,t]$ stands for a general integral transform introduced and studied by Goyal [4] and the conditions easily obtainable from the main theorem are assumed to be satisfied.

(ii) Further, putting $n=p=0$, $p=\eta=1$, $m=\alpha=q=\delta=2$ in (3.1) and applying the known relation [4, p. 132, Eq. (2.2)], we arrive at the following interesting result under the conditions directly obtainable from the main theorem.

Corollary 2

$$(3.2)\quad W\left[e^{-ax-by}L_r^{(\nu)}(ax+by)f(x,y);\begin{matrix}\lambda_1+\frac{1}{2}\,;\,k_1+\frac{1}{2}\,,r_1 \\ \lambda_2+\frac{1}{2}\,;\,k_2+\frac{1}{2}\,,r_2\end{matrix}\quad s,t\right]$$

$$=\sum_{k=0}^{r}\frac{(r-k)\,!\,(\nu+2k)\,\Gamma(\nu+k)\,(-ab)^k}{k\,!\,\Gamma(\nu+k+r+1)}\quad W\Bigg[(xy)^k\,e^{-ax-by}$$

$$.L_{r-k}^{(\nu+2k)}(ax)\,L_{r-k}^{(\nu+2k)}(by)\,f(x,y);\begin{matrix}\lambda_1+\frac{1}{2}\,;\,k_1+\frac{1}{2}\,,r_1 \\ \lambda_2+\frac{1}{2}\,;\,k_2+\frac{1}{2}\,,r_2\end{matrix}\,;s,t\Bigg]$$

where

$$(3.3)\ W\left[f(x,y); \begin{matrix} \lambda_1+\frac{1}{2};\ k_1+\frac{1}{2},\ r_1 \\ \lambda_2+\frac{1}{2};\ k_2+\frac{1}{2},\ r_2 \end{matrix} ;\ s,\ t \right]$$

$$= s\,t \int_0^\infty \int_0^\infty (sx)^{-\lambda_1-\frac{1}{2}} (t\,y)^{-\lambda_2-\frac{1}{2}} e^{-\frac{1}{2}(sx+ty)}$$

$$\cdot W_{k_1+\frac{1}{2}, r_1}(sx)\, W_{k_2+\frac{1}{2}, r_2}(ty)\, f(x,y)\, dx\, dy$$

is the generalized Laplace transform of two variables introduced by Nigam [11, p. 331].

Lastly, if we put $\lambda_1 = k_1 = -r_1$, $\lambda_2 = k_2 = -r_2$ in (3.2) it will reduce to corresponding result for the well-known Laplace transform of two variables.

4. APPLICATIONS

In terms of Fox's H-function [2], let

$$(4.1)\ f(x,y) = x^{\rho-1} y^{\sigma-1} H_{p,q}^{m,0}\left[c x^\lambda y^\delta \,\middle|\, \begin{matrix} (g_j, G_j)_{1,p} \\ (h_j, H_j)_{1,q} \end{matrix} \right]$$

in the main theorem, use (1.1) and known results [13, p. 201, Eq. (1); p. 125, Eq. (2)] ; we find that

$$(4.2)\ \int_0^\infty \int_0^\infty x^{\rho-1} y^{\sigma-1} e^{-ax-by} L_r^{(\nu)}(ax+by)$$

$$.H_{p,q}^{m,0}\left[c x^\lambda y^\delta \,\middle|\, \begin{matrix} (g_j, G_j)_{1,p} \\ (h_j, H_j)_{1,q} \end{matrix} \right] H[\, sx,\ ty \,]\, dx\, dy$$

$$= \sum_{k=0}^{r} \frac{(\nu+2k)\,\Gamma(\nu+k)\,(-ab)^k\,\Gamma(1+\nu+k+r)}{k\,.\,(r-k)\ \{\Gamma(1+\nu+2k)\}^2}$$

$$.\int_0^\infty \int_0^\infty x^{\rho+k-1} y^{\sigma+k-1}\ {}_1F_1(1+\nu+k+r;\ 1+\nu+2k;\ -ax)$$

$$\cdot\, {}_1F_1\,(1+\nu+k+r;\, 1+\nu+2k;\, -by)\, H^{m,\,0}_{p\ \ q}\left[cx^{\lambda} y^{\delta} \,\middle|\, \begin{matrix} (g_j,\, G_j)_{1,p} \\ (h_j,\, H_j)_{1,q} \end{matrix} \right]$$

$$.\, H_1\,[\, sx + ty \,]\; dx\; dy.$$

Now, using series expansions for ${}_1F_1$-- functions, changing the order of integrations and summations (which is justified under the conditions imposed with the main integral given below), and applying a known integral due to Panda [12, p. 158, Eq. (1.2)],we arrive at the following double integral (which is believed to be new) :

Main Integral

$$(4.3)\quad \int_0^{\infty}\int_0^{\infty} x^{\rho-1}\, y^{\sigma-1}\, e^{-ax-by}\, L_r^{(\nu)}\,(ax+by)\, H^{m,\,0}_{p,\,q}\left[cx^{\lambda} y^{\delta} \,\middle|\, \begin{matrix} (g_j,\, G_j)_{1,p} \\ (h_j,\, H_j)_{1,q} \end{matrix} \right]$$

$$.\, H_1\,[\, sx,\, ty\,]\; dx\; dy$$

$$= s^{-\rho}\, t^{-\sigma} \sum_{u,\,v=0}^{\infty} \sum_{k=0}^{r} \phi(u,v,k)\, H^{m+n_2+n_3,\; m_2+m_3}_{p+q_1+q_2+q_3,\; q+p_1+p_2+p_3}\Bigg[cs^{-\lambda}\, t^{-\delta} \,\Bigg|$$

$$(d_j';\, \lambda\delta_j)_{1\, m_2},\, (f_j', \delta F_j)_{1,\, q_3},\, (d_j',\, \lambda\delta_j)_{m_2+1,\, q_2},\, (g_j,\, G_j)_{1,\, p}$$

$$(h_j;\, H_j)_{1,\, m},\, (c_j',\, \lambda\varepsilon_j)_{1,n_2},\, (e_j\, ,\delta E_j)_{1,p_3},\, (c_j',\, \lambda\varepsilon_j)_{n_2+1,\, p_2},\, (h_j, H_j)_{m+1,\, q}$$

$$\left.\begin{matrix} (b_j'\ \beta_j')_{1,\ q_1} \\ (a_j'.\, a_j')_{1,\ p_1} \end{matrix}\right]$$

where

$$(4.4)\quad \phi(u,v,k) = \frac{(\nu+2k)\ \Gamma(\nu+k)\Gamma(1+\nu+k+r+u)\Gamma(1+\nu+k+r+v)}{k\ !\ (r-k)\ !\ \Gamma(1+\nu+k+r)\Gamma(1+\nu+2k+u)}$$

$$\frac{(-a/s)^{u+w}\ (-b/t)^{v+k}}{\Gamma(1+\nu+2k+v)\ u!\ v!},$$

$$d_j' = 1 - d_j - (\rho + u + k)\ \delta_j\ ;\ f_j' = 1 - f_j - (\sigma + v + k)\ F_j$$

$c_j' = 1 - c_j - (\rho + u + k)\, \varepsilon_j \; ; \; e_j' = 1 - e_j - (\sigma + v + k)\, E_j$

$a_j' = 1 - a_j - (\rho + u + k)\alpha_j - (\sigma + v + k)\, A_j \; ; \; \alpha_j' = \lambda\alpha_j + \delta A_j$

$b_j' = 1 - b_j - (\rho + u + k)\, \beta_j - (\sigma + v + k) B_j \; ; \; \beta_j' = \beta_j + \delta B_j$

The integral (4.3) is valid under the following conditions :

$Re\,(a) > 0,\; Re\,(b) > 0,\; U > 0,\; V > 0 \; | \arg s | < \frac{1}{2} U\pi,$

$| \arg t | < \frac{1}{2} V\pi,\; \lambda > 0,\; \delta > 0,\; | \arg c | < \frac{1}{2} A\pi,$

$$- \lambda \min_{1 \leq j \leq m} \left[Re \left(\frac{h_j}{H_j} \right) \right] - \min_{1 \leq j \leq m_2} \left[Re \left(\frac{d_j}{\delta_j} \right) \right] < R\,(\rho) -$$

$$- \delta \min_{1 \leq j \leq m} \left[Re \left(\frac{h_j}{H_j} \right) \right] - \min_{1 \leq j \leq m_3} \left[Re \left(\frac{f_j}{F_j} \right) \right] < Re\,(\sigma),$$

and the series occurring on the right-hand side is assumed to oonverge absolutely, A being given by

$$A = \sum_{j=1}^{m} H_j - \sum_{j=m+1}^{q} H_j - \sum_{j=1}^{p} G_j .$$

5. PARTICULAR CASES OF (4 4)

(i) Taking $m = q = 2,\; p = 0,\; H_1 = H_2 = 1,\; h_1 = \mu/2,\;\; h_2 = -\mu/2$ in (4.4) and replacing λ by 2λ, δ by 2δ, and c by $c^2/4$, we get (by virtue of a known formula [8, p. 145. Eq. 20]) The following integral :

$$(5.1) \quad \int_0^\infty \int_0^\infty x^{\rho-1} y^{\sigma-1} e^{-ax-by} L_r^{(\nu)}(ax+by)\, K_\mu(cx^\lambda y^\delta)\, H_1[sx, ty]\, dx\, dy$$

$$= s^{-\rho} t^{-\sigma} \sum_{u,v=0}^{\infty} \sum_{k=0}^{r} \phi(u,v,k) H^{n_2+n_3+2,\; m_2+m_3}_{q_1+q_2+q_3,\; p_1+p_2+p_3+2} \left[\frac{cs^{-\lambda} t^{-\delta}}{2} \,\middle|\, \begin{matrix} (d_j', \delta_j)_{1, m_2}, (f_j', \delta F_j)_{1, m_3}, (d_j', \delta_j)_{m_2+1, q_2}, (b_j', \beta_j')_{1, q_1} \\ (\pm\mu/2, \frac{1}{2}), (c_j', \lambda\varepsilon_j)_{1, n_2}, (e_j', \delta E_j)_{1, n_3}, (c_j', \lambda\varepsilon_j)_{n_2+1, p_2}\; (a_j', \alpha_j')_{1, p_1} \end{matrix} \right]$$

(ii) If we put $m=q=1$, $p=0$, $n_1=0$, $H_1=1$, $\lambda=\delta=1$ in (4.4), we get the following result ;

$$(5.2)\quad \int_0^\infty \int_0^\infty x^{\rho-1} y^{\sigma-1} e^{-ax-by} L_r^{(\nu)}(ax+by) e^{-cxy} H_1[sx, ty]\, dx\, dy$$

$$= s^{-\rho} t^{-\sigma} \sum_{u,v=0}^{\infty} \sum_{k=0}^{r} \phi(u, v, k) H^{n_2+n_3+1,\, m_2+m_3}_{q_1+q_2+q_3,\, p_1+p_2+p_3+1}$$

$$\left[-\frac{c}{st} \left| \begin{matrix} (d_j',\delta_j)_{1,m_2}, (f_j', F_j)_{1,q_3}, (d_j',\delta_j)_{m_2+1,q_2}, (b_j',\beta_j+B_j)_{1,q_1} \\ (0,1)(c_j',\varepsilon_j)_{1,n_2}, (e_j',E_j)_{1,p_3}, (c_j',\varepsilon_j)_{n_2+1,p_2}, (a_j',\alpha_j+A_j)_{1,p_1} \end{matrix} \right. \right].$$

The conditions of validity of (5.1) and (5.2) are easily obtainable from those given with (4.4).

Several other double integrals involving simpler special functions of one and two variables can also be obtained from (4.4) simply by specializing the parameters of various functions involved.

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