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ON INTEGRALS INVOLVING THE *H*-FUNCTION OF SEVERAL VARIABLES

by

S. N. AGAL AND C. L. KOUL

Department of Mathematics, M. R. Engineering College, Jaipur-302004, India

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ABSTRACT

Two multiple integrals of a very general nature are first evaluated, and integration with respect to a parameter is then carried out to establish four other integrals. In the last section two double integrals are evaluated. All these integrals involve the *H*-function of several variables defined by H M. Srivastava and R. Panda.

1. Introduction

The *H*-function of several complex variables $x_1, \ldots x_r$ difined by Srivastava and Panda [5] by means of a multiple Mellin-Barnes type integral is represented in the following form [5, p. 168, Eq. (1.3) **et seq.**] :

(1.1) $H(x_1, \ldots, x_r)$

$$=H^{0,\, l\, :\, (m',\, n');\, \ldots\, ;\, (m^{(r)},\, n^{(r)})}_{A,\, C\, :\, [B',\, D'];\, \ldots;\, [B^{(r)},\, D^{(r)}]}\left(\begin{matrix}[(a):\alpha',\ldots,\alpha^{(r)}] : [(b'):\phi'] ;\ldots; [(b^{(r)}):\phi^{(r)}]; \\ [(c):\beta',\ldots,\beta^{(r)}] : [(d'):\delta'] ;\ldots; [(d^{(r)}):\delta^{(r)}];\end{matrix}\; x_1,\ldots,x_r\right)$$

$$=\frac{1}{(2\pi w)^r}\int_{L_1}\cdots\int_{L_r}\Psi(s_1,\ldots,s_r)\prod_{i=1}^{r}\left\{\Phi_i(s_i)x_i^{s_i}\,ds_i\right\},w=\sqrt{-1}\,.$$

For the values of the function Φ and Ψ, the various conditions on the parameters and conditions of absolute convergence of the multiple contour integrals on the R. H. S. of (1.1), and for its particular cases, etc., we refer to Srivastava and Panda [5]. These conditions are assumed to be satisfied by the various H-functions occurring in this paper.

Here (a) denotes the set of A parameters $a_1, \ldots, a_A$ and $(b^{(i)})$ denotes the set of $B^{(i)}$ parameters $b_1^{(i)}, \ldots, b^{(i)}_{B^{(i)}}$, etc. Also, the appearance of the asterisk (*) indicates that the parameters at those places are the same as the parameters of the H-function of r variables in the L. H. S. of (1.1) at the corresponding places. In case $l=0$ in (1.1), we denote the resuliting function by H_1.

2. Multiple integrals

If R is the region $0 \leq x_1, \ldots, 0 \leq x_r, x_1 + \ldots + x \leq 1$ and

$$F(X) = (1-X)^{b-S-1}(1+kX)^{-b}, X = \sum_{i=1}^{r} x_i, S = \sum_{i=1}^{r} \rho_i$$

then, for $i=1, \ldots, r$, $j=1, \ldots, m^{(i)}$, $k > -1$,

$$(2.1) \quad \int_R \cdots \int H(z_1 X_1, \ldots, z_r X_r) F(X) \prod_{i=1}^{r} (x_i^{\rho_i - 1} dx_i)$$

$$= (1+k)^{-S} H^{0,\, l+r+2\, :\, *}_{A+r+2,\, C+2\, :\, *} \left(\begin{matrix} P : * ; \\ Q : * ; \end{matrix} \; z_1, \ldots, z_r \right)$$

where

$$P : [\, 1-\rho_j : \xi_1', \ldots, \xi_r^{(r)} \,]_{j=1,r},\ [\, 1-b+S : R_1-u_1, \ldots, R_r-u_r \,],$$

$$[\, 1-S : R_1, \ldots, R_r \,],\ [\, (a) : a', \ldots, a^{(r)} \,],$$

$$Q : [\, 1-S : R_1-u_1, \ldots, R_r-u_r \,],\ [\, 1-b : 2R_1-u_1, \ldots, 2R_r-u_r \,],$$

$$[\, (c) : \beta', \ldots, \beta^{(r)} \,],$$

$$X_m = \frac{\prod_{i=1}^{r} (X_i)^{\mu_i} \; X^{u_m}(1-X)^{\mu'}}{(1+kX)^{u_m+2\mu'}}, \quad m = 1, \ldots, r$$

$$Z_i = z_i\,(1+k)^{-R_i}, \quad R_i = u_i + \sum_{m=1}^{r} \xi_m^{(i)}$$

$$\lambda_j^{(i)} = d_j^{(i)} \,/\, \delta_j^{(i)}, \quad \mu_i = \xi_i^{(m)}, \quad \mu' = \sum_{i=1}^{r} (\mu_i),$$

$$\min Re\,(\rho_i, b-S, \rho_i + \sum_{m=1}^{r} \lambda_j^{(i)} \mu_i), \; S + \sum_{i=1}^{r} \lambda_j^{(i)} R_i,$$

$$b - S + \sum_{i=1}^{r} \lambda_j^{(i)} (R_i - u_i) > 0$$

$$(2.2) \quad \int_R \cdots \int H(z_1 X_1', \ldots, z_r X_r') \; F(X) \prod_{i=1}^{r} (x_i^{\rho_i - 1}\, dx_i)$$

$$= \frac{\prod_{i=1}^{r} (\rho_i)}{\Gamma(S)(1+k)^S} H^{0,\; l+2 \;:\; *}_{A+2, C+1 \;:\; *} \left(\begin{matrix} [1-S: u_1, \ldots, u_r], \; [1-b+S: v_1, \ldots, v_r], \\ [1-b: u_1+v_1, \ldots, u_r+v_r], \end{matrix} \right.$$

$$\left. \begin{matrix} [(a): \alpha', \ldots, \alpha^{(r)}] \;:\; * \;; \\ [(c): \beta', \ldots, \beta^{(r)}] \;:\; * \;; \end{matrix} \; Z_1', \ldots, Z_r' \right)$$

where

$$X_i' = X_i^{u_i} (1-X)^{v_i} (1+kX)^{-u_i - v_i} \quad Z_i' = z_i (1+k)^{-u_i},$$

$$\min Re\,(\rho_i, b-S \;\; b-S + \sum_{i=1}^{r} \lambda_j^{(i)} v_i, S + \sum_{i=1}^{r} \lambda_j^{(i)} u_i) > 0.$$

3. Integration with respect to parameter

For $k > 0$, we have

$$(3.1)\qquad \frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty} H\left(\begin{matrix}[a_1+u,a_2-u,a_3,\ldots,a_A:\alpha',\ldots,\alpha^{(r)}]:*; \\ * \qquad :*;\end{matrix}\; z_1,\ldots,z_r\right)du$$

$$=2^{a_1+a_2-2}\, H^{0,\,l-1\;:\;*}_{A-1,\;C\;:\;*}\left(\begin{matrix}T:* \\ *:*\end{matrix}\;;\;2^{-2\alpha'}z_1,\ldots,2^{-2\alpha^{(r)}}z_r\right)$$

where

$T:[a_1+a_2-1;\,2\alpha',\ldots,2\alpha^{(r)}],\;[a_3,\ldots,a_A:\alpha',\ldots,\alpha^{(r)}]$

$l\geqq 2,\ Re\,(a_1+a_2)<2\,;$

$$(3.2)\qquad \frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty} H\left(\begin{matrix}[a_1+u,\,a_2-u,\,a_3,\ldots,a_A:\alpha',\ldots,\alpha^{(r)}]:*; \\ * \qquad :*;\end{matrix}\; z_1,\ldots,z_r\right)$$

$$\times\Gamma(a_3+u)\,\Gamma(a_4-u)du$$

$$=\Gamma(a_3+a_4)H^{0,\;l+1\;:\;*}_{A+1,C+1\;:\;*}\left(\begin{matrix}U:*; \\ V:*;\end{matrix}\;z_1,\ldots,z_r\right)$$

where

$U:[a_1-a_3,a_2-a_4,a_3,\ldots,a_A:\alpha',\ldots,\alpha^{(r)}],\;[a_1+a_2-1:2\alpha',\ldots,2\alpha^{(r)}]$

$V:[a_1+a_2-a_3-a_4-1:2\alpha',\ldots,2\alpha^{(r)}],\;[(c);\beta',\ldots,\beta^{(r)}]$

$l\geqq 2\;\;Re\,(a_3+a_4-a_1-a_2+2)>0;$

$$(3.3)\qquad \frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty} H\left(\begin{matrix}[a_1+u,a_2,\ldots,a_A;\alpha',\ldots,\alpha^{(r)}]:*; \\ * \qquad :*;\end{matrix}\; z_1,\ldots,z_r\right)$$

$$\times\Gamma(a_2+u)\,\Gamma(a_3-u)\,\Gamma(a_4+u)du$$

$$=\Gamma(a_2+a_3)\,\Gamma(a_3+a_4)H^{0,\;l+1\;:\;*}_{A+1,C+1\;:\;*}\left(\begin{matrix}P:; \\ Q:;\end{matrix}\;z_1,\ldots,z_r\right)$$

where

$P:[a_1-a_2,\,a_1-a_3,\,a_2,\ldots,a_A:\alpha',\ldots,\alpha^{(r)}]$

$Q:[a_1-a_2-a_3-a_4:\alpha',\ldots,\alpha^{(r)}],\;[(c):\beta',\ldots,\beta^{(r)}]$

$l \geqq 1,\ R\colon (a_2+a_3+a_4-a_1+1) > 0$;

(3.4) $$2^{c_1+c_2-2} \int_{-\infty}^{\infty} H\left(\begin{matrix} * & :*; \\ [c_1+u, c_2-u, c_3, \ldots, c_C : \beta', \ldots, \beta^{(r)}] & :*; \end{matrix}\ z_1 \ldots, z_r \right) du$$

$$= H^{0,\, l\, :\, *}_{A, C-1 : *}\left(\begin{matrix} * & :*; \\ [c_1+c_2-1 : 2\beta', \ldots, 2\beta^{(r)}], [c_3, \ldots, c_C : \beta', \ldots, \beta^{(r)}] & :*; \end{matrix}\ z_1 \ldots, z_r \right)$$

where $C \geqq 2,\ Re\ (c_1+c_2) < 2$.

4. Double Integrals

(4.1) $$\frac{2^{\alpha+\beta}}{2\pi i} \int_{k-i\infty}^{k+i\infty} \int_{0}^{\infty} \Gamma(\beta-u) e^{-x} x^{\alpha+u-1} H_1(z_1 x^{\sigma_1}, \ldots, z_r x^{\sigma_r}) dx\, du$$

$$= H^{0\ l\ +\ 1\ :\ *}_{A+1, C+1 : *}\left(\begin{matrix} [1-\alpha-\beta : \sigma_1, \ldots, \sigma_r], [(a) : a', \ldots, \alpha^{(r)}] : *; \\ :*; \end{matrix}\ 2^{-\sigma_1} z_1, \ldots, 2^{-\sigma_r} z_r \right)$$

where

$$min\ Re\ (\alpha, \sigma_i\ a+\beta, a+ \sum_{i=1}^{r} \lambda_j^{(i)}\ \sigma_i) > 0,\ \lambda_j^{(i)} = d_j^{(i)} / \delta_j^{(i)}$$

$$(j=1, \ldots,\ m^{(i)});$$

(4.2) $$\frac{1}{2\pi i} \int_{k-i\infty}^{k+i\infty} \int_{0}^{\infty} \Gamma(\alpha+u)\Gamma(\gamma-u)\Gamma(\phi-u)\ e^{-x} x^{\beta+u-1}$$

$$H_1(z_1 x^{\sigma^1}, \ldots, z_r x^{\sigma^r}) dx\ du$$

$$- \Gamma(a+\gamma)\, \Gamma(\alpha+\phi)\ H^{0,\ l\ +\ 2\ ;\ *}_{A+2,\ C+1 ; *}\left(\begin{matrix} P' & ; * ; \\ Q' & ; * ; \end{matrix}\ z_1, \ldots, z_r \right)$$

where

P' : $[1-\gamma-\beta : \sigma_1, \ldots, \sigma_r]$, $[1-\phi-\beta ; \sigma_1, \ldots, \sigma_r]$, $[(a) ; a', \ldots, \alpha^{(r)}]$

Q' ; $[1-\alpha-\gamma-\phi-\beta : \sigma_1, \ldots, \sigma_r]$, $[(c) ; \beta', \ldots, \beta^{(r)}]$

$$min\ Re\ (\beta,\ \sigma_i,\ \alpha+\beta+\gamma+\phi, \beta+ \sum_{i=1}^{r} \lambda_j^{(i)} \sigma_i) > 0.$$

Method of Derivation. To establish (2.1), we substitute for the H-function on the left of (2.1)from (1.1), change the order of integration (which is justified under the conditions mentioned with (2.1) and use [6, p. 258, § 12.5], [2, p. 10, Eq. (11)] and (1.1) to arrive at (2.1). The integral (2.2) is established similarly by using [6, p. 258, § 12.5] and [1]. The integrals (3.1)—(3.4) and (4.1)--(4.2) are established easily by using [3, pp. 81,84, Eqs. (4. 2. 2), (4. 2. 3) and (4. 2. 4. 1)].

Remarks. The integrals established in this paper are capable of yielding a number of integrals because of the general nature of the H-function of several variables. For example, these reduce readily to a number of elegant integrals given by Exton recently (see [3], Chapters 4 and 5). Also, integrals involving the products of r H-functions of one variable can be derived by taking $A=G=0$. In case $r=2$ in (4.1) and (4.2), we get the corrected versions of some known results due to Munot and Mathur [4].

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