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SOME INEQUALITIES FOR POLLACZEK POLYNOMIALS*

by

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ABSTRACT

A connection is found between Pollaczek polynomials with different parameters and ultraspherical polynomials, and then used to obtain inequalities for Pollaczek polynomials.

1. INTRODUCTION

F. Pollaczek ([3], [4]) introduced some remarkable orthogonal polynomials which generalize the Legendre and ultraspherical polynomials. They can be defined from the recurrence relation

$$(1.1) \qquad (n+1)P_n(x) = 2[(n+\lambda+a)x+b]P_n(x)-(n+2\lambda-1)P_{n-1}(x),$$

$$P_{-1}(x) = 0,\ P_0(x) = 1,$$

or from the generating function

$$(1.2) \qquad \sum_{n=0}^{\infty} P_n(x)\, r^n = (1-re^{i\theta})^{-\lambda+it}\,(1-re^{-i\theta})^{-\lambda-it}$$

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where $x = \cos\theta$ and $t = (a\cos\theta + b)/\sin\theta = (ax+b) \quad (a-x^2)^{-\frac{1}{2}}$. When we wish to consider the dependence on the parameters λ, a, b, these polynomials will be denoted by $C_n^\lambda(x;a,b)$. *Szegö* [5,*Appendix*] gave another form of the generating function :

$$(1.3) \qquad \sum_{n=0}^{\infty} C_n^\lambda(x;a,b)\, r^n = (1-2xr+r^2)^{-\lambda}$$

$$\cdot \exp\left\{ (ax+b) \sum_{k=1}^{\infty} \frac{r^k}{k} U_{k-1}(x) \right\}.$$

Here $U_k(x)$ is the Chebychev polynomial of the second kind defined by

$$U_k(\cos\theta) = \sin(k+1)\theta/\sin\theta.$$

When $a = b = 0$, the polynomials are the ultraspherical polynomials, which are denoted by $C_n^\lambda(x)$.

From either (1.2) or 1.3 it is obvious that

$$(1.4) \qquad C_n^{\lambda+\mu}(x;a,b) = \sum_{k=0}^{n} C_{n-k}^\lambda(x;a,b)\, C_k^\mu(x)$$

and, more generally, that

$$(1.5) \qquad C_n^{\lambda+\mu}(x;a_1+a_2,b_1+b_2) = \sum_{k=0}^{n} C_{n-k}^\lambda(x;a_1,b_1)\, C_k^\mu(x;a_2,b_2).$$

2. Inequalities For Pollaczek Polynomials

Askey ([1], [2]) showed that

$$(2.1) \qquad \left| C_n^\lambda(x;a,0) \right| \leqq C_n^\lambda(1;a,0)$$

when $a \geqq 0$ and $\lambda > 0$. He first proved this when $\lambda \geqq 1$, and for $a \geqq (\lambda - \lambda^2)/(1+\lambda)$ when $0 \leqq \lambda < 1$ [1], and later [2] used a different argument to complete the proof when $0 < \lambda < 1$. This inequality is much easier to prove than either of his proofs suggests. Notice that from (1.4) we have

$$(2.2) \qquad \left| C_n^{\lambda}(x; a, b) \right| \leqq \sum_{k=0}^{n} \left| C_{n-k}^{0}(x; a, b)\, C_k^{\lambda}(x) \right| .$$

Szegö [5] gave a number of proofs of

$$(2.3) \qquad \left| C_k^{\lambda}(x) \right| \leqq 1, \ \lambda > 0, \quad -1 \leqq x \leqq 1.$$

From the generating function (1.2) we have

$$\sum_{n=0}^{\infty} C_n^{0}(x; a, b)\, r^n = \sum_{j=0}^{\infty} \frac{(ax+b)^j}{j!} \left[\sum_{k=1}^{\infty} r^k \frac{u_{k-1}(x)}{k} \right]^j .$$

It is complicated to write an explicit formula for $C_n^{0}(x; a, b)$, but it has the form

$$\sum a(j, l_1, \ldots, l_j, m_1, \ldots, m_j)\, (ax+b)^j \left[U_{l_1-1}(x) \right]^{m_1} \cdots \left[U_{l_j-1}(x) \right]^{m_j}$$

for some nonnegative coefficients $a(j, l_1, \ldots, l_j, m_1, \ldots, m_j)$. The sum is over nonnegative integers i, $l_i - 1$ and m_i with $j + \sum_{i=1}^{j} m_i (l_i - 1) = n$. For $-1 \leqq x \leqq 1$, if $a \geqq 0$, $b \geqq 0$, then $|ax+b| \leqq a+b$ and $|U_k(x)| \leqq U_k(1)$ is obvious from the definition, so

(2.4) $$\left| C_n^0 (x; a, b) \right| \leq C_n^0 (1; a, b), \quad -1 \leq x \leq 1 ,$$

when $a \geqq 0$, $b \geqq 0$. The inequalities (2.2), (2.3) and (2.4) combine to give

(2.5) $$\left| C_n^\lambda (x; a, b) \right| \leq C_n^\lambda (1; a, b), \quad -1 \leq x \leq 1$$

when $\lambda \geqq 0$, $a \geqq 0$, $b \geqq 0$. This is an extension of (2.1) to $b > 0$.

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