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ON THREE DIMENSIONAL LEGENDRE STURM LIOUVILLE DIFFUSION AND WAVE PROBLEM GENERATED DUE TO FRACTIONAL TIME DERIVATIVE

By

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Abstract

In this work, we establish and discuss three dimensional Legendre Sturm Liouville diffusion and wave problem generated due to fractional time derivative and with initial and boundary value conditions. Also, we obtain and analyze its solution by Duhamel principle and verify it by convolution theory.
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1 Introduction

The Sturm Liouville system, consisting of initial and boundary values, helps us to solve the problems of Physical, Chemical and Engineering sciences that on using eigenvalues make developments in an important theory which essentially is an extension of the spectral theory from discretized vector spaces into the continuous function spaces ([2], [18], [19], [31]). This theory have continued to provide new ideas and major advances in the field of spectral analysis, solutions to separable partial differential equation and to various transforms and many other applications in the field of Physics ([4], [6], [31]). The one dimensional Sturm Liouville system consists of a second order ordinary differential equation with initial and boundary conditions and yet, it has no unique solution. In this system, the differential equation problems may be regular or singular at each endpoint of the underlying interval ([1], [7], [8], [29]). Duhamel's principle has been utilized for solving some such type of initial and boundary value problems ([4], [6], [30]).

On the other hand, the partial differential equations of fractional order have been occurred in evolution of constructing and modeling of various processes and systems that helping for exploring and stimulating enlargement of Mathematical theory ([3], [7], [9], [10], [12], [13], [14]-[16], [17], [21] and [22]). See also, books of the authors [5], [11], [23], [24], [26] and [27] for fractional calculus evolution and study of various problems of fractional derivatives and integrals.

To relate above theories in this work, we consider the Sturm Liouville system in the form

$$\frac{d}{dx}\left[p(x)\frac{d}{dx}\right]Y(x) - q(x)Y(x) = \lambda\omega(x)Y(x), \tag{1.1}$$

where, $p(x) > 0, q(x) \geq 0, \omega(x) > 0$ and $\forall\lambda, x \in \mathbb{R}, \mathbb{R}$ is the set of real numbers.

The system (1.1), under following boundary conditions, converts into one dimensional Sturm Liouville problem and to solve this problem a representation theorem is followed by (see Churchill [4, p. 291]):

If

$$\frac{d}{dx}\left[p(x)\frac{d}{dx}\right]Y(x)-[q(x)+\lambda\omega(x)]Y(x)=0, \tag{1.2}$$

where, $p(x)>0, q(x)\geq 0, \omega(x)>0, \forall\lambda\in\mathbb{R}$, and $(a<x<b), A_1Y(a)+A_2Y^{'}(a)=0, B_1Y(b)+B_2Y^{'}(b)=0; A_1, A_2, B_1, B_2\in\mathbb{R}$, and independent of λ. Then, Eqn. (1.2) under given conditions is satisfied by $Y(x)=CY_n(x), Y_n(x)$ is the real eigenfunctions $(\forall n\in\mathbb{N}=\{1,2,3,\ldots\})$ and for all $(a\leq x\leq b)$ and in respect of the nonzero real eigenvalues $\lambda_n(n=1,2,3,\ldots)$, C is any nonzero constant. Here,$\omega(x), q(x), p(x), p^{'}(x), (\omega(x)p(x))^{''}$ are continuous real valued functions of $x, \forall(a\leq x\leq b)$.

Again the Eqn. (1.2) has the normalized eigenfunctions on the interval $a\leq x\leq b$ as

$$\psi_n(x)=\frac{Y_n(x)}{\|Y_n(x)\|},\ \text{where, } \|Y_n(x)\|=\left(\int_a^b\omega(x)[Y_n(x)]^2dx\right)^{\frac{1}{2}} \quad \forall n\in\mathbb{N}=\{1,2,3,\ldots\}. \tag{1.3}$$

Further, in Eqn. (1.3), the orthogonal condition of the eigenfunctions $\psi_n(x)$ on the interval (a,b) with weight function $\omega(x)$, may be written by

$$\int_a^b\omega(x)\psi_m(x)\psi_n(x)dx=\begin{cases}0, \text{ if } m\neq n\\ 1, \text{ if } m=n\end{cases}\quad (m,n=1,2,3,\ldots). \tag{1.4}$$

Also, by Eqn. (1.4), the generalized Fourier series on the interval (a,b) due to a sectionally continuous function f with its sectional continuous derivatives f' and f'' is expressed as

$$f(x)=\sum_{n=1}^{\infty}A_n\psi_n(x),\ \text{where },A_n=\int_a^b\omega(\xi)f(\xi)\psi_n(\xi)d\xi. \tag{1.5}$$

To add new ideas in our investigation, we put $A_1=B_1=1, A_2=B_2=0$, $p(x)=1-x^2, q(x)=\frac{\mu^2}{1-x^2}, \omega(x)=1, \lambda=\lambda_n=-n(n+1)(n=1,2,3,\ldots), \forall x\in[-1,1]$, in the Eqn. (1.2) to get the Legendre Sturm Liouville problem due to following differential equation with boundary conditions

$$(1-x^2)\frac{d^2}{dx^2}Y(x)-2x\frac{d}{dx}Y(x)+\left\{n(n+1)-\frac{\mu^2}{1-x^2}\right\}Y(x)=0, (-1<x<1), \tag{1.6}$$

along with the boundary conditions $Y(-1)=0$ and $Y(1)=0$.

The orthogonal condition of the associated Legendre functions $P_n^{\mu}(x)\forall n,\ \mu\in\mathbb{N}=(1,2,3,\ldots)$ is found by (see [28])

$$\int_{-1}^{1}P_m^{\mu}(x)P_n^{\mu}(x)dx=\begin{cases}0, m\neq n;\\ \frac{2}{2n+1}(-1)^{\mu}\frac{\Gamma(n+\mu+1)}{\Gamma(n-\mu+1)}, m=n;\mu\in\mathbb{N}.\end{cases} \tag{1.7}$$

Hence, on making an appeal to the Eqns. (1.2)-(1.5), and (1.7), the solution of the Eqn. (1.6) is written by

$$Y(x) = \frac{2n+1}{2}(-1)^{\mu}\frac{\Gamma(n-\mu+1)}{\Gamma(n+\mu+1)}P_n^{\mu}(x)(\forall n, \mu = 1, 2, 3, \ldots). \tag{1.8}$$

Again, due to the Eqns. (1.6)-(1.8), we prove that:

Proposition 1.1. *For $n, \mu = 1, 2, 3, \ldots$, the solution of the Sturm Liouville problem*

$$\left[\frac{d}{dx}\left[(1-x^2)\frac{d}{dx}\right] + \left(n(n+1) - \frac{\mu^2}{1-x^2}\right)\right]\bar{y}(x) = 0, \tag{1.9}$$

under the conditions , $\bar{y}(-1) = 0$ and $\bar{y}(1) = 0$, is equal to

$$\bar{y}(x) = \frac{2n+1}{2}(-1)^{\mu}\frac{\Gamma(n-\mu+1)}{\Gamma(n+\mu+1)}P_n^{\mu}(x)(\forall n, \mu = 1, 2, 3, \ldots), (-1 < x < 1). \tag{1.10}$$

Proof. The equation (1.9) given in the Proposition 1.1 may be written by

$$(1-x^2)\frac{d^2}{dx^2}\bar{y}(x) - 2x\frac{d}{dx}\bar{y}(x) + \left(n(n+1) - \frac{\mu^2}{1-x^2}\right)\bar{y}(x) = 0. \tag{1.11}$$

Then on making an appeal to the Eqns. (1.6)-(1.8) and with the aid of the representation theorem (1.2)-(1.5) in Eqn. (1.11), we obtain the result (1.10). □

Motivated by above work of the Eqns. (1.1)-(1.11), the k-dimensional Sturm Liouville system is presented in the form

$${}_t^{\mathsf{C}}D_{0+}^{\alpha}Y(x,t) = -divF(DY) + [q(x) + \lambda\omega(x)]Y(x,t) + f(x,t), 0 < \alpha \leq 2, t > 0, \tag{1.12}$$

$x = (x_1, \ldots, x_k) \in \mathbb{R}^k, \forall k \in \mathbb{N}, F(DY)$ is the function of displacement gradient DY, (see Evans [6, p. 66]) and for small DY, the linearization, $F(DY) \approx -p(x)gradY(x,t)$, $p(x) > 0 \forall x \in \mathbb{R}^k, t > 0$, and the Caputo fractional derivative ${}_t^{\mathsf{C}}D_{0+}^{\alpha}$ of $Y(x,t)$ is with respect to time variable t, $q(x) \geq 0 \forall x \in \mathbb{R}^k$, $f : G \times [0, \infty) \to \mathbb{R}, G \subset \mathbb{R}^k$, along with some of its important generalizations (multi-term equation and equation of distributed order) with some initial and boundary conditions considered in open bounded k-dimensional domain G.

In the Eqn. (1.12), the Caputo fractional derivative ${}_t^{\mathsf{C}}D_{0+}^{\alpha}$, $m-1 < \alpha \leq m$, of function $f(t)$ is defined by [5, Definition 3.1, p. 49]

$$({}_t^{\mathsf{C}}D_{0+}^{\alpha}f)(t) = (I^{m-\alpha}f^{(m)})(t), \ \forall m \in \mathbb{N}, \tag{1.13}$$

where, $f^{(m)}(t) = \frac{d^m f}{dt^m}(t)$, $I^{m-\alpha}$ being the Riemann-Liouville fractional integral defined by

$$(I^{m-\alpha}f)(t) = \begin{cases} \frac{1}{\Gamma(m-\alpha)}\int_0^t (t-\tau)^{m-\alpha-1}f(\tau)d\tau, t > 0, m-1 < \alpha \leq m, \\ f(t), \alpha = m, \forall m \in \mathbb{N}. \end{cases}$$

For making further extensions in this field, we appeal the Eqns. (1.12) and (1.13) and then obtain some results of the Legendre Sturm Liouville problem of one dimensional in space and fractional time derivative on introducing inhomogeneous boundary conditions in Proposition 2.1 presented in Section 2, also derive the solution of the three dimensional in space and one dimensional fractional time derivative problem with an application of Duhamel principle and discuss some of its examples in next sections.

2 Formulation of One Dimensional in Space and Generated by Fractional Time Derivative Legendre Sturm Liouville Diffusion and Wave Problem with inhomogeneous conditions

To formulate the one dimensional in space and fractional time derivative Legendre Sturm Liouville diffusion and wave problem with inhomogeneous conditions, first we state initial value problem with fractional time derivative (1.13). Thus in Eqn. (1.12), we set $k = 1, p(x) = 1 - x^2, q(x) = 0, \lambda\omega(x) = -\frac{\mu^2}{1-x^2}, \forall x \in (-1,1), \mu \in \mathbb{N}$ and then, find the one dimensional Legendre Sturm Liouville diffusion wave initial value problem generated by one dimensional fractional time derivative (1.13):

Proposition 2.1. *If*

$$\left[\frac{d}{dx}\left[(1-x^2)\frac{d}{dx}\right] - \frac{\mu^2}{1-x^2}\right]\bar{y}(x,t) = {}_t^{\mathsf{C}}D_{0+}^{\alpha}\bar{y}(x,t), 0 < \alpha \leq 2, \tag{2.1}$$

with initial condition $\bar{y}(x,0) = f(x)$.

Then,

$$\bar{y}(x,t) = \sum_{n=1}^{\infty}\left(\frac{2n+1}{2}\right)(-1)^{\mu}\frac{\Gamma(n-\mu+1)}{\Gamma(n+\mu+1)}P_n^{\mu}(x)E_{\alpha}(-n(n+1)(t)^{\alpha}) \times \int_{-1}^{1} f(\xi)P_n^{\mu}(\xi)d\xi, \forall\mu = 1,2,3,\dots . \tag{2.2}$$

Proof. To prove above facts, consider the elementary function

$$\bar{y}(x,t) = C_n P_n^{\mu}(x)E_{\alpha}(-n(n+1)(t)^{\alpha})\forall n,\mu \in \mathbb{N},\ (-1 < x < 1). \tag{2.3}$$

Here, $P_n^{\mu}(.)$ is the associated Legendre function (see Sneddon [28]) and $E_{\alpha}(.)$ is the one parameter Mittag- Leffler function (see Mathai and Haubold [23]).

Use the formula (2.3) in both sides of Eqn. of (2.1), and apply the Theorem of Diethelm [5, p. 70], and on consequence of the Proposition 1.1 and the Eqn. (1.11), we find the identity

$$C_n E_{\alpha}(-n(n+1)(t)^{\alpha})\left[\frac{d}{dx}\left[(1-x^2)\frac{d}{dx}\right] + \left(n(n+1) - \frac{\mu^2}{1-x^2}\right)\right]P_n^{\mu}(x) = 0. \tag{2.4}$$

Again, on considering the given initial condition given in Eqn (2.1) in formula (2.3) and then in it applying the Eqn. (1.7), we find that

$$C_n = \left(\frac{2n+1}{2}\right)(-1)^{\mu}\frac{\Gamma(n-\mu+1)}{\Gamma(n+\mu+1)}\int_{-1}^{1} f(\xi)P_n^{\mu}(\xi)d\xi. \tag{2.5}$$

Thus, on using (2.3) and (2.5), we find the result of problem (2.1) in the form

$$\bar{y}(x,t) = \left(\frac{2n+1}{2}\right)(-1)^{\mu}\frac{\Gamma(n-\mu+1)}{\Gamma(n+\mu+1)}P_n^{\mu}(x)E_{\alpha}(-n(n+1)(t)^{\alpha}) \times \int_{-1}^{1} f(\xi)P_n^{\mu}(\xi)d\xi\forall n,\mu \in \mathbb{N}. \tag{2.6}$$

Finally, with the help of Eqn. (2.6), we obtain the solution (2.2). □

Now to explore new ideas and enlargement of above work, we construct one dimensional time-fractional and Legendre Sturm Liouville diffusion and wave problem and with inhomogeneous boundary conditions, given by:

$$ {}_t^{\mathsf{C}}D_{0+}^{\alpha}Y(x,t) = \left[\frac{d}{dx}\left[(1-x^2)\frac{d}{dx}\right] - \frac{\mu^2}{1-x^2}\right]Y(x,t) + f(x,t), \tag{2.7} $$

$\forall(x,t) \in (-1,1)\times(0,\infty)$, where, $f : (-1,1)\times[0,\infty) \to \mathbb{R}$, $0 < \alpha \leq 2$, subjected to the initial and inhomogeneous boundary values

$$ Y(x,0) = \left(\frac{1-x}{2}\right)\varphi_1(0) + \left(\frac{1+x}{2}\right)\varphi_2(0), \forall(x,t) \in [-1,1]\times(0), \tag{2.8} $$

$Y(-1,t) = \varphi_1(t), \forall(x,t) \in (-1)\times(0,\infty), Y(1,t) = \varphi_2(t), \forall(x,t) \in (1)\times(0,\infty).$

It is remarkable that for $\alpha = 1$, the equation (2.7) converts into a linear second order parabolic partial differential equation and a diffusion problem with initial and boundary conditions (2.8). On the other hand, when $0 < \alpha \leq 1$, above problem (2.7)-(2.8) becomes identical to the initial-boundary value problem for the one dimensional time fractional diffusion equation with inhomogeneous boundary conditions. For $\alpha = 2$, the equation (2.7) changed into a linear second order parabolic partial differential equation and a wave problem with initial and boundary conditions (2.8), found in the literature.

3 Solution of the Problem (2.7)-(2.8)

To solve the problem (2.7)-(2.8), we set $Y(x,t) = y(x,t) + (\frac{1-x}{2})\varphi_1(t) + (\frac{1+x}{2})\varphi_2(t)$, and find that in the form

$$ {}_t^{\mathsf{C}}D_{0+}^{\alpha}y(x,t) = \left[\frac{d}{dx}\left[(1-x^2)\frac{d}{dx}\right] - \frac{\mu^2}{1-x^2}\right]y(x,t) + f_1(x,t), \tag{3.1} $$

with the initial and boundary conditions

$$ \begin{aligned} y(-1,t) = 0, y(1,t) = 0, y(x,0) = 0, f_1(x,t) = f(x,t) - \left(\frac{1-x}{2}\right){}_t^{\mathsf{C}}D_{0+}^{\alpha}\varphi_1(t) \\ -\left(\frac{1+x}{2}\right){}_t^{\mathsf{C}}D_{0+}^{\alpha}\varphi_2(t) + \left(\left(\frac{x^2+x-\frac{\mu^2}{2}}{1+x}\right)\varphi_1(t) - \left(\frac{\frac{\mu^2}{2}+x-x^2}{1-x}\right)\varphi_2(t)\right). \end{aligned} \tag{3.2} $$

Then to solve the problem (3.1)-(3.2), for fixed τ and $\sigma \in (-1,1)$, we define the discrete function

$$ y(x,t,\tau) \quad = \quad \begin{cases} \left(\frac{2n+1}{2}\right)(-1)^{\mu}\frac{\Gamma(n-\mu+1)}{\Gamma(n+\mu+1)}P_n^{\mu}(x)\int_{-1}^{1}E_{\alpha}(-n(n+1)(t-\tau)^{\alpha})f_1(\sigma,\tau) \\ \times P_n^{\mu}(\sigma)d\sigma \forall n,\mu \in \mathbb{N}, \text{ and } t > \tau > 0, -1 \leq \sigma < x \leq 1; \\ f_1(x,t), \text{ at } t = \tau. \end{cases} \tag{3.3} $$

and prove that:

Theorem 3.1. *For fixed τ and σ, if the problem is given by*

$$ {}_t^{\mathsf{C}}D_{0+}^{\alpha}y(x,t,\tau) - \left[\frac{d}{dx}\left[(1-x^2)\frac{d}{dx}\right] - \frac{\mu^2}{1-x^2}\right]y(x,t,\tau) = 0, \tag{3.4} $$

where, $t > \tau > 0, -1 \leq \sigma < x \leq 1$*; and*

$$y(x,t,\tau) = f_1(x,t), \ \text{at } t = \tau.$$

Then, the function (3.3) satisfies the Eqn. (3.4) and is not a solution of the problem (3.1)-(3.2) and the solution of the problem (2.7)-(2.8) is

$$Y(x,t) = \sum_{n=1}^{\infty}\left(\frac{2n+1}{2}\right)(-1)^{\mu}\frac{\Gamma(n-\mu+1)}{\Gamma(n+\mu+1)}P_n^{\mu}(x)\int_0^t\int_{-1}^1 E_{\alpha}(-n(n+1)(t-\tau)^{\alpha}) \times f_1(\sigma,\tau)P_n^{\mu}(\sigma)d\sigma d\tau + \left(\frac{1-x}{2}\right)\varphi_1(t) + \left(\frac{1+x}{2}\right)\varphi_2(t), \forall \mu \in \mathbb{N}, \quad (3.5)$$

provided that at initial time $t = \tau$, $\left(\frac{2n+1}{2}\right)(-1)^{\mu}\frac{\Gamma(n-\mu+1)}{\Gamma(n+\mu+1)}P_n^{\mu}(x)\int_{-1}^1 f_1(\sigma,\tau)P_n^{\mu}(\sigma)d\sigma = f_1(x,t)$.

Proof. For $t > \tau > 0, -1 \leq \sigma < x \leq 1$, make an use of the function (3.3) in first term of left hand side of the Eqn. (3.4) and then due to Diethelm [5, Theorem 4.3, p. 70] to get

$${}_t^{\mathsf{C}}D_{0+}^{\alpha}y(x,t,\tau) = n(n+1)\left(\frac{2n+1}{2}\right)(-1)^{\mu+1}\frac{\Gamma(n-\mu+1)}{\Gamma(n+\mu+1)}P_n^{\mu}(x) \times \int_{-1}^1 E_{\alpha}(-n(n+1)(t-\tau)^{\alpha})f_1(\sigma,\tau)P_n^{\mu}(\sigma)d\sigma \forall n,\mu \in \mathbb{N}. \quad (3.6)$$

Further in the same conditions, use the function (3.3) in second term of the left hand side of the Eqn. (3.4), we find

$$\left[\frac{d}{dx}\left[(1-x^2)\frac{d}{dx}\right] - \frac{\mu^2}{1-x^2}\right]y(x,t,\tau) = \left(\frac{2n+1}{2}\right)(-1)^{\mu}\frac{\Gamma(n-\mu+1)}{\Gamma(n+\mu+1)} \times \int_{-1}^1 E_{\alpha}(-n(n+1)(t-\tau)^{\alpha})f_1(\sigma,\tau)P_n^{\mu}(\sigma)d\sigma\left[\frac{d}{dx}\left[(1-x^2)\frac{d}{dx}\right] - \frac{\mu^2}{1-x^2}\right]P_n^{\mu}(x). \quad (3.7)$$

Then, make an appeal to the Proposition 1.1 into the Eqns. (3.6) and (3.7), we obtain following identity

$${}_t^{\mathsf{C}}D_{0+}^{\alpha}y(x,t,\tau) - \left[\frac{d}{dx}\left[(1-x^2)\frac{d}{dx}\right] - \frac{\mu^2}{1-x^2}\right]y(x,t,\tau) = \left(\frac{2n+1}{2}\right)(-1)^{\mu+1} \times \frac{\Gamma(n-\mu+1)}{\Gamma(n+\mu+1)}\int_{-1}^1 E_{\alpha}(-n(n+1)(t-\tau)^{\alpha})f_1(\sigma,\tau)P_n^{\mu}(\sigma)d\sigma \times \left[\frac{d}{dx}\left[(1-x^2)\frac{d}{dx}\right] + \left(n(n+1) - \frac{\mu^2}{1-x^2}\right)\right]P_n^{\mu}(x) = 0. \quad (3.8)$$

Again, $\forall n,\mu \in \mathbb{N}$ and for $t > \tau > 0, -1 \leq \sigma < x \leq 1$, make an appeal to the definition (3.3) and use the results (1.7), we may write

$$\left(\frac{2n+1}{2}\right)(-1)^{\mu}\frac{\Gamma(n-\mu+1)}{\Gamma(n+\mu+1)}P_n^{\mu}(x)$$

$$\times \int_{-1}^{1} E_\alpha(-n(n+1)(t-\tau)^\alpha) f_1(\sigma,\tau) P_n^\mu(\sigma) d\sigma = f_1(x,t),$$

when $t=\tau$, and then at this time $t=\tau$, the condition given in equation (3.5) is satisfied.

Then at $t=\tau$, to get

$$\begin{aligned} y(x,t,\tau) - f_1(x,t) \\ = \left(\frac{2n+1}{2}\right)(-1)^\mu \frac{\Gamma(n-\mu+1)}{\Gamma(n+\mu+1)} P_n^\mu(x) \int_{-1}^{1} E_\alpha(-n(n+1)(t-\tau)^\alpha) f_1(\sigma,\tau) P_n^\mu(\sigma) d\sigma \\ - \left(\frac{2n+1}{2}\right)(-1)^\mu \frac{\Gamma(n-\mu+1)}{\Gamma(n+\mu+1)} P_n^\mu(x) \int_{-1}^{1} E_\alpha(-n(n+1)(t-\tau)^\alpha) f_1(x,t) P_n^\mu(\sigma) d\sigma. \end{aligned} \tag{3.9}$$

Thus, the Eqn. (3.9) for $t=\tau$ gives us

$$y(x,t,\tau) - f_1(x,t) = 0. \tag{3.10}$$

Hence the formula defined in the Eqn. (3.3) satisfies the Eqn. (3.4) and it does not satisfy all conditions of Eqns. (3.1) - (3.2).

Therefore, due to the Duhamel principle $y(x,t) = \int_0^t y(x,t,\tau)d\tau$ for $-1 \le x \le 1$ and $t > \tau > 0$, we obtain the solution of the problem (3.1)-(3.2) in the form

$$\begin{aligned} y(x,t) = \left(\frac{2n+1}{2}\right)(-1)^\mu \frac{\Gamma(n-\mu+1)}{\Gamma(n+\mu+1)} P_n^\mu(x) \\ \times \int_0^t \int_{-1}^{1} E_\alpha(-n(n+1)(t-\tau)^\alpha) f_1(\sigma,\tau) P_n^\mu(\sigma) d\sigma d\tau, \forall n,\mu \in \mathbb{N}. \end{aligned} \tag{3.11}$$

Finally, Eqn. (3.11) gives us the solution (3.5). Again, since $P_n^\mu(-1) = 0 = P_n^\mu(1)$, so that the Eqn. (3.5) satisfies all conditions of the problem (2.7)-(2.8) and hence it is the solution of the problem (2.7)-(2.8). □

4 Formulation of Three Dimensional in Space Legendre Sturm Liouville Diffusion Problem Generated by Fractional Time Derivative

In this section, on making an appeal to the Equation (1.12) for $k=3$, and then $\forall x = (x_1,x_2,x_3) \in [-1,1]\times[-1,1]\times[-1,1]$,we formulate Legendre Sturm Liouville in three dimensional space generated by fractional time derivative in terms of diffusion and wave problem with initial and homogeneous boundary conditions, given by:

Theorem 4.1. *For the function defined by*

$$f : (-1,1)\times(-1,1)\times(-1,1)\times[0,\infty) \to \mathbb{R},$$

such that $f(x_1,x_2,x_3,t) = f(x_1,t) + f(x_2,t) + f(x_3,t)$, *and* $\forall (x_1,x_2,x_3,t) \in (-1,1)\times(-1,1)\times(-1,1)\times(0,\infty), t > \tau_3 > \tau_2 > \tau_1 > 0$, *if*

$$\begin{aligned} {}_t^{\mathsf{C}}D_{0+}^\alpha Y(x_1,x_2,x_3,t) \\ = \left\{ \left[\frac{d}{dx_1}\left[(1-x_1^2)\frac{d}{dx_1}\right] - \frac{\mu^2}{1-x_1^2}\right] + \left[\frac{d}{dx_2}\left[(1-x_2^2)\frac{d}{dx_2}\right] - \frac{\mu^2}{1-x_2^2}\right] \right. \end{aligned}$$

$$+\left[\frac{d}{dx_3}\left[(1-x_3^2)\frac{d}{dx_3}\right]-\frac{\mu^2}{1-x_3^2}\right]\Bigg\}Y(x_1,x_2,x_3,t)+f(x_1,x_2,x_3,t), \quad (4.1)$$

where, $0<\alpha\leq 2$, subject to the initial and homogeneous boundary values, given by

$$\begin{aligned}
&Y(x_1,x_2,x_3,0)=0,\forall(x_1,x_2,x_3,t)\in[-1,1]\times[-1,1]\times[-1,1]\times(0),\\
&\qquad Y(-1,-1,-1,t)=0,\forall(x_1,x_2,x_3,t)\in(-1)\times(-1)\times(-1)\times(0,\infty),\\
&\qquad\qquad Y(1,1,1,t)=0,\forall(x_1,x_2,x_3,t)\in(1)\times(1)\times(1)\times(0,\infty). \quad (4.2)
\end{aligned}$$

Then, for $P_n^\mu(-1)=0=P_n^\mu(1)$ and $\left(\frac{2n+1}{2}\right)(-1)^\mu\frac{\Gamma(n-\mu+1)}{\Gamma(n+\mu+1)}P_n^\mu(x)$ $\times\int_{-1}^1 f(\sigma_i,\tau_i)P_n^\mu(\sigma_i)d\sigma_i=f(x_i,t)$, at $t=\tau_i,\forall i=1,2,3$, the result is found by

$$\begin{aligned}
Y(x_1,x_2,x_3,t)=&\sum_{n=1}^{\infty}\left(\left(\frac{2n+1}{2}\right)(-1)^\mu\frac{\Gamma(n-\mu+1)}{\Gamma(n+\mu+1)}\right)^3\prod_{i=1}^{3}P_n^\mu(x_i)\\
&\times\int_0^t\int_{-1}^1 E_\alpha(-n(n+1)(t-\tau_i)^\alpha)f(\sigma_i,\tau_i)P_n^\mu(\sigma_i)d\sigma_i d\tau_i \quad (4.3)
\end{aligned}$$

Proof. To solve the problem (4.1)-(4.2), for $t>\tau_3>\tau_2>\tau_1>0$, and $\forall(x_1,x_2,x_3)\in(-1,1)\times(-1,1)\times(-1,1)$ we consider that $Y(x_1,x_2,x_3,t)=u(x_1,t)v(x_2,t)w(x_3,t)$. Also, $u(x_1,t)=\int_0^t u(x_1,t,\tau_1)d\tau_1$, $v(x_2,t)=\int_0^t v(x_2,t,\tau_2)d\tau_2$, $w(x_3,t)=\int_0^t w(x_3,t,\tau_3)d\tau_3$, and make an appeal to the techniques of Theorem 3.1, under certain time, we suppose that the homogeneous problem of Eqn. (4.1) is

$$\begin{aligned}
&{}_t^{\mathsf{C}}D_{0+}^{\alpha}Y(x_1,x_2,x_3,t,\tau_1,\tau_2,\tau_3)\\
&\quad=\Bigg\{\left[\frac{d}{dx_1}\left[(1-x_1^2)\frac{d}{dx_1}\right]-\frac{\mu^2}{1-x_1^2}\right]+\left[\frac{d}{dx_2}\left[(1-x_2^2)\frac{d}{dx_2}\right]-\frac{\mu^2}{1-x_2^2}\right]\\
&\qquad+\left[\frac{d}{dx_3}\left[(1-x_3^2)\frac{d}{dx_3}\right]-\frac{\mu^2}{1-x_3^2}\right]\Bigg\}Y(x_1,x_2,x_3,t,\tau_1,\tau_2,\tau_3), \quad (4.4)
\end{aligned}$$

Again, the Eqn. (4.4) has the separable solution

$$Y(x_1,x_2,x_3,t,\tau_1,\tau_2,\tau_3)=u(x_1,t,\tau_1)v(x_2,t,\tau_2)w(x_3,t,\tau_3), \quad (4.5)$$

when $t>\tau_3>\tau_2>\tau_1>0$, and $\forall(x_1,x_2,x_3)\in(-1,1)\times(-1,1)\times(-1,1)$; and for $t=\tau_1$, or $t=\tau_2$, or $t=\tau_3$, there exists $Y(x_1,\ x_2,\ x_3,\ t,\ \tau_1,\ \tau_2,\ \tau_3)=f(x_1,\ x_2,\ x_3,\ t,\ \tau_1,\ \tau_2,\ \tau_3)$ $=(\frac{(t-\tau_2)(t-\tau_3)}{(t_1-\tau_2)(t_1-\tau_3)})f(x_1,t)+(\frac{(t-\tau_1)(t-\tau_3)}{(t_2-\tau_1)(t_2-\tau_3)})f(x_2,t)+(\frac{(t-\tau_1)(t-\tau_2)}{(t_3-\tau_1)(t_3-\tau_2)})f(x_3,t)$, so that by the step conditions we find

$$Y(x_1,x_2,x_3,t,\tau_1,\tau_2,\tau_3)=\begin{cases}f(x_1,\tau_1), & \text{at } t=\tau_1,\\ f(x_2,\tau_2), & \text{at } t=\tau_2,\\ f(x_3,\tau_3), & \text{at } t=\tau_3.\end{cases} \quad (4.6)$$

Then, from Eqns. (4.4) and (4.5), we may write

$$v(x_2,t,\tau_2)w(x_3,t,\tau_3){}_t^{\mathsf{C}}D_{0+}^{\alpha}u(x_1,t,\tau_1)+u(x_1,t,\tau_1)w(x_3,t,\tau_3){}_t^{\mathsf{C}}D_{0+}^{\alpha}v(x_2,t,\tau_2)$$

$$+ u(x_1,t,\tau_1)v(x_2,t,\tau_2){}_t^{\mathsf{C}}D_{0+}^{\alpha} w(x_3,t,\tau_3) = v(x_2,t,\tau_2)w(x_3,t,\tau_3)$$
$$\times \left[\frac{d}{dx_1}\left[(1-x_1^2)\frac{d}{dx_1}\right] - \frac{\mu^2}{1-x_1^2}\right]u(x_1,t,\tau_1) + u(x_1,t,\tau_1)w(x_3,t,\tau_3)$$
$$\times \left[\frac{d}{dx_2}\left[(1-x_2^2)\frac{d}{dx_2}\right] - \frac{\mu^2}{1-x_2^2}\right]v(x_2,t,\tau_2) + u(x_1,t,\tau_1)v(x_2,t,\tau_2)$$
$$\times \left[\frac{d}{dx_3}\left[(1-x_3^2)\frac{d}{dx_3}\right] - \frac{\mu^2}{1-x_3^2}\right]w(x_3,t,\tau_3) \quad (4.7)$$

Now, apply the Eqns. (4.2) and (4.6) into the Eqn. (4.7), the identity holds, when it is provided that following equations are satisfied

$${}_t^{\mathsf{C}}D_{0+}^{\alpha} u(x_1,t,\tau_1) = \left[\frac{d}{dx_1}\left[(1-x_1^2)\frac{d}{dx_1}\right] - \frac{\mu^2}{1-x_1^2}\right]u(x_1,t,\tau_1),$$
$$u(-1,t,\tau_1) = u(1,t,\tau_1) = 0, \text{ when } t > \tau_1 > 0, 1 \geq x_1 > \sigma_1 > -1,$$
$$u(x_1,t,\tau_1) = f(x_1,\tau_1), \text{ at } t = \tau_1, \quad (4.8)$$

${}_t^{\mathsf{C}}D_{0+}^{\alpha} v(x_2,t,\tau_2) = \left[\frac{d}{dx_2}\left[(1-x_2^2)\frac{d}{dx_2}\right] - \frac{\mu^2}{1-x_2^2}\right]v(x_2,t,\tau_2), v(-1,t,\tau_2) = v(1,t,\tau_2) = 0,$ when $t > \tau_2 > 0, 1 \geq x_2 > \sigma_2 > -1, v(x_2,t,\tau_2) = f(x_2,\tau_2)$, at $t = \tau_2$,

${}_t^{\mathsf{C}}D_{0+}^{\alpha} w(x_3,t,\tau_3) = \left[\frac{d}{dx_2}\left[(1-x_2^2)\frac{d}{dx_2}\right] - \frac{\mu^2}{1-x_2^2}\right]w(x_3,t,\tau_3), w(-1,t,\tau_3) = w(1,t,\tau_3) = 0,$ when $t > \tau_3 > 0, 1 \geq x_3 > \sigma_3 > -1, v(x_3,t,\tau_3) = f(x_3,\tau_3)$, at $t = \tau_3$.

Then, make an appeal to the Theorem 3.1 into the Eqns. of (4.8), for $P_n^{\mu}(-1) = 0 = P_n^{\mu}(1)$,their solutions may be written by

$$u(x_1,t) = \sum_{n=1}^{\infty}\left(\frac{2n+1}{2}\right)(-1)^{\mu}\frac{\Gamma(n-\mu+1)}{\Gamma(n+\mu+1)}P_n^{\mu}(x_1)$$
$$\times \int_0^t \int_{-1}^1 E_{\alpha}(-n(n+1)(t-\tau_1)^{\alpha})f(\sigma_1,\tau_1)P_n^{\mu}(\sigma_1)d\sigma_1 d\tau_1 \forall \mu \in \mathbb{N};$$
$$v(x_2,t) = \sum_{n=1}^{\infty}\left(\frac{2n+1}{2}\right)(-1)^{\mu}\frac{\Gamma(n-\mu+1)}{\Gamma(n+\mu+1)}P_n^{\mu}(x_2)$$
$$\times \int_0^t \int_{-1}^1 E_{\alpha}(-n(n+1)(t-\tau_2)^{\alpha})f(\sigma_2,\tau_2)P_n^{\mu}(\sigma_2)d\sigma_2 d\tau_2 \forall \mu \in \mathbb{N};$$
$$w(x_3,t) = \sum_{n=1}^{\infty}\left(\frac{2n+1}{2}\right)(-1)^{\mu}\frac{\Gamma(n-\mu+1)}{\Gamma(n+\mu+1)}P_n^{\mu}(x_3)$$
$$\times \int_0^t \int_{-1}^1 E_{\alpha}(-n(n+1)(t-\tau_3)^{\alpha})f(\sigma_3,\tau_3)P_n^{\mu}(\sigma_3)d\sigma_3 d\tau_3 \forall \mu \in \mathbb{N}; \quad (4.9)$$

respectively.

Finally, on using the results given in (4.9)), in the relation $Y(x_1,x_2,x_3,t) = u(x_1,t)v(x_2,t)w(x_3,t)$, we obtain the solution (4.3), under the given conditions of Eqn. (4.2). □

5 Examples

It is desired to solve

$$
\begin{aligned}
&{}_t^{\mathsf{C}}D_{0^+}^{\alpha} Y(x_1,x_2,t) = \\
&\quad\left\{\left[\frac{d}{dx_1}\left[(1-x_1^2)\frac{d}{dx_1}\right]-\frac{\mu^2}{1-x_1^2}\right]+\left[\frac{d}{dx_2}\left[(1-x_2^2)\frac{d}{dx_2}\right]-\frac{\mu^2}{1-x_2^2}\right]\right\}Y(x_1,x_2,t) \\
&\qquad + 2sin\left(\frac{x_1+x_2}{2}t\right)\cos\left(\frac{x_1-x_2}{2}t\right), \forall(x_1,x_2,t)\in(-1,1) \\
&\qquad\qquad \times(-1,1)\times(0,\infty), t>\tau_2>\tau_1>0, 0<\alpha\le 2, \quad (5.1)
\end{aligned}
$$

subject to the initial and homogeneous boundary values, given by

$$
\begin{aligned}
&Y(x_1,x_2,0)=0, \forall(x_1,x_2,t)\in[-1,1]\times[-1,1]\times(0), Y(-1,-1,t)=0, \forall(x_1,\\
&\qquad x_2,t)\in(-1)\times(-1)\times(0,\infty), \quad (5.2)
\end{aligned}
$$

$Y(1,1,t)=0, \forall(x_1,x_2,t)\in(1)\times(1)\times(0,\infty)$.

Then, for $P_n^{\mu}(-1)=0=P_n^{\mu}(1)$ and on making an appeal to the Theorem 4.1, we find

$$
\begin{aligned}
Y(x_1,x_2,t) &= \sum_{n=1}^{\infty}\left(\left(\frac{2n+1}{2}\right)(-1)^{\mu}\frac{\Gamma(n-\mu+1)}{\Gamma(n+\mu+1)}\right)^2\prod_{i=1}^{2}P_n^{\mu}(x_i) \\
&\times\int_0^t\int_{-1}^{1}E_{\alpha}(-n(n+1)(t-\tau_i)^{\alpha})\sin(\sigma_i\tau_i)P_n^{\mu}(\sigma_i)d\sigma_i d\tau_i, \quad (5.3)
\end{aligned}
$$

provided that $\left(\frac{2n+1}{2}\right)(-1)^{\mu}\frac{\Gamma(n-\mu+1)}{\Gamma(n+\mu+1)}P_n^{\mu}(x)\int_{-1}^{1}\sin(\sigma_i\tau_i)P_n^{\mu}(\sigma_i)d\sigma_i=\sin(x_i t)$, at $t=\tau_i, \forall i=1,2$. That obviously, on using Eqn. (1.7) gives the following identity $\forall\ m=n(m,n\in\mathbb{N})$ and $t=\tau_i\forall i=1,2$;

$\int_{-1}^{1}\sin(\sigma_i\tau_i)P_n^{\mu}(\sigma_i)d\sigma_i=\int_{-1}^{1}\sin(x_i t)P_n^{\mu}(x_i)dx_i$.

Verification: To verify our formula by separation of variables rule (see Pipe [25, p. 510]), we consider the homogeneous equation of (5.1), given by

$$
\begin{aligned}
&{}_t^{\mathsf{C}}D_{0^+}^{\alpha} Y(x_1,x_2,t) = \\
&\quad\left\{\left[\frac{d}{dx_1}\left[(1-x_1^2)\frac{d}{dx_1}\right]-\frac{\mu^2}{1-x_1^2}\right]+\left[\frac{d}{dx_2}\left[(1-x_2^2)\frac{d}{dx_2}\right]-\frac{\mu^2}{1-x_2^2}\right]\right\}Y(x_1,x_2,t) \quad (5.4)
\end{aligned}
$$

which has the solution $Y(x_1,x_2,t)=U(x_1,t)V(x_2,t)$, where, $U(x_1,t)=U(x_1)E_{\alpha}(-n(n+1)(t)^{\alpha}), V(x_2,t)=V(x_2)E_{\alpha}(-n(n+1)(t)^{\alpha})$ and we have $2\sin\left(\frac{x_1+x_2}{2}t\right)\cos\left(\frac{x_1-x_2}{2}t\right)=\sin(x_1t)+sin(x_2t)$, such that at $t=\tau_1, U(x_1,t)=\sin(\sigma_1\tau_1)$ and $V(x_2,t)=1, \forall 0<\tau_1<t, 1>x_1>\sigma_1>-1$ and at $t=\tau_2, U(x_1,t)=1$ and $V(x_2,t)=\sin(\sigma_2\tau_2), \forall 0<\tau_2<t, 1>x_2>\sigma_2>-1$, then from Eqns. (5.1) -(5.2) to get

$$
\left[\frac{d}{dx_1}\left[(1-x_1^2)\frac{d}{dx_1}\right]+\left(n(n+1)-\frac{\mu^2}{1-x_1^2}\right)\right]U(x_1)=0,
$$

$$U(-1,t)=0, U(1,t)=0, \text{ at } t=\tau_1,$$
$$U(x_1,t)=\sin(\sigma_1\tau_1), V(x_2,t)=1, \forall 0<\tau_1<t, 1>x_1>\sigma_1>-1, \quad (5.5)$$

and

$$\left[\frac{d}{dx_2}\left[(1-x_2^2)\frac{d}{dx_2}\right]+\left(n(n+1)-\frac{\mu^2}{1-x_2^2}\right)\right]V(x_2)=0,$$
$$V(-1,t)=0, V(1,t)=0, \text{ at } t=\tau_2, U(x_1,t)=1,$$
$$V(x_2,t)=\sin(\sigma_2\tau_2), 0<\tau_2<t, 1>x_2>\sigma_2>-1. \quad (5.6)$$

In the Eqns. (5.5) and (5.6), use the techniques of Proposition 2.1 and use Eqns. (1.6)-(1.8) $\forall 0<\tau_1<t, 1>x_1>\sigma_1>-1$ and $\forall 0<\tau_2<t,\ 1>x_2>\sigma_2>-1$,to get

$$U(x_1,t-\tau_1)=\left(\frac{2n+1}{2}\right)(-1)^{\mu}\frac{\Gamma(n-\mu+1)}{\Gamma(n+\mu+1)}P_n^{\mu}(x_1)E_{\alpha}(-n(n+1)(t-\tau_1)^{\alpha}),$$
$$t>\tau_1>0, \text{ and at } t=\tau_1, U(x_1,t)=\sin(\sigma_1\tau_1), V(x_2,t)=1 \quad (5.7)$$

and $V(x_2,(t-\tau_2))=\left(\frac{2n+1}{2}\right)(-1)^{\mu}\frac{\Gamma(n-\mu+1)}{\Gamma(n+\mu+1)}P_n^{\mu}(x_2)E_{\alpha}(-n(n+1)(t-\tau_2)^{\alpha}), t>\tau_2>0$ and at $t=\tau_2, U(x_1,t)=1, V(x_2,t)=\sin(\sigma_2\tau_2)$.

Thus, from Eqns. (5.4) and (5.7) and by convolution theory (see Evans [6, p. 47]) and with the aid of integral equation theory, $\forall \mu, n\in\mathbb{N}$, we get

$$U(x_1,t)=P_n^{\mu}(x_1)\int_0^t\int_{-1}^1 U(\sigma_1,t-\tau_1)\sin(\sigma_1\tau_1)d\sigma_1 d\tau_1 \text{ and}$$
$$V(x_2,t)=P_n^{\mu}(x_2)\int_0^t\int_{-1}^1 U(\sigma_2,t-\tau_2)\sin(\sigma_2\tau_2)d\sigma_2 d\tau_2. \quad (5.8)$$

Then, make an appeal to the Eqns. (5.7) and (5.8), $\forall \mu, n\in\mathbb{N}$, we get

$$U(x_1,t)=\left(\frac{2n+1}{2}\right)(-1)^{\mu}\frac{\Gamma(n-\mu+1)}{\Gamma(n+\mu+1)}P_n^{\mu}(x_1)$$
$$\times\int_0^t\int_{-1}^1 E_{\alpha}(-n(n+1)(t-\tau_1)^{\alpha})P_n^{\mu}(\sigma_1)\sin(\sigma_1\tau_1)d\sigma_1 d\tau_1 \quad (5.9)$$

and

$$V(x_2,t)=\left(\frac{2n+1}{2}\right)(-1)^{\mu}\frac{\Gamma(n-\mu+1)}{\Gamma(n+\mu+1)}P_n^{\mu}(x_2)$$
$$\times\int_0^t\int_{-1}^1 E_{\alpha}(-n(n+1)(t-\tau_2)^{\alpha})P_n^{\mu}(\sigma_2)\sin(\sigma_2\tau_2)d\sigma_2 d\tau_2.$$

Finally, make an appeal to the Eqn. (5.9), we obtain the solution (5.3).

Conclusion

These investigation may be very helpful to approximate and compute of many problems consisting of second order differential equations of different special functions. Also, these results may be useful in various sciences and theories like spectral theory and approximation theories.

By Theorem 4.1, the general formula may be found for the general variable problems of symmetrical Sturm-Liouville systems as

$$Y(x_1,\dots,x_k,t)=\sum_{n=1}^{\infty}\left(\left(\frac{2n+1}{2}\right)(-1)^{\mu}\frac{\Gamma(n-\mu+1)}{\Gamma(n+\mu+1)}\right)^k \times\prod_{i=1}^{k}P_n^{\mu}(x_i)\int_0^t\int_{-1}^1 E_{\alpha}(-n(n+1)(t-\tau_i)^{\alpha})f(\sigma_i,\tau_i)P_n^{\mu}(\sigma_i)d\sigma_i d\tau_i,$$

in the symmetric Sturm Liouville systems. In further directions, our above theory may be helpful to solve the problems consisting of unsymmetrical Sturm-Liouville systems with Caputo type fractional time derivative.

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