Jñānābha, Special Issue, 2018, 61-71
Proceedings : MSSCID-2017, 20^{th} Annual Conference of VPI, Jaipur, India

ON A FRACTIONAL TIME DERIVATIVE AND MULTI-DIMENSIONAL SPACE EVOLUTION BESSEL STURM LIOUVILLE DIFFUSION AND WAVE PROBLEM

By
[1]**Hemant Kumar and** [2]**Surya Kant Rai**
[1&2]Department of Mathematics
D. A-V. (Postgraduate) College Kanpur-208001, Uttar Pradesh, India
E-mail: [1]palhemant2007@rediffmail.com, [2]suryakantrai@gmail.com

(*Received : November* 24, 2017 ; *Revised: October* 18, 2018)

Abstract

In this work, we formulate a fractional time derivative and one dimensional space evolution initial value diffusion and wave problem and then, generalize it in multi-dimensional space and with the fractional time derivative diffusion and wave initial and boundary values problems. Again, we discuss their solutions and special cases as an application of Bessel Sturm Liouville problem and by Duhamel principle.

2010 Mathematics Subject Classification: 26A33, 35M33, 35M12, 44A30.
Keywords and phrases: Caputo fractional derivative, multi-dimensional space and fractional time derivative, diffusion-wave problem, initial and boundary values, Bessel Sturm Liouville problems, Duhamel principle.

1 Introduction

About 170 years ago, the Sturm-Liouville problem has been studied to solve various problems in different area of science for example, physics, chemistry, engineering and mathematics. The one dimensional Sturm-Liouville system consists of a second order ordinary differential equation with initial and boundary conditions and yet, it has no unique solution. In this system, the differential equation problems may be regular or singular at each endpoint of the underlying interval ([4], [7], [24]).

Fractional calculus is the emerging mathematical area to study convolution type pseudo-differential operators, specifically integral and derivatives of any arbitrary real or complex order that generalize the ordinary integrals and derivatives ([5], [10], [19], [22]). On application of fractional calculus, various workers have constructed many models and problems in physical, biological, mechanical, ecosystem and chemical sciences for example earth's free oscillations, the vibrations of strings, an interaction of atomic particles, or the steady state flow in a bar, population growth, anomalous diffusion problems etc. ([8], [9], [12], [13], [14], [16], [17]). In the present time, several researchers have made interest in construction of fractional Sturm Liouville problems and to solve them ([1], [2], [3], [11], [21]) through the eigenvalues.

Duhamel's principle is one of the methods for solving of the initial and boundary value problems ([4], [6]). Recently, Kumar [15] has considered this method to derive the solution of three dimensional Legendre Sturm Liouville diffusion and wave problem generated by fractional time derivative. In this sense, we study multi-dimensional Bessel Sturm Liouville diffusion and wave problem generated by fractional time derivative and find its solution in terms of the non- zero zeros of Bessel function on using Duhamel principle.

Now, we present the systematic development of Sturm Liouville problem generated from the fractional time derivative and through regular space derivative and then put its conversion to classical Bessel Sturm Liouville problem:

1.1 Conversion of fractional time derivative problem into the Sturm Liouville problem and representation of its solution

To present our work, we consider the fractional time derivative problem in the form

$$\frac{d}{dx}[p(x)\frac{d}{dx}-\mu q(x)]Y(x,t)+\omega(x)_tD^{\alpha}_{0+}Y(x,t)=\varphi(x,t), \quad (1.1)$$

here in Eqn. (1.1), $0<\alpha\leq 2, t>0, \mu\geq 0, p(x)>0, q(x)\geq 0, \omega(x)>0, \forall x\in\mathbb{R}$, $\mathbb{R}$ is the set of real numbers, $\varphi:\mathbb{R}\times[0,\infty)\to\mathbb{R}$ and the Caputo fractional derivative ${}_tD^{\alpha}_{0^+}$, $m-1<\alpha\leq m$, of function $y(t)$ is defined by [5, Definition 3.1, p. 49]

$({}_tD^{\alpha}_{0+}y)(t)=(I^{m-\alpha}y^{(m)})(t)$, $\forall m\in\mathbb{N}$, where, $y^{(m)}(t)=\frac{d^my}{dt^m}(t)$, $I^{m-\alpha}$being the Riemann-Liouville fractional integral given by

$$(I^{m-\alpha}y)(t)=\begin{cases}\frac{1}{\Gamma(m-\alpha)}\int_0^t(t-\tau)^{m-\alpha-1}y(\tau)d\tau, t>0,\\ m-1<\alpha\leq m, y(t), \alpha=m, \forall m\in\mathbb{N}.\end{cases} \quad (1.2)$$

Again, in the Eqn. (1.1), on considering $Y(x,t)=y(x)E_{\alpha}(-(\lambda_n)^2(t)^{\alpha})\forall n\in\mathbb{N}$, and applying the Theorem of Diethelm [5, p. 70] as on action of Caputo derivative (1.2) to the Mittag-Leffler function $E_{\alpha}(-(\lambda_n)^2(t)^{\alpha})$ and thus to take the Sturm-Liouville homogeneous problem

$$\frac{d}{dx}[p(x)\frac{d}{dx}]y(x)-(\mu q^{'}(x)+\omega(x)(\lambda_n)^2)y(x)=0 \quad (1.3)$$

where, $(a<x<b), A_1y(a)+A_2y^{'}(a)=0, B_1y(b)+B_2y^{'}(b)=0; A_1,A_2,B_1,B_2\in\mathbb{R}$, and independent of λ_n. Then, it is satisfied by $y(x)=Cy_n(x)$, $y_n(x)$is the real eigenfunctions $(\forall n\in\mathbb{N}=\{1,2,3,\ldots\})$ and for all $(a\leq x\leq b)$ and in respect of the nonzero real eigenvalues $\lambda_n(n=1,2,3,\ldots)$, C is any nonzero constant. Here,$\omega(x), q^{'}(x)$, $p(x)$, $p^{'}(x)$, $(\omega(x)p(x))^{''}$are continuous real valued functions of $x, \forall(a\leq x\leq b)$.

So that due to Churchill [4, p. 291] and by Eqn. (1.3), the normalized eigenfunctions on $a\leq x\leq b$ and for all $(n=1,2,3,\ldots)$, are given by

$\psi_n(x)=\frac{y_n(x)}{\|y_n(x)\|}$, where, $\|y_n(x)\|=(\int_a^b w(x)[y_n(x)]^2dx)^{\frac{1}{2}}$.

The normalized eigenfunctions $\psi_n(x)$ are orthonormal on the interval (a,b) with weight function $w(x)$, so that the orthogonal condition is given by

$$\int_a^b w(x)\psi_m(x)\psi_n(x)dx=\begin{cases}0, & m\neq n\\ 1, & m=n\end{cases} \quad (m,n=1,2,3,\ldots). \quad (1.4)$$

Also, on the interval (a,b) a sectionally continuous function f with sectionally continuous derivatives $f^{'}$ and $f^{''}$ is represented into the generalized Fourier series

$$f(x)=\sum_{n=1}^{\infty}A_n\psi_n(x), A_n=\int_a^b w(\xi)f(\xi)\psi_n(\xi)d\xi. \quad (1.5)$$

1.2 Conversion of the Sturm Liouville problem into the Bessel Sturm Liouville problem and its solution

In this sub-section, we use the Sturm-Liouville theory in the given Eqns. (1.3)-(1.5) and then to find Bessel type problem. So that we introduce $A_1 = B_1 = 1, A_2 = B_2 = 0$, $p(x) = x, q(x) = x^2, \mu = -\frac{1}{2}, \omega(x) = x^{-1}$, and $\forall n = 1, 2, 3, \ldots, \lambda_n = \nu(\nu > -1), \forall x \in [0, b]$ in the Eqns. (1.1)- (1.3), to find that in the form

$$x^2 \frac{d^2}{dx^2} y(x) + x \frac{d}{dx} y(x) + (x^2 - \nu^2) y(x) = 0, (0 < x < b), y(0) = 0, y(b) = 0.$$

Or here, on replacing x by $\frac{a}{b}x$, it can be written as

$$x^2 \frac{d^2}{dx^2} y\left(\frac{a}{b}x\right) + x \frac{d}{dx} y\left(\frac{a}{b}x\right) + \left((\frac{a}{b})^2 x^2 - \nu^2\right) y\left(\frac{a}{b}x\right) = 0,$$
$$(0 < x < b), y(0) = 0, y(b) = 0. \quad (1.6)$$

The orthogonal condition of the Bessel functions is found by [23] as

$$\int_0^b x J_\nu^2(\frac{a}{b}x) dx = b^2/2J_{\nu+1}^2(a), \text{ provided that } J_\nu(a) = 0, b > 0, \nu > -1. \quad (1.7)$$

Hence, on making an appeal to the Eqns. (1.2)-(1.5), and (1.7), the solution of the Eqn. (1.6), in the interval $(0 < x < b)$ and for the weight function $w(x) = x$. and in form of the eigenfunctions are found by

$$y_n(x) = \frac{\sqrt{2}}{b} \frac{J_\nu(\frac{\alpha_n}{b}x)}{J_{\nu+1}(\alpha_n)}, (J_\nu(\alpha_n) = 0 \forall n = 1, 2, 3, \ldots). \quad (1.8)$$

In our investigation, we formulate a fractional time derivative diffusion and wave problem with homogeneous initial and boundary conditions. Then, we generalize it in multi-dimensional space and fractional time derivative diffusion and wave problem for $0 < \alpha \leq 2, t > 0$ and again to discuss their solutions on application of Bessel Sturm Liouville problem and by Duhamel principle ([4], [6]) and by the techniques of Kumar [15].

2 Fractional time derivative evolution and in one dimensional space homogeneous initial value problem

In this section, we present following fractional time derivative and one dimensional in space diffusion and wave initial value problem, on setting $p(x) = x, q(x) = (\frac{\alpha_n}{l})^2 x^2, \mu = -\frac{1}{2}, \omega(x) = x^{-1}$in Eqn. (1.1):

Proposition 2.1. *If*

$$\frac{d}{dx}[x \frac{d}{dx} + (\frac{\alpha_n}{l})^2 \frac{x^2}{2}] y(x,t) + x^{-1} {}_tD_{0+}^\alpha y(x,t) = 0, 0 < \alpha \leq 2, 0 < x < l, \forall n \in \mathbb{N}, \quad (2.1)$$

with initial condition

$$y(x, 0) = f(x). \quad (2.2)$$

Then, for $J_\nu(\alpha_n) = 0$,

$$y(x,t) = \sum_{n=1}^{\infty} \frac{2E_\alpha(-\nu^2(t)^\alpha)}{[lJ_{\nu+1}(\alpha_n)]^2} J_\nu\left(\alpha_n \frac{x}{l}\right) \int_0^l \xi f(\xi) J_\nu\left(\alpha_n \frac{\xi}{l}\right) d\xi. \quad (2.3)$$

Proof. In accordance of the Sturm Liouville theory stated in section 1, we prove the Proposition 2.1, to consider that

$$y(x,t) = C_n J_\nu \left(\alpha_n \frac{x}{l}\right) E_\alpha(-\nu^2 (t)^\alpha) \forall n \in \mathbb{N}. \tag{2.4}$$

Here, $J_\nu(.)$ is the ν ($\nu > -1$) order classical Bessel function (see, Sneddon [23]) and $E_\alpha(.)$ is the one parameter Mittag-Leffler function (see Mathai and Haubold [18]).

Use the formula (2.4) in left hand side of fractional differential equation of Eqn. (2.1), and apply the Theorem of Diethelm [5, p. 70], and then on using Eqn. (1.6) to find right hand side of Eqn. (2.1) as

$$C_n E_\alpha(-\nu^2(t)^\alpha)[x^2 \frac{d^2}{dx^2} + x\frac{d}{dx} + ((\frac{\alpha_n}{l})^2 x^2 - \nu^2)] J_\nu(\alpha_n \frac{x}{l}) = 0. \tag{2.5}$$

Again, due to initial condition (2.2), given in Proposition 2.1, and Eqn. (2.4) and then by orthogonal property (1.7), we find

$$C_n = \frac{2}{[l J_{\nu+1}(\alpha_n)]^2} \int_0^l \xi f(\xi) J_\nu(\alpha_n \frac{\xi}{l}) d\xi. \tag{2.6}$$

Therefore, on using (2.4) and (2.6), we find the result of problem (2.1)- (2.2), in the form.

$$y(x,t) = \frac{2E_\alpha(-\nu^2(t)^\alpha)}{[l J_{\nu+1}(\alpha_n)]^2} J_\nu(\alpha_n \frac{x}{l}) \int_0^l \xi f(\xi) J_\nu(\alpha_n \frac{\xi}{l}) d\xi, \forall n \in \mathbb{N}. \tag{2.7}$$

Finally, with the help of Eqn. (2.7), we find the result (2.3). □

3 Fractional time derivative evolution and in one dimensional space non-homogeneous initial and boundary values problem

In this section, on applying the Proposition 2.1, we present a fractional time derivative evolution and in one dimensional space non-homogeneous initial and boundary values problem in the form:

Theorem 3.1. *If* $\forall n \in \mathbb{N}$, *and for the function defined by* $\varphi : (0,l) \times [0,\infty) \to \mathbb{R}, (0,l) \subset \mathbb{R}$, *if the boundary value problem is given by*

$$\frac{d}{dx}[x\frac{d}{dx} + (\frac{\alpha_n}{l})^2 \frac{x^2}{2}] y(x,t) + x^{-1} {}_t D^\alpha_{0+} y(x,t) = \varphi(x,t), 0 < \alpha \leq 2,$$
$$\forall (x,t) \in (0,l) \times (0,\infty) \tag{3.1}$$

subjected to the initial and boundary values $y(x,0) = 0, \forall (x,t) \in [0,l] \times \{0\}, [0,l] \subset \mathbb{R}$,

$$y(0,t) = 0, \forall (x,t) \in \{0\} \times (0,\infty), y(l,t) = 0, \forall (x,t) \in \{l\} \times (0,\infty). \tag{3.2}$$

Then,

$$y(x,t) = \sum_{n=1}^{\infty} \frac{2}{[l J_{\nu+1}(\alpha_n)]^2} \int_0^t \int_0^l J_\nu(\alpha_n \frac{x}{l}) J_\nu(\alpha_n \frac{\rho}{l}) \rho \varphi(\rho,\sigma) E_\alpha(-\nu^2(t-\sigma)^\alpha) d\rho d\sigma \tag{3.3}$$

Proof. To solve Eqns. (3.1)-(3.2), we write the homogeneous problem as

$$[x^2\frac{d^2}{dx^2}+x\frac{d}{dx}+(\frac{\alpha_n}{l})^2x^2+{}_tD^{\alpha}_{0+}]y(x,t)=0, 0<\alpha\leq 2, \forall(x,t)\in(0,l)\times(0,\infty), \tag{3.4}$$

subjected to the initial and homogeneous boundary values $y(x,0)=0, \forall(x,t)\in[0,l]\times\{0\}, [0,l]\subset\mathbb{R}$,

$$y(0,t)=0, \forall(x,t)\in\{0\}\times(0,\infty), y(l,t)=0, \forall(x,t)\in\{l\}\times(0,\infty). \tag{3.5}$$

Now, for fixed σ, ρ such that $\rho\in(0,l)$, and $\forall n\in\mathbb{N}$, we define a function $y(x,t;\sigma)$

$$y(x,t;\sigma)=\frac{2J_\nu(\alpha_n\frac{x}{l})}{[lJ_{\nu+1}(\alpha_n)]^2}E_\alpha(-\nu^2(t-\sigma)^\alpha)\int_0^l\rho J_\nu(\alpha_n\frac{\rho}{l})\varphi(\rho,\sigma)d\rho \tag{3.6}$$

Then, on application of the Proposition 2.1, the function (3.6) solves following equations (3.7)-(3.8) as

$$[x^2\frac{d^2}{dx^2}+x\frac{d}{dx}+(\frac{\alpha_n}{l})^2x^2+{}_tD^{\alpha}_{0+}]y(x,t;\sigma)=0, 0<\alpha\leq 2, \forall(x,t)\in(0,l)\times(\sigma,\infty) \tag{3.7}$$

with initial conditions,

$y(x,t;\sigma)|_{t=\sigma}=\varphi(x,\sigma), \forall(x,t)\in(0,l)\times\{t=\sigma\}$, which implies that the equality, given as

$$\int_0^l xJ_\nu(\alpha_n\frac{x}{l})y(x,\sigma;\sigma)dx=\int_0^l\rho J_\nu(\alpha_n\frac{\rho}{l})\varphi(\rho,\sigma)d\rho,$$

and also with the boundary values

$$y(0,t;\sigma)=0, \forall(x,t)\in\{0\}\times(\sigma,\infty), y(l,t;\sigma)=0, \forall(x,t)\in\{l\}\times(\sigma,\infty). \tag{3.8}$$

The function $y(x,t;\sigma)$, given in Eqn. (3.6), does not solve the Eqns. (3.4)-(3.5).

But then, due to the *Duhamel's principle* (see also, Evans [6, p. 49]), we find the solution of the Eqns. (3.1)-(3.2), by the formula $y(x,t)=\int_0^t y(x,t;\sigma)d\sigma$, $\forall n\in\mathbb{N}$, in the form

$$y(x,t)=\frac{2J_\nu(\alpha_n\frac{x}{l})}{[lJ_{\nu+1}(\alpha_n)]^2}\int_0^t\int_0^l\rho J_\nu(\alpha_n\frac{\rho}{l})\varphi(\rho,\sigma)E_\alpha(-\nu^2(t-\sigma)^\alpha)d\rho d\sigma \tag{3.9}$$

The Eqn. (3.9) immediately gives the result (3.3).

It is notify that for $\alpha=1$, the equation (3.1) converts into a linear second order parabolic partial differential equation and a diffusion problem with initial and boundary conditions given in Eqn. (3.2). On the other hand, when $0<\alpha\leq 1$, above problem becomes identical to the initial-boundary value problem for the one dimensional time fractional diffusion equation due to Luchko ([16], [17]) with some additional boundary conditions. □

4 Fractional time derivative evolution and in three dimensional space diffusion and wave problem

In this section, we make an extension of our above problem (3.1)-(3.2) via fractional time derivative and in three dimensional space, $\forall x = (x_1, x_2, x_3) \in [0, l] \times [0, l] \times [0, l]$,in the form:

Theorem 4.1. *For the function defined by $\varphi : (0, l) \times (0, l) \times (0, l) \times [(0, \infty) \to \mathbb{R}$, $(0, l) \times (0, l) \times (0, l) \subset \mathbb{R}^3$, and if*

$$
\begin{aligned}
&-(x_1x_2x_3)^{-1}{}_tD^{\alpha}_{0+}y(x_1, x_2, x_3, t) = \{\frac{\partial}{\partial x_1}[x_1\frac{\partial}{\partial x_1} + (\frac{\alpha^1_n}{l})^2\frac{x_1^2}{2}] + \frac{\partial}{\partial x_2}[x_2\frac{\partial}{\partial x_2} + (\frac{\alpha^2_n}{l})^2\frac{x_2^2}{2}] \\
&\quad + \frac{\partial}{\partial x_3}[x_3\frac{\partial}{\partial x_3} + (\frac{\alpha^3_n}{l})^2\frac{x_3^2}{2}]\}y(x_1, x_2, x_3, t) + \varphi(x_1, x_2, x_3, t), \text{ where, } \varphi(x_1, x_2, x_3, t) \\
&= \{\frac{(t-\tau_2)(t-\tau_3)}{(t_1-\tau_2)(t_1-\tau_3)}\}\varphi(x_1, t) + \{\frac{(t-\tau_1)(t-\tau_3)}{(t_2-\tau_1)(t_2-\tau_3)}\}\varphi(x_2, t) + \{\frac{(t-\tau_1)(t-\tau_2)}{(t_3-\tau_1)(t_3-\tau_2)}\}\varphi(x_3, t), \\
&\quad \forall (x_1, x_2, x_3, t) \in (0, l) \times (0, l) \times (0, l) \times (0, \infty), t > \tau_3 > \tau_2 > \tau_1 > 0, 0 < \alpha \le 2, \quad (4.1)
\end{aligned}
$$

subject to the initial and homogeneous boundary values are

$$
\begin{aligned}
&y(x_1, x_2, x_3, 0) = 0, \forall (x_1, x_2, x_3, t) \in [0, l] \times [0, l] \times [0, l] \times \{0\}, \\
&\quad y(0, 0, 0, t) = 0, \forall (x_1, x_2, x_3, t) \in \{0\} \times \{0\} \times \{0\} \times (0, \infty), \\
&\quad\quad y(l, l, l, t) = 0, \forall (x_1, x_2, x_3, t) \in \{l\} \times \{l\} \times \{l\} \times (0, \infty). \quad (4.2)
\end{aligned}
$$

Then, for $J_\nu(\alpha^i_n) = 0$ and $J_{\nu+1}(\alpha^i_n) \neq 0 \forall i = 1, 2, 3,$

$$
y(x_1, x_2, x_3, t) = \sum_{n=1}^{\infty}\prod_{i=1}^{3}\frac{2J_\nu(\alpha^i_n\frac{x_i}{l})}{[lJ_{\nu+1}(\alpha^i_n)]^2}\int_0^t\int_0^l \rho_i J_\nu(\alpha^i_n\frac{\rho_i}{l})\varphi(\rho_i, \tau_i)E_\alpha(-\nu^2(t-\tau_i)^\alpha)d\rho_i d\tau_i. \quad (4.3)
$$

Proof. To solve the problem (4.1)-(4.2), for $t > \tau_3 > \tau_2 > \tau_1 > 0$, and $\forall (x_1, x_2, x_3) \in (0, l) \times (0, l) \times (0, l)$ we consider the separable solution that

$$
\begin{aligned}
y(x_1, x_2, x_3, t) &= u(x_1, t)v(x_2, t)w(x_3, t) \\
&= \int_0^t u(x_1, t, \tau_1)d\tau_1 \int_0^t v(x_2, t, \tau_2)d\tau_2 \int_0^t w(x_3, t, \tau_3)d\tau_3
\end{aligned}
$$

where, for fixed τ_1, τ_2, τ_3 and $\rho_1 \in (0, l)$, $\rho_2 \in (0, l), \rho_3 \in (0, l)$ and $\forall n \in \mathbb{N}$, the functions, $u(x_1, t; \tau_1), v(x_2, t, \tau_2)$, $w(x_3, t, \tau_3)$, are given by

$$
\begin{aligned}
u(x_1, t; \tau_1) &= \frac{2J_\nu(\alpha^1_n\frac{x_1}{l})E_\alpha(-\nu^2(t-\tau_1)^\alpha)}{[lJ_{\nu+1}(\alpha^2_n)]^2}\int_0^l \rho_1 J_\nu(\alpha^1_n\frac{\rho_1}{l})\varphi(\rho_1, \tau_1)d\rho_1, \\
v(x_2, t; \tau_2) &= \frac{2J_\nu(\alpha^2_n\frac{x_2}{l})E_\alpha(-\nu^2(t-\tau_2)^\alpha)}{[lJ_{\nu+1}(\alpha^2_n)]^2}\int_0^l \rho_2 J_\nu(\alpha^2_n\frac{\rho_2}{l})\varphi(\rho_2, \tau_2)d\rho_2, \\
w(x_3, t; \tau_3) &= \frac{2J_\nu(\alpha^3_n\frac{x_3}{l})E_\alpha(-\nu^2(t-\tau_3)^\alpha)}{[lJ_{\nu+1}(\alpha^2_n)]^2}\int_0^l \rho_3 J_\nu(\alpha^3_n\frac{\rho_3}{l})\varphi(\rho_3, \tau_3)d\rho_3. \quad (4.4)
\end{aligned}
$$

Then on introducing the functions given in Eqn. (4.4) in the Eqns. (4.1)- (4.2), and then there exists following separate equations (see also, Pipe [20])

$$[x_1^2\frac{d^2}{dx_1^2}+x_1\frac{d}{dx_1}+(\frac{\alpha_n^1}{l})^2x_1^2+{}_tD_{0+}^{\alpha}]u(x_1,t;\tau_1)=0, 0<\alpha\leq 2, \forall(x_1,t)\in(0,l)\times(\tau_1,\infty),$$

with initial conditions, at $t=\tau_1, u(x_1,t;\tau_1)=\varphi(x_1,\tau_1), \forall(x_1,t)\in(0,l)\times\{t=\tau_1\}$, and due to

$$\int_0^l x_1J_\nu(\alpha_n^1\frac{x_1}{l})u(x_1,\tau_1;\tau_1)dx_1=\int_0^l \rho_1J_\nu(\alpha_n^1\frac{\rho_1}{l})\varphi(\rho_1,\tau_1)d\rho_1 \tag{4.5}$$

is followed.

$$[x_2^2\frac{d^2}{dx_2^2}+x_2\frac{d}{dx_2}+(\frac{\alpha_n^2}{l})^2x_2^2+{}_tD_{0+}^{\alpha}]v(x_2,t;\tau_2)=0, 0<\alpha\leq 2, \forall(x_2,t)\in(0,l)\times(\tau_2,\infty),$$

with initial conditions, at $t=\tau_2, v(x_2,t;\tau_2)=\phi(x_2,\tau_2), \forall(x_2,t)\in(0,l)\times\{t=\tau_2\}$, and due to

$$\int_0^l x_2J_\nu(\alpha_n^2\frac{x_2}{l})v(x_2,\tau_2;\tau_2)dx_2=\int_0^l \rho_2J_\nu(\alpha_n^2\frac{\rho_2}{l})\varphi(\rho_2,\tau_2)d\rho_2 \tag{4.6}$$

is followed.

$$[x_3^2\frac{d^2}{dx_3^2}+x_3\frac{d}{dx_3}+(\frac{\alpha_n^3}{l})^2x_3^2+{}_tD_{0+}^{\alpha}]w(x_3,t;\tau_3)=0, 0<\alpha\leq 2, \forall(x_3,t)\in(0,l)\times(\tau_3,\infty),$$

with initial conditions, at $t=\tau_3,\ w(x_3,t;\sigma)=\phi(x_3,\tau_3), \forall(x_3,t)\in(0,l)\times\{t=\tau_3\}$ and due to

$$\int_0^l \rho_3J_\nu(\alpha_n^3\frac{x_3}{l})w(x_3,\tau_3;\tau_3)dx_3=\int_0^l \rho_3J_\nu(\alpha_n^3\frac{\rho_3}{l})\varphi(\rho_3,\tau_3)d\rho_3 \tag{4.7}$$

is followed.

Now, use the functions given in Eqn. (4.4), and then on making an appeal to the techniques of the Eqns. (3.4)-(3.9) of the Theorem 3.1, in the Eqns. (4.5), (4.6), (4.7) to get the solutions

$$\begin{aligned}
u(x_1,t)&=\frac{2J_\nu(\alpha_n^1\frac{x_1}{l})}{[lJ_{\nu+1}(\alpha_n^2)]^2}\int_0^t\int_0^l \rho_1J_\nu(\alpha_n^1\frac{\rho_1}{l})\varphi(\rho_1,\tau_1)E_\alpha(-\nu^2(t-\tau_1)^\alpha)d\rho_1d\tau_1,\\
v(x_2,t)&=\frac{2J_\nu(\alpha_n^2\frac{x_2}{l})}{[lJ_{\nu+1}(\alpha_n^2)]^2}\int_0^t\int_0^l \rho_2J_\nu(\alpha_n^2\frac{\rho_2}{l})\varphi(\rho_2,\tau_2)E_\alpha(-\nu^2(t-\tau_2)^\alpha)d\rho_2d\tau_2,\\
w(x_3,t)&=\frac{2J_\nu(\alpha_n^3\frac{x_3}{l})}{[lJ_{\nu+1}(\alpha_n^2)]^2}\int_0^t\int_0^l \rho_3J_\nu(\alpha_n^3\frac{\rho_3}{l})\varphi(\rho_3,\tau_3)E_\alpha(-\nu^2(t-\tau_3)^\alpha)d\rho_3d\tau_3,
\end{aligned} \tag{4.8}$$

respectively.

Finally, on application of the results given in (4.8) in the relation

$$y(x_1,x_2,x_3,t)=u(x_1,t)v(x_2,t)w(x_3,t),$$

we obtain the solution (4.3). □

5 Fractional time derivative evolution and in multi-dimensional space diffusion and wave problem

Now, we present a problem involving fractional time derivative and in multi-dimensional space, $\forall x = (x_1, \ldots, x_k) \in [0, l] \times \ldots \times [0, l]$, and discuss its solution through following theorem:

Theorem 5.1. *For the function defined by* $\varphi : (0, l) \times \ldots \times (0, l) \times [0, \infty) \to \mathbb{R}, (0, l) \times \ldots \times (0, l) \subset \mathbb{R}^k, k \in \mathbb{N}$ *and if*

$$-(x_1...x_k)^{-1}{}_tD^{\alpha}_{0+}y(x_1, ..., x_k, t) = \left\{\frac{\partial}{\partial x_1}\left[x_1\frac{\partial}{\partial x_1} + \left(\frac{\alpha^1_n}{l}\right)^2\frac{x_1^2}{2}\right] + ... \right.$$
$$\left. + \frac{\partial}{\partial x_k}\left[x_k\frac{\partial}{\partial x_k} + \left(\frac{\alpha^k_n}{l}\right)^2\frac{x_k^2}{2}\right]\right\}y(x_1, \ldots, x_k, t) + \varphi(x_1, \ldots, x_k, t), \quad (5.1)$$

where,

$$\varphi(x_1, \ldots, x_k, t) = \sum_{r=1}^{k}[\{\frac{\prod_{i=1,i\neq r}^{k}(t - \tau_i)}{\prod_{i=1,i\neq r}^{k}(t_r - \tau_i)}\}\varphi(x_r, t)],$$
$$\forall(x_1, \ldots, x_k, t) \in (0, l) \times \ldots \times (0, l) \times (0, \infty), t > \tau_k > \ldots > \tau_1 > 0, 0 < \alpha \leq 2, \quad (5.2)$$

subject to the initial and homogeneous boundary values are

$$y(x_1, \ldots, x_k, 0) = 0, \forall(x_1, \ldots, x_k, t) \in [0, l] \times \ldots \times [0, l] \times \{0\},$$
$$y(0, \ldots, 0, t) = 0, \forall(x_1, \ldots, x_3, t) \in \{0\} \times \ldots \times \{0\} \times (0, \infty),$$
$$y(l, \ldots, l, t) = 0, \forall(x_1, \ldots, x_3, t) \in \{l\} \times \ldots \times \{l\} \times (0, \infty). \quad (5.3)$$

Then, for $J_\nu(\alpha^i_n) = 0$ *and* $J_{\nu+1}(\alpha^i_n) \neq 0 \forall i = 1, 2, 3, \ldots, k$,

$$y(x_1, \ldots, x_k, t) = \sum_{n=1}^{\infty}\prod_{i=1}^{k}\frac{2J_\nu(\alpha^i_n\frac{x_i}{l})}{[lJ_{\nu+1}(\alpha^i_n)]^2}\int_0^t\int_0^l \rho_i J_\nu(\alpha^i_n\frac{\rho_i}{l})\varphi(\rho_i, \tau_i)E_\alpha(-\nu^2(t - \tau_i)^\alpha)d\rho_i d\tau_i. \quad (5.4)$$

Proof. We apply the techniques of the Theorem 4.1 to separate the solutions due to Eqn. (4.5)-(4.7) and then the techniques of Theorem 3.1, we obtain the result (5.4). □

6 Special Cases

Corollary 6.1. *Let all conditions of the Theorem 5.1 are followed and* $J_\nu(\alpha^i_n) = 0$*, for all positive zeros* α^i_n*, for all* ($i = 1, 2, \ldots, k$; *and* $n = 1, 2, 3, \ldots$) *and* $\varphi(x_1, \ldots, x_k, t) = \sum_{r=1}^{k}[\{\frac{\prod_{i=1,i\neq r}^{k}(t-\tau_i)}{\prod_{i=1,i\neq r}^{k}(t_r-\tau_i)}\}\varphi(\frac{x_r}{l})\phi(t)]$, $\forall(x_1, \ldots, x_k, t) \in (0, 1) \times \ldots \times (0, 1) \times (0, \infty)$, $t > \tau_k > \ldots > \tau_1 > 0, 0 < \alpha \leq 2$, *then* $\forall n \in \mathbb{N}$, *there exists following results*

$$y(x_1, \ldots, x_k, t) = \prod_{i=1}^{k}\frac{2J_\nu(\alpha^i_n\frac{x_i}{l})}{\{J_{\nu+1}(\alpha^i_n)\}^2}\int_0^t \phi(\tau_i)E_\alpha(-\nu^2(t - \tau_i)^\alpha)d\tau_i \int_0^1 \rho_i\varphi(\rho_i)J_\nu(\alpha^i_n\rho_i)d\rho_i \quad (6.1)$$

and

$$y(x_1,\ldots,x_k,t)=\frac{\{\prod_{i=1}^{k}\int_0^t\phi(\tau_i)E_\alpha(-\nu^2(t-\tau_i)^\alpha)d\tau_i\int_0^1(\rho_i)^{\nu+1}\varphi(\rho_i)d\rho_i\}}{\{\sum_{n=1}^{\infty}\prod_{i=1}^{k}\frac{J_{\nu+1}(\alpha_n^i)}{\alpha_n^i J_\nu(\alpha_n^i\frac{x_i}{l})}\}}\tag{6.2}$$

Proof. Consider the techniques from the Theorems 3.1, 4.1 and then the Theorem 5.1, write $\varphi(x_1,\ldots,x_k,t)=\sum_{r=1}^{k}[\{\frac{\prod_{i=1,i\neq r}^{k}(t-\tau_i)}{\prod_{i=1,i\neq r}^{k}(t_r-\tau_i)}\}\varphi(\frac{x_r}{l})\phi(t)]\ \forall n\in\mathbb{N}$ and thus to find the result (6.1).

Again, $\forall n\in\mathbb{N}$, write the result (6.1) in the form

$$y(x_1,\ldots,x_k,t)\prod_{i=1}^{k}\frac{J_{\nu+1}(\alpha_n^i)}{J_\nu(\alpha_n^i\frac{x_i}{l})}=\prod_{i=1}^{k}\frac{2}{J_{\nu+1}(\alpha_n^i)}\int_0^t\phi(\tau_i)E_\alpha(-\nu^2(t-\tau_i)^\alpha)d\tau_i \times\int_0^1\rho_i\varphi(\rho_i)J_\nu(\alpha_n^i\rho_i)d\rho_i\tag{6.3}$$

Now, sum up the results in both sides of Eqn. (6.3) from $n=1$ to $n=\infty$ and then make an application of the result due to Sneddon [23, Example (4.3), p. 148], we obtain the formula (6.2). □

Corollary 6.2. *Let all conditions of the Theorem 5.1 are followed and* $J_\nu(\alpha_n^i)=0$, *for all positive zeros* α_n^i, *for all* $(i=1,2,\ldots,k;$ *and* $n=1,2,3,\ldots)$ *and* $\varphi(x_1,\ldots,x_k,t)=\sum_{r=1}^{k}[\{\frac{\prod_{i=1,i\neq r}^{k}(t-\tau_i)}{\prod_{i=1,i\neq r}^{k}(t_r-\tau_i)}\}(\frac{x_r}{l})^{2\mu}\phi(t)]$, $\forall(x_1,\ldots,x_k,t)\in(0,1)\times\ldots\times(0,1)\times(0,\infty)$, $t>\tau_k>\ldots>\tau_1>0, 0<\alpha\leq 2$, *then* $\forall n\in\mathbb{N},\mu\in\mathbb{R}$, *by Eqn. (6.2), there exists following result*

$$y(x_1,\ldots,x_k,t)=(\frac{1}{\nu++2\mu+2})^k\frac{\{\prod_{i=1}^{k}\int_0^t\phi(\tau_i)E_\alpha(-\nu^2(t-\tau_i)^\alpha)d\tau_i\}}{\{\sum_{n=1}^{\infty}\prod_{i=1}^{k}\frac{J_{\nu+1}(\alpha_n^i)}{\alpha_n^i J_\nu(\alpha_n^i\frac{x_i}{l})}\}},\mu\in\mathbb{R}\tag{6.4}$$

Again, by Eqn. (6.1), to find

$$y(x_1,\ldots,x_k,t)=\frac{(2I_\mu)^k\prod_{i=1}^{k}\int_0^t\phi(\tau_i)d\tau_i}{\{\prod_{i=1}^{k}\frac{J_1(\alpha_n^i)}{J_0(\alpha_n^i\frac{x_i}{l})}\}},$$

where, I_μ *is defined due to Sneddon [23, Example (4.2), p. 147], given by*

$$I_\mu=\frac{1}{J_1(\alpha_n^i)}\int_0^1(\rho_i)^{2\mu+1}J_0(\alpha_n^i\rho_i)d\rho_i.\forall i=1,2,\ldots,k.\tag{6.5}$$

Conclusion

The techniques given in the Eqns. (3.1)-(3.9) are very simple and straight forward to solve the Bessel Sturm Liouville problems, involving fractional time derivative and multi-dimensional space evolution with initial and boundary values, in terms of non-zero zeros of Bessel function. These investigations may be very helpful to approximate and compute of many problems of dynamical astronomy, in cylindrical coordinates of Laplace equation [6] consisting second order differential equations of different special functions. Also, these results may be useful in various sciences and theories like spectral theory and approximation theories.

References

[1] S. Abbasbandy and A. Shirzadi, Homotopy analysis method for multiple solutions of the fractional Sturm-Liouville problems, *Numerical Algorithms,* **54** (2010), 521-532.

[2] Q. M. Al-Mdallal, An efficient method for solving fractional Sturm-Liouville problems, *Chaos Solitons and Fractals,* **40** (2009), 183-189.

[3] Q. M. Al-Mdallal, On the numerical solution of fractional Sturm-Liouville problems, *International Journal of Computer Mathematics,* **87** (2010), 2837-2845.

[4] R. V. Churchill, *Operational Mathematics*, International Student Edition, McGraw-Hill Inc. London, 1972.

[5] K. Diethelm, *The Analysis of Fractional Differential Equations: An Application-Oriented Exposition Using Differential Operators of Caputo Type*, Springer-Verlag Berlin Heidelberg, 2010.

[6] L. C. Evans, *Partial Differential Equations*, AMS, Graduate Studies in Mathematics vol. **19,** Berkeley, 1997.

[7] W. N. Everrit, *A Catalogue of Sturm Liouville Differential Equations*, *Sturm-Liouville Theory*, Springer Link, Birkhäuser Basel, 2005, 271-331.

[8] R. Gorenflo and F. Mainardi, *Fractional Calculus: Integral and Differential Equations of Fractional Order*, vol. **378** of CISM courses and lectures, Springer-Verlag, Berlin, 1997, 223-276.

[9] R. Gorenflo, F. Mainardi and H. M. Srivastava, Special functions in fractional relaxation-oscillation and fractional diffusion wave phenomena, in : *The Eight International Colloquium on differential equations (Plovdiv,* 1997*),* VSP Publishing Company, Utrecht, 1998, 195-202.

[10] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier, New York, 2006.

[11] M. Klimek, Fractional Sturm-Liouville Problem in Terms of Riesz Derivatives. In: Theoretical Developments and Applications of Non-Integer Order Systems (Editors: Stefan Domek, Pawel Dworak), Series: *Lecture notes in Electrical Engineering* **357**, Springer International Publishing, Heidelberg (2016), 3-16.

[12] H. Kumar, M. A. Pathan and Harish Srivastava, Application of H-function for obtaining an analytic solution of the space-and-time fractional initial value problem, *Jnānābha*, **39** (2009), 67-76.

[13] H. Kumar, M. A. Pathan and Harish Srivastava, A general solution of a space-time fractional anomalous diffusion problem using the series of bilateral eigenfunctions, *Communication Korean Mathematical Society,* **29(1),** (2014), 173-185.

[14] H. Kumar, M. A. Pathan and Priyanka Yadav, Series Solution for initial value problems of time fractional generalized anomalous diffusion equations, *Le Matematiche* **LXVII** (2012) -fasc II, 217-229.

[15] H. Kumar, On three dimensional Legendre Sturm Liouville diffusion and wave problem generated due to fractional time derivative, *Jnānābha*, **48** (1) (2018), 121-133.

[16] Yu. Luchko, Initial-boundary-value problems for the one dimensional time fractional diffusion equation, *arXiv:*1111.2961v1 [math.AP] **12** (Nov. 2011), 1-21.

[17] Yu. Luchko, Initial-boundary-value problems for the generalized multiterm time-fractional diffusion equation, *Journal of Mathematical Analysis and Applications,* **374** (2) (2011), 538-548.

[18] A. M. Mathai and H. J. Haubold, *Special Functions for Applied Scientists*, Springer, New York, 2008.
[19] K. B. Oldham and J. Spanier, *The Fractional Calculus, Theory and applications of Differentiation and Integration to arbitrary order*, Dover Publication, Inc., New York, 2006.
[20] L. A. Pipe, *Applied Mathematics for Engineers and Physicists*, McGraw-Hill Book Company Inc., New York, 1958.
[21] M. Rivero, J. J. Trujillo, M. P. Velasco, A fractional approach to the Sturm-Liouville problem, *Central European Journal of Physics,* **11** (10) (2013), 1246-1254.
[22] S. G. Samko, A. A. Kilbas and O. I. Marichev, *Fractional Integrals and Derivatives*, translated from the 1987 Russian original, Gordon and Breach, Yverdon, 1993.
[23] I. N. Sneddon, *Special Functions for Mathematical Physics and Chemistry*, Oliver and Boyd Ltd, London, New York, 1961.
[24] A. Zettl, Sturm-Liouville Theory, *Mathematical Surveys and Monographs*, **121**, AMS, 2005.