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CERTAIN RESULTS OF (p,q)-ANALOGOUE OF I-FUNCTION WITH (p,q)-DERIVATIVE

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Abstract

In this paper, the authors have derived the (p,q)-analogue of I-function by using the (p,q) generalization of the Gamma and Beta functions. Some particular cases of these results in terms of (p,q)-analogue of H-Function which appear to be new, and G-function, defined earlier by Swati et. al. has also been obtained.

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1 Introduction and Preliminaries

The (p,q)-shifted factorial is based on the concept of twin-basic number $[n]_{(p,q)} = \frac{(p^n-q^n)}{(p-q)}$. The basic number occurs in the theory of two parameter quantum algebras and has also been introduced in combinatorics by Jagannathan et.al. [8]. Several properties of this number were studied briefly in [5]. Around the same time as [5], Brodimas et. al. [3] and Arik, el. al. [2] also independently introduced the (p,q)-number in the physics literature, but in a very much less detailed manner. The (p,q)-identities thus derived, with doubling of the number of parameters, offer more choices for applications. It has been observed that many of the q-results can be generalized directly to (p,q)-results. If we have the (p,q)-results, the q-results can be obtained more easily by mere substitutions for the parameters instead of any limiting process as required in the usual q-theory [10]. This also provides a new look for the q-identities.

The q-deformed algebra [11,13] and their generalization to (p,q)-analogue [4,5] have attracted much attention of the researchers to increase the accessibility of different dimensions of (p,q)-analogue algebra. The main reason is that these topics stand for real life problems, in mathematics and physics, later to the theory of quantum calculus.

In the present paper, the authors attention is towards defining the (p,q)-analogue of I-function with (p,q)-derivative by using the (p,q) generalization of the Gamma and Beta functions.

From the theory of basic hypergeometric series [6], some basic definitions are given below:

The q-shifted factorial is given by

$$(a,q)_n = \left[\begin{array}{ll} 1, & n=0 \\ (1-a)(1-aq)...(1-aq^{n-1}), & n=1,2,3,... \end{array} \right. \tag{1.1}$$

with $(a_1,a_2,...,a_k;q)_n = (a_1;q)_n(a_2;q)_n...(a_k;q)_n$.

The q-gamma function was first introduced by Thomae and later by Jackson.

The q-analogue of gamma function which is defined by F.H. Jackson [7] is given by

$$\Gamma_q(x) = \frac{(q;q)_\infty(1-q)^{1-x}}{(q^x;q)_\infty}, 0<q<1.$$

Jackson gave the general definition which is given below

$$\int_a^b f(t)d_qt = \int_0^b f(t)d_qt - \int_0^a f(t)d_qt$$

where

$$\int_0^a f(t)d_qt = a(1-q)\sum_{n=0}^\infty f(aq^n)q^n.$$

Jackson also defined an integral i.e.

$$\int_0^\infty f(t)d_qt = (1-q)\sum_{n=-\infty}^\infty f(aq^n)q^n.$$

P. Njionou Sadjang [15] introduced the so-called the shifted factorial as follows:

$$(x\ominus a)_{p,q}^n = (x-a)(px-aq)(p^2x-aq^2)...(xp^{n-1}-aq^{n-1}), \tag{1.2}$$

$$(x\oplus a)_{p,q}^n = (x+a)(px+aq)(p^2x+aq^2)...(xp^{n-1}+aq^{n-1}). \tag{1.3}$$

These definitions are extended as follows

$$(a\ominus b)_{p,q}^n = \prod_{k=0}^\infty (ap^k-bq^k), \tag{1.4}$$

$$(a\oplus b)_{p,q}^n = \prod_{k=0}^\infty (ap^k+bq^k). \tag{1.5}$$

Let x be a complex number, the (p,q)-Gamma function is defined by P. Njionou Sadjang [14]

$$\Gamma_{p,q}(x) = \frac{(p\ominus q)_{p,q}^\infty}{(p^x\ominus q^x)_{p,q}^\infty}(p-q)^{1-x}, 0<q<p. \tag{1.6}$$

If we put $p=1$, then $\Gamma_{p,q}$ reduces to Γ_q.

The (p,q)-Gamma function fulfils the following fundamental relation

$$\Gamma_{p,q}(x+1) = [x]_{p,q}\Gamma_{p,q}(x).$$

If n is a nonnegative integer, it follows from above that

$$\Gamma_{p,q}(x+1) = [x]_{p,q}!.$$

It can be also easily seen from the definition that

$$\Gamma_{p,q}(n+1) = \frac{(p \ominus q)^n_{p,q}}{(p-q)^n_{p,q}}.$$

P. Njionou Sadjang [14] also defined the (p,q)-Beta function as

$$B_{p,q}(x,y) = \frac{\Gamma_{p,q}(x)\Gamma_{p,q}(y)}{\Gamma_{p,q}(x+y)}.$$

The (p,q)-derivative of the function $f(x)$ is defined as follows [5,11] :

$$D_{p,q}f(x) = \frac{f(px) - f(qx)}{(p-q)x}, x \neq 0, \tag{1.7}$$

where $D_{p,q}f(0) = f'(0)$, provided that $f(x)$ is differentiable at $x = 0$.

The (p,q)-numbers $[n]_{p,q}$ and (p,q) factorials $[n]_{p,q}!$ are defined [15] as:

$$[n]_{p,q} = \frac{p^n - q^n}{(p-q)}$$

and

$$[n]_{p,q}! = [1]_{p,q}[2]_{p,q}...[n]_{p,q}$$

respectively.

Also it happens that $D_{p,q}(x^n) = [n]_{p,q}x^{n-1}$.

Remark 1. $D_{p,q}(x)$ *reduces to Hahn Derivative* $d_q f(x)$ *iff* $p \to 1$.

Remark 2. $[n]_{p,q} = [n]_q$ *(Hahn Basic Number) iff* $p \to 1$. *where* $[n]_q = \frac{1-q^n}{1-q}, q \neq 1$.

Note:

$$D^n_{p,q}(x^\mu) = \frac{\Gamma_{p,q}(\mu+1)}{\Gamma_{p,q}(\mu-n+1)} x^{\mu-n}, Re(\mu) + 1 > 0, \tag{1.8}$$

$$I^n_{p,q}(x^\mu) = \frac{\Gamma_{p,q}(\mu+1)}{\Gamma_{p,q}(\mu+n+1)} x^{\mu+n}, Re(\mu) + 1 > 0. \tag{1.9}$$

Saxena et. al. [16] introduced the following basic analogue of I-Function in terms of the Mellin-Barnes type basic contour integral as:

$$I(z) = I^{m,n}_{A_i,B_i;r}\left[\left(z;q\left|\begin{matrix}(a_j,\alpha_j)_{1,n};(a_{ji},\alpha_{ji})_{n+1,A_i}\\(b_j,\beta_j)_{1,m};(b_{ji},\beta_{ji})_{m+1,B_i}\end{matrix}\right.\right)\right]$$
$$= \frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{m} G(q^{b_j-\beta_j s}) \prod_{j=1}^{n} G(q^{1-a_j+\alpha_j s})\pi z^s}{\sum_{i=1}^r[\prod_{j=m+1}^{B_i} G(q^{1-b_{ji}+\beta_{ji}s}) \prod_{j=n+1}^{A_i} G(q^{a_{ji}-\alpha_{ji}s})G(q^s)G(q^{1-s})sin\pi s]} ds \tag{1.10}$$

where $\alpha_j, \beta_j, \alpha_{ji}, \beta_{ji}$ are real and positive, a_j, b_j, a_{ji}, b_{ji} are complex numbers and

$$G(q^{\alpha}) = \prod_{n=0}^{\infty}(1 - q^{\alpha+n})^{-1} = \frac{1}{(q^{\alpha};q)_{\infty}} \tag{1.11}$$

where L is contour of integration running from $-i\infty$ to $+i\infty$ in such a manner so that all poles of$G(q^{b_j-\beta_j s}); 1 \leq j \leq m$ are to right of the path and those of $G(q^{(1-a_j+\alpha_j s)}); 1 \leq j \leq n$,are to left. The integral converges if $Re[slog(x) - logsin\pi s] < 0$, for large values of $|s|$ on the contour L.

Setting $r = 1, A_i = A, B_i = B$, in equation (1.10) we get q-analogue of H-Function defined by Saxena et.al. [16] as follows:

$$H_{A,B}^{m,n}\left[\left(z;q\left|\begin{matrix}(a_j,\alpha_j)_{1,A}\\(b_j,\beta_j)_{1,B}\end{matrix}\right.\right)\right]$$
$$= \frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{m} G(q^{b_j-\beta_j s}) \prod_{j=1}^{n} G(q^{1-a_j+\alpha_j s})\pi z^s}{\prod_{j=m+1}^{B} G(q^{1-b_j+\beta_j s}) \prod_{j=n+1}^{A} G(q^{a_j-\alpha_j s})G(q^s)G(q^{1-s})sin\pi s}ds. \tag{1.12}$$

Further if we put $\alpha_j = \beta_j = 1$, equation (1.12) reduces to the basic analogue of Meijer's G-Function given by Saxena et. al. [16].

$$G_{A,B}^{m,n}\left[\left(z;q\left|\begin{matrix}a_1,a_2,...,a_A\\b_1,b_2,...,b_B\end{matrix}\right.\right)\right] =$$
$$\frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{m} G(q^{b_j-s}) \prod_{j=1}^{n} G(q^{1-a_j+s})\pi z^s}{\prod_{j=m+1}^{B} G(q^{1-b_j+s}) \prod_{j=n+1}^{A} G(q^{a_j-s})G(q^s)G(q^{1-s})sin\pi s}ds. \tag{1.13}$$

2 Main Results

(A) In this section we will find the (p,q)-analogue of I-function. Swati Pathak and Renu Jain [12] defined the (p,q)-Gamma function as follows:

$$\Gamma_{p,q}(x) = \frac{((p,q);(p,q))_{\infty}(p-q)^{1-x}}{((p^x,q^x);(p,q))_{\infty}}, (0 < p < q). \tag{2.1}$$

When $x = n+1$ with a non-negative integer this definition reduces to

$$\Gamma_{p,q}(n+1) = [n]_{p,q}!, \tag{2.2}$$

and

$$\Gamma_{p,q}(x+1) = [x]_{p,q}\Gamma_{p,q}(x).$$

We can deduce that $\Gamma_{p,q}(1) = 1$.

Now we shall make use of (p,q)-Gamma function for defining (p,q)-analogue of I-function which is as follows:

$$I_{A_i,B_i+1;r}^{m,n}\left[\left(z(p-q)^{\sum_{t=1}^{m}\beta_t-\sum_{t=1}^{n}\alpha_t+\sum_{i=1}^{r}[\sum_{t=m+1}^{q_i}\beta_{ti}-\sum_{t=n+1}^{p_i}\alpha_{ti}]};(p,q)\left|\begin{matrix}(a_j,\alpha_j)_{1,n};(a_{ji},\alpha_{ji})_{n+1,A_i}\\(b_j,\beta_j)_{1,m};(b_{ji},\beta_{ji})_{m+1,B_i}(1,1)\end{matrix}\right.\right)\right]$$

$$= \frac{1}{2\pi i} \int_L \frac{\prod_{j=1}^{m} G(p^{b_j-\beta_j s}, q^{b_j-\beta_j s}) \prod_{j=1}^{n} G(p^{1-a_j+\alpha_j s}, q^{1-a_j+\alpha_j s})(p-q)^{s[\sum_{t=1}^{m}\beta_t - \sum_{t=1}^{n}\alpha_t + \sum_{i=1}^{r}[\sum_{t=m+1}^{q_i}\beta_{ti} - \sum_{t=n+1}^{p_i}\alpha_{ti}]]}\pi z^s}{\sum_{i=1}^{r}[\prod_{j=m+1}^{B_i} G(p^{1-b_{ji}+\beta_{ji}s}, q^{1-b_{ji}+\beta_{ji}s}) \prod_{j=n+1}^{A_i} G(p^{a_{ji}-\alpha_{ji}s}, q^{a_{ji}-\alpha_{ji}s}) G(p^s,q^s) G(p^{1-s},q^{1-s}) sin\pi s]} \tag{2.3}$$

On multiplying equation (1.9) by:

$$(p-q)^{\sum_{t=1}^{n} a_t - \sum_{t=1}^{m} b_t + m+n-1+\sum_{i=1}^{r}[\sum_{t=n+1}^{A_i}\alpha_{ti} - \sum_{t=m+1}^{B_i}\beta_{ti} - A_i]} G(p,q)^{\sum_{t=1} nA_t + B_t - (m+n-1)}$$

and then making use of identity

$$\Gamma_{p,q}(x) = \frac{G(p^x,q^x)}{(p-q)^{x-1}G(p,q)}, |\frac{q}{p}| < 1, \tag{2.4}$$

where

$$G(p^\alpha, q^\alpha) = \prod_{n=0}^{\infty}(p^{\alpha+n}, q^{\alpha+n})^{-1} = \frac{1}{((p^\alpha,q^\alpha);(p,q))_\infty}. \tag{2.5}$$

The R.H.S. of equation (2.3) becomes

$$= \frac{1}{2\pi i} \int_L \frac{\prod_{j=1}^{m} \Gamma_{p,q}(b_j - \beta_j s) \prod_{j=1}^{n} \Gamma_{p,q}(1-a_j-\alpha_j s)\pi z^s}{\sum_{i=1}^{r}[\prod_{j=m+1}^{B_i} \Gamma_{p,q}(1-b_{ji}+\beta_{ji}s) \prod_{j=n+1}^{A_i} \Gamma_{p,q}(a_{ji}-\alpha_{ji}s)\Gamma_{p,q}(s)\Gamma_{p,q}(1-s) sin\pi s]} ds$$

where L is contour of integration running from $-i\infty$ to $+i\infty$ in such a manner so that all poles of $\Gamma_{p,q}(b_j - \beta_j s); 1 \le j \le m$ are to right of the path and those of $\Gamma_{p,q}(1-a_j+\alpha_j s); 1 \le j \le n$,are to left. The integral converges if $Re[slog(x) - logsin\pi s] < 0$, for large values of $|s|$ on the contour L.

Special Cases: Taking $r = 1$, in above equation we will get the (p,q)-analogue of Fox's H-function as follows:

$$H_{A,B}^{m,n}\left[\left(z;(p,q)\left|\begin{matrix}(a_j,\alpha_j)_{1,A}\\(b_j,\beta_j)_{1,B}\end{matrix}\right.\right)\right].$$

$$= \frac{1}{2\pi i} \int_L \frac{\prod_{j=1}^{m} \Gamma_{p,q}(b_j - \beta_j s) \prod_{j=1}^{n} \Gamma_{p,q}(1-a_j+\alpha_j s)\pi z^s ds}{\prod_{j=m+1}^{B} \Gamma_{p,q}(1-b_j+\beta_j s) \prod_{j=n+1}^{A} \Gamma_{p,q}(a_j-\alpha_j s)\Gamma_{p,q}(s)\Gamma_{p,q}(1-s) sin\pi s}. \tag{2.6}$$

If we take $\alpha_j = \beta_j = 1$, in equation (2.6) we will get the (p,q)-analogue of Meijer's G-function defined by Swati Pathak et.al [12] as follows:

$$G_{A,B}^{m,n}\left[\left(z;(p,q)\left|\begin{matrix}a_1,a_2,...,a_A\\b_1,b_2,...,b_B\end{matrix}\right.\right)\right]$$

$$= \frac{1}{2\pi i} \int_L \frac{\prod_{j=1}^{m} \Gamma_{p,q}(b_j - s) \prod_{j=1}^{n} \Gamma_{p,q}(1-a_j+s)\pi z^s ds}{\prod_{j=m+1}^{B} \Gamma_{p,q}(1-b_j+s) \prod_{j=n+1}^{A} \Gamma_{p,q}(a_j-s)\Gamma_{p,q}(s)\Gamma_{p,q}(1-s) sin\pi s}. \tag{2.7}$$

(B) In this section, we will evaluate the (p, q)-derivative operator involving (p, q)-analogue of I-function.

Theorem 2.1. *Let the parameters A_i, B_i are non-negative integers satisfying the inequality $0 \leq n \leq A_i, 0 \leq m \leq B_i$ and a_j, b_j, a_{ji}, b_{ji} are complex numbers, then*

$$zD_{p,q}[z^{1-a_1} I^{m,n}_{A_i,B_i;r}\left[\left(z;(p,q)\left|\begin{matrix}(a_j,1)_{1,n};(a_{ji},1)_{n+1,A_i}\\(b_j,1)_{1,m};(b_{ji},1)_{m+1,B_i}\end{matrix}\right.\right)\right]]$$
$$= z^{1-a_1} I^{m,n}_{A_i,B_i;r}\left[\left(z;(p,q)\left|\begin{matrix}(a_1-1,1)(a_j,1)_{2,n};(a_{ji},1)_{n+1,A_i}\\(b_j,1)_{1,m};(b_{ji},1)_{m+1,B_i}\end{matrix}\right.\right)\right], \quad (2.8)$$

where $z \neq 0, 0 < q < p$ and $\omega = \sqrt{-1}$.

Proof. To prove theorem 2.1 when $a_1 > 0$, we apply equation (1.8)

$$L.H.S. = zD_{p,q}[z^{1-a_1}\frac{1}{2\pi\omega}\int_L \frac{\prod_{j=1}^{m}\Gamma_{p,q}(b_j-s)\prod_{j=1}^{n}\Gamma_{p,q}(1-a_j+s)\pi z^s ds}{\sum_{i=1}^{r}[\prod_{j=m+1}^{B_i}\Gamma_{p,q}(1-b_{ji}+s)\prod_{j=n+1}^{A_i}\Gamma_{p,q}(a_{ji}-s)\Gamma_{p,q}(s)\Gamma_{p,q}(1-s)sin\pi s]}]$$

$$= z\frac{1}{2\pi\omega}\int_L \frac{\prod_{j=1}^{m}\Gamma_{p,q}(b_j-s)\prod_{j=1}^{n}\Gamma_{p,q}(1-a_j+s)\pi D_{p,q}[z^{1-a_1+s}]ds}{\sum_{i=1}^{r}[\prod_{j=m+1}^{B_i}\Gamma_{p,q}(1-b_{ji}+s)\prod_{j=n+1}^{A_i}\Gamma_{p,q}(a_{ji}-s)\Gamma_{p,q}(s)\Gamma_{p,q}(1-s)sin\pi s]}$$

$$= \frac{1}{2\pi\omega}\int_L \frac{\prod_{j=1}^{m}\Gamma_{p,q}(b_j-s)\prod_{j=1}^{n}\Gamma_{p,q}(1-a_j+s)\pi[1-a_1+s]_{p,q}z^{1-a_1+s}ds}{\sum_{i=1}^{r}[\prod_{j=m+1}^{B_i}\Gamma_{p,q}(1-b_{ji}+s)\prod_{j=n+1}^{A_i}\Gamma_{p,q}(a_{ji}-s)\Gamma_{p,q}(s)\Gamma_{p,q}(1-s)sin\pi s]}.$$

Since,

$$\Gamma_{p,q}(1+a) = \frac{p^a-q^a}{p-q}\Gamma_{p,q}(a) = [a]_{p,q}\Gamma_{p,q}(a),$$

therefore

$$[1-a_1+s]_{p,q}\Gamma_{p,q}(1-a_1+s) = \Gamma_{p,q}(1-(a_1-1)+s).$$

Thus

$$L.H.S. = \frac{1}{2\pi\omega}\int_L \frac{\prod_{j=1}^{m}\Gamma_{p,q}(b_j-s)\Gamma_{p,q}(1-(a_1-1)+s)\prod_{j=2}^{n}\Gamma_{p,q}(1-a_j+s)\pi z^s z^{1-a_1}ds}{\sum_{i=1}^{r}[\prod_{j=m+1}^{B_i}\Gamma_{p,q}(1-b_{ji}+s)\prod_{j=n+1}^{A_i}\Gamma_{p,q}(a_{ji}-s)\Gamma_{p,q}(s)\Gamma_{p,q}(1-s)sin\pi s]}$$
$$= z^{1-a_1}I_{A_i,B_i;r}m,n\left[\left(z;(p,q)\left|\begin{matrix}(a_1-1,1)(a_j,1)_{2,n};(a_{ji},1)_{n+1,A_i}\\(b_j,1)_{1,m};(b_{ji},1)_{m+1,B_i}\end{matrix}\right.\right)\right].$$

Hence the result. □

Theorem 2.2. *Let the parameters A_i, B_i are non-negative integers satisfying the inequality $0 \leq n \leq A_i, 0 \leq m \leq B_i$ and $\alpha_j, \beta_j, \alpha_{ji}, \beta_{ji}$ are positive real numbers and a_j, b_j, a_{ji}, b_{ji} are complex numbers, then*

$$D^{\mu}_{p,q}[I^{m,n}_{A_i,B_i;r}\left[\left(px^{\lambda};(p,q)\left|\begin{matrix}(a_j,\alpha_j)_{1,n};(a_{ji},\alpha_{ji})_{n+1,A_i}\\(b_j,\beta_j)_{1,m};(b_{ji},\beta_{ji})_{m+1,B_i}\end{matrix}\right.\right)\right]]$$
$$= x^{-\mu} I^{m,n+1}_{A_i,B_i+1;r}\left[\left(px^{\lambda};(p,q)\left|\begin{matrix}(0,\lambda)(a_j,\alpha_j)_{1,n};(a_{ji},\alpha_{ji})_{n+1,A_i}\\(b_j,\beta_j)_{1,m};(b_{ji},\beta_{ji})_{m+1,B_i}(\mu,\lambda)\end{matrix}\right.\right)\right], \quad (2.9)$$

where $z \neq 0, 0 < q < p$ and $\omega = \sqrt{-1}$.

Proof. To prove theorem (2.2) when $\lambda > 0$, we apply equation (1.8)

$$L.H.S. = D^{\mu}_{p,q}[I^{m,n}_{A_i,B_i;r}\left[\left(px^{\lambda};(p,q)\left|\begin{matrix}(a_j,\alpha_j)_{1,n};(a_{ji},\alpha_{ji})_{n+1,A_i}\\(b_j,\beta_j)_{1,m};(b_{ji},\beta_{ji})_{m+1,B_i}\end{matrix}\right.\right)\right]]$$
$$= \frac{1}{2\pi\omega}\int_L \frac{\prod_{j=1}^{m}\Gamma_{p,q}(b_j-\beta_j s)\prod_{j=1}^{n}\Gamma_{p,q}(1-a_j+\alpha_j s)\pi D_{p,q}[(px^{\lambda})^s]ds}{\sum_{i=1}^{r}[\prod_{j=m+1}^{B_i}\Gamma_{p,q}(1-b_{ji}+\beta_{ji}s)\prod_{j=n+1}^{A_i}\Gamma_{p,q}(a_{ji}-\alpha_{ji}s)\Gamma_{p,q}(s)\Gamma_{p,q}(1-s)sin\pi s]}$$
$$= \frac{1}{2\pi\omega}\int_L \frac{\prod_{j=1}^{m}\Gamma_{p,q}(b_j-\beta_j s)\prod_{j=1}^{n}\Gamma_{p,q}(1-a_j+\alpha_j s)\pi\Gamma_{p,q}(1-0+\lambda s)p^s x^{\lambda s-\mu}ds}{\sum_{i=1}^{r}[\prod_{j=m+1}^{B_i}\Gamma_{p,q}(1-b_{ji}+\beta_{ji}s)\prod_{j=n+1}^{A_i}\Gamma_{p,q}(a_{ji}-\alpha_{ji}s)\,\Gamma_{p,q}(1-\mu+\lambda s)\Gamma_{p,q}(s)\Gamma_{p,q}(1-s)sin\pi s]}$$
$$= x^{-\mu} I^{m,n+1}_{A_i,B_i+1;r}\left[\left(px^{\lambda};(p,q)\left|\begin{matrix}(0,\lambda)(a_j,\alpha_j)_{1,n};(a_{ji},\alpha_{ji})_{n+1,A_i}\\(b_j,\beta_j)_{1,m};(b_{ji},\beta_{ji})_{m+1,B_i}(\mu,\lambda)\end{matrix}\right.\right)\right]. \quad (2.10)$$

Hence the result. □

Special Cases:
Put $r = 1$ in (2.7) and (2.8) we get the (p,q)-analogue of H-function as mentioned below:

$$zD_{p,q}\left[z^{1-a_1}H^{m,n}_{P,Q}\left[\left(z;(p,q)\left|\begin{matrix}(a_j,1)_{1,P}\\(b_j,1)_{1,Q}\end{matrix}\right.\right)\right]\right] = z^{1-a_1}H^{m,n}_{P,Q}\left[\left(z;(p,q)\left|\begin{matrix}(a_1-1,1)(a_j,1)_{2,P}\\(b_j,1)_{1,Q}\end{matrix}\right.\right)\right], \quad (2.11)$$

$$D^{\mu}_{p,q}\left[H^{m,n}_{P,Q}\left[\left(px^{\lambda};(p,q)\left|\begin{matrix}(a_j,A_j)_{1,P}\\(b_j,B_j)_{1,Q}\end{matrix}\right.\right)\right]\right] = x^{-\mu}H^{m,n+1}_{P,Q+1;r}\left[\left(px^{\lambda};(p,q)\left|\begin{matrix}(0,\lambda)(a_j,A_j)_{1,P}\\(b_j,B_j)_{1,Q}(\mu,\lambda)\end{matrix}\right.\right)\right]. \quad (2.12)$$

Put $p = 1$, the above results of (p,q)-analogue change into well known results of basic analogue of H-function [9] as mentioned

$$zD_q\left[z^{1-a_1}H^{m,n}_{P,Q}\left[\left(z;q\left|\begin{matrix}(a_j,1)_{1,P}\\(b_j,1)_{1,Q}\end{matrix}\right.\right)\right]\right] = z^{1-a_1}H^{m,n}_{P,Q}\left[\left(z;q\left|\begin{matrix}(a_1-1,1)(a_j,1)_{2,P}\\(b_j,1)_{1,Q}\end{matrix}\right.\right)\right], \quad (2.13)$$

$$D^{\mu}_{q}\left[H^{m,n}_{P,Q}\left[\left(x^{\lambda};q\left|\begin{matrix}(a_j,A_j)_{1,P}\\(b_j,B_j)_{1,Q}\end{matrix}\right.\right)\right]\right] = x^{-\mu}H^{m,n+1}_{P,Q+1;r}\left[\left(x^{\lambda};q\left|\begin{matrix}(0,\lambda)(a_j,A_j)_{1,P}\\(b_j,B_j)_{1,Q}(\mu,\lambda)\end{matrix}\right.\right)\right] \quad (2.14)$$

Conclusion
The results proved in this paper give some contribution to the (p, q)-algebra and are believed to be new and fruitful for (p, q) analogue and are likely to find certain applications to the solution of the (p, q)-integral equations involving various (p, q)-hypergeometric functions.

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