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A STUDY OF GENERALIZED FRACTIONAL DERIVATIVE FORMULAS ASSOCIATED WITH GENERALIZED M-SERIES

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Abstract

A remarkably large number of generalized fractional derivative formulas involving a variety of special functions have been developed by many authors. In the present paper the authors have established two generalized fractional derivative formulas involving generalized M-series, which is expressed in terms of the generalized Wright hypergeometric function. Some special cases of the results presented here are also pointed out.

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1 Introduction and Preliminaries

The generalized Wright hypergeometric function introduced by Wright [12] and given by

$$ {}_p\Psi_q(z) = {}_p\Psi_q\left[z \mid \begin{array}{c}(a_1,\alpha_1),...,(a_p,\alpha_p)\\(b_1,\beta_1),...,(b_q,\beta_q)\end{array}\right] = \sum_{n=0}^{\infty}\frac{\prod_{i=1}^{p}\Gamma(a_i+\alpha_i n)}{\prod_{j=1}^{q}\Gamma(b_j+\beta_j n)}\frac{z^n}{n!}, \tag{1.1}$$

where $z, a_i, b_j \in C$ and $\alpha_i, \beta_j \in \Re - \{0\}, \ (i = 1, ..., p; \ j = 1, ..., q)$. Wright proved several theorems on the asymptotic expansion of generalized Wright function ${}_p\Psi_q(z)$ for all values of the argument z under the condition

$$\sum_{i=1}^{p}\alpha_i - \sum_{j=1}^{q}\beta_j \leq 1. \tag{1.2}$$

When $\alpha_1 = \ldots = \alpha_p = \beta_1 \ldots = \beta_q = 1$, then (1.1) reduces to the generalized hypergeometric function ${}_pF_q(.)$ as shown below

$${}_p\Psi_q\left[z \mid \begin{array}{c}(a_1,1),...,(a_p,1)\\(b_1,1),...,(b_q,1)\end{array}\right] = \frac{\prod_{i=1}^{p}\Gamma(a_i)}{\prod_{j=1}^{q}\Gamma(b_j)} \ {}_pF_q(a_1,\ldots,a_p;b_1,\ldots,b_q;z), \tag{1.3}$$

where $z, a_i, b_j \in C \, (i = 1, ..., p; \ j = 1, ..., q)$; and $\Re\left(\sum_{j=1}^{q} b_j - \sum_{i=1}^{p} a_i\right) > 0$.

Properties of this generalized Wright function were investigated in [2]. In particular, it was proved that ${}_p\Psi_q(z), z \in C$ is an entire function under the condition (1.2).

Recently, a new generalization of M-series introduced by Faraj et al. [1] in the following manner:

$$M^{\alpha,\beta}_{p,q;m,n}(a_1,...,a_p;b_1,...,b_q;z) = M^{\alpha,\beta}_{p,q;m,n}(z) = \sum_{k=0}^{\infty} \frac{(a_1)_{km} \cdots (a_p)_{km}}{(b_1)_{kn} \cdots (b_q)_{kn}} \frac{z^k}{\Gamma(\alpha k+\beta)}, \tag{1.4}$$

where z,$\alpha, \beta \in C, \Re(\alpha) > 0$ and m,n are non-negative real numbers and

$$(\gamma)_{kn} = \frac{\Gamma(\gamma+kn)}{\Gamma(\gamma)}, \tag{1.5}$$

denotes the generalized Pochhammer symbol.

The series in (1.4) is absolutely convergent for all values of z provided that $pm < qn + Re(\alpha)$, moreover if $pm = qn + Re(\alpha)$, the series converges for $|z| < \delta = \alpha^\alpha$.

Some special cases of the generalized M-series $M^{\alpha,\beta}_{p,q;m,n}(z)$ are as follows

(i) For $m = n = 1$ in (1.4), we get the generalized M-series introduced by Sharma and Jain [11] as follows

$$M^{\alpha,\beta}_{p,q;1,1}(z) = M^{\alpha,\beta}_{p,q}(z) = \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_q)_k} \frac{z^k}{\Gamma(\alpha k+\beta)}. \tag{1.6}$$

(ii) When $m = n = 1$ and $\beta = 1$ in (1.4), then result will be the M-series defined by Sharma [10] in the following manner

$$M^{\alpha,1}_{p,q;1,1}(z) = M^{\alpha}_{p,q}(z) = \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_q)_k} \frac{z^k}{\Gamma(\alpha k+1)}. \tag{1.7}$$

(iii) If we take $p = q = 1$, (1.4) reduces to generalized Mittag-Leffler function introduced by Salim and Faraj [8] as

$$M^{\alpha,\beta}_{1,1;m,n}(z) = E^{a_1,b_1,m}_{\alpha,\beta,n}(z) = \sum_{k=0}^{\infty} \frac{(a_1)_{km}}{(b_1)_{kn}} \frac{z^k}{\Gamma(\alpha k+\beta)}. \tag{1.8}$$

(iv) Further, if we put $\alpha = \beta = 1$ and $m = n = 1$ with arbitrary p, q we have the generalized hypergeometric function ${}_pF_q(.)$ ([4], [5], [6]) as follows

$$M^{1,1}_{p,q;1,1}(z) = {}_pF_q\left[\begin{array}{c} a_1,...,a_p \\ b_1,...,b_q \end{array}; z\right] = \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_q)_k} \frac{z^k}{k!}. \tag{1.9}$$

The generalized M-Series extend the Wright generalized hypergeometric function ${}_p\Psi_q(z)$ stated in definition (1.1). According to some well-chosen specific values of the parameters, the generalized M-Series can be easily reduced to classical generalized special functions.

2 Generalized fractional differential operators

The fractional calculus operators involving various special functions have been found significant importance and applications in various subfield of mathematical analysis.

Let $\alpha, \alpha', \beta, \beta', \gamma \in C, x > 0$ and $\Re(\gamma) > 0$, then the generalized fractional differential operators involving Appell's function $F_3(.)$ as a kernel are defined by Saigo and Maeda [7] (see also [3]) as follows:

$$\left(D_{0+}^{\alpha,\alpha',\beta,\beta',\gamma} f\right)(x) = \left(I_{0+}^{-\alpha',-\alpha,-\beta',-\beta,-\gamma} f\right)(x) = \frac{d^n}{dx^n}\left(I_{0+}^{-\alpha',-\alpha,-\beta'+n,-\beta,-\gamma+n} f\right)(x) \tag{2.1}$$

$$= \frac{1}{\Gamma(n-\gamma)} \frac{d^n}{dx^n}\left(x^{\alpha'}\right) \int_0^x (x-t)^{n-\gamma-1}\; t^{\alpha}$$
$$\times F_3\left(-\alpha', -\alpha, n-\beta', -\beta, n-\gamma; 1-t/x, 1-t/x\right) f(t)\, dt, \tag{2.2}$$

$$\left(\Re(\gamma) > 0, n = [\Re(\gamma)+1]\right),$$

and

$$\left(D_{-}^{\alpha,\alpha',\beta,\beta',\gamma} f\right)(x) = \left(I_{-}^{-\alpha',-\alpha,-\beta',-\beta,-\gamma} f\right)(x) = (-1)^n \frac{d^n}{dx^n}\left(I_{-}^{-\alpha',-\alpha,-\beta',-\beta+n,-\gamma+n} f\right)(x) \tag{2.3}$$

$$= \frac{1}{\Gamma(n-\gamma)} (-1)^n \frac{d^n}{dx^n}(x^{\alpha}) \int_x^{\infty} (t-x)^{n-\gamma-1}\; t^{\alpha'}$$
$$\times F_3(-\alpha', -\alpha, -\beta', n-\beta, n-\gamma; 1-x/t, 1-t/x) f(t) dt, \tag{2.4}$$

$$\left(\Re(\gamma) > 0, n = [\Re(\gamma)+1]\right),$$

where $I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}$ and $I_{-}^{\alpha,\alpha',\beta,\beta',\gamma}$ are the Saigo-Maeda fractional integral operators.

If we take $\alpha = \alpha+\beta,\ \alpha' = \beta' = 0,\ \beta = -\gamma$ and $\gamma = \alpha$, then above operators reduce to the Saigo fractional calculus operators [6] as

For $\alpha, \beta, \gamma \in C, x > 0$ and $\Re(\alpha) > 0$, then

$$\left(D_{0+}^{\alpha,\beta,\gamma} f\right)(x) = \left(I_{0+}^{-\alpha,-\beta,\alpha+\gamma} f\right)(x) = \left(\frac{d}{dx}\right)^k \left(I_{0+}^{-\alpha+k,-\beta-k,\alpha+\gamma-k} f\right)(x), \tag{2.5}$$

and

$$\left(D_{-}^{\alpha,\beta,\gamma} f\right)(x) = \left(I_{-}^{-\alpha,-\beta,\alpha+\gamma} f\right)(x) = \left(-\frac{d}{dx}\right)^k \left(I_{-}^{-\alpha+k,-\beta-k,\alpha+\gamma} f\right)(x), \tag{2.6}$$

here $k = [\Re(\gamma)] + 1$.If we set $\beta = -\alpha$, then operators (2.5) and (2.6) reduce to Riemann-Liouville fractional differential operators [9].

Further, we also have (see [7], p-394, eq. (4.18) and (4.19))

$$\left(I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\rho-1}\right)(x) = x^{\rho-\alpha-\alpha'+\gamma-1}\, \Gamma\left[\begin{array}{c} \rho, \rho+\gamma-\alpha-\alpha'-\beta, \rho+\beta'-\alpha' \\ \rho+\gamma-\alpha-\alpha', \rho+\gamma-\alpha'-\beta, \rho+\beta' \end{array}\right], \tag{2.7}$$

where, $\Re(\gamma) > 0$, $\Re(\rho) > max\,[0, \Re(\alpha + \alpha' + \beta - \gamma), \Re(\alpha' - \beta')]$, and

$$\left(I_{-}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\rho-1}\right)(x) = x^{\rho-\alpha-\alpha'+\gamma-1}\,\Gamma\left[\begin{array}{c} 1+\alpha+\alpha'-\gamma-\rho, 1+\alpha+\beta'-\gamma-\rho, 1-\beta-\rho \\ 1-\rho, 1+\alpha+\alpha'+\beta'-\gamma-\rho, 1+\alpha-\beta-\rho \end{array}\right]. \tag{2.8}$$

where, $\Re(\gamma) > 0$, $\Re(\rho) < 1 + min\,[\Re(-\beta), \Re(\alpha + \alpha' - \gamma), \Re(\alpha + \beta' - \gamma)]$.

Here, the Symbol $\Gamma\left[\begin{array}{c} a, b, c \\ d, e, f \end{array}\right]$ will be used to represent the ratio of product of gamma functions as $\frac{\Gamma(a)\Gamma(b)\Gamma(c)}{\Gamma(d)\Gamma(e)\Gamma(f)}$.

3 Main results

In this section, we establish the left-sided and right-sided generalized fractional derivative formulas involving generalized M-series. These formulas are given by the following theorems:

Theorem 3.1. *Let $\mu, \mu', \eta, \eta', \delta, \alpha, \beta, \sigma \in C$, $x, \nu > 0$ and $m, n > 0$ be such that $\Re(\delta) > 0, \Re(\nu) > 0, \Re(\alpha) > 0$ and $pm \leq qn + \Re(\alpha)$, then there holds the formula*

$$\left\{D_{0+}^{\mu,\mu',\eta,\eta',\delta}\left[t^{\sigma-1} M_{p,q;m,n}^{\alpha,\beta}\left(wt^{\nu}\right)\right]\right\}(x) = x^{\sigma+\mu+\mu'-\delta-1}\,\frac{\prod_{j=1}^{q}\Gamma\left(b_j\right)}{\prod_{i=1}^{p}\Gamma\left(a_i\right)}$$
$$\times\, p+4\psi_{q+4}\left[\begin{array}{c} (a_1, m), \ldots, (a_p, m), (1,1), (\sigma, \nu), (\sigma+\mu+\mu'+\eta'-\delta, \nu), \\ (b_1, n), \ldots, (b_q, n), (\beta, \alpha), (\sigma+\mu+\mu'-\delta, \nu), (\sigma+\mu+\eta'-\delta, \nu), \end{array}\right.$$
$$\left.\begin{array}{c} (\sigma+\mu-\eta, \nu) \\ (\sigma-\eta, \nu) \end{array} ; wx^{\nu}\right] \tag{3.1}$$

Proof. By using series representation of generalized M-Series (1.4), (2.1) and (2.7), we have

$$\left\{D_{0+}^{\mu,\mu',\eta,\eta',\delta}\left[t^{\sigma-1} M_{p,q;m,n}^{\alpha,\beta}\left(wt^{\nu}\right)\right]\right\}(x)$$
$$= \left\{D_{0+}^{\mu,\mu',\eta,\eta',\delta}\left[t^{\sigma-1}\sum_{k=0}^{\infty}\frac{(a_1)_{mk}\cdots(a_p)_{mk}}{(b_1)_{nk}\cdots(b_q)_{nk}}\frac{(wt^{\nu})^k}{\Gamma(\alpha k+\beta)}\right]\right\}(x),$$

by interchanging the order of differentiation and summation, we arrive at

$$= \sum_{k=0}^{\infty}\frac{(a_1)_{mk}\cdots(a_p)_{mk}}{(b_1)_{nk}\cdots(b_q)_{nk}}\frac{w^k}{\Gamma(\alpha k+\beta)}\left(D_{0+}^{\mu,\mu',\eta,\eta',\delta} t^{\sigma+\nu k-1}\right)(x)$$
$$= \sum_{k=0}^{\infty}\frac{(a_1)_{mk}\cdots(a_p)_{mk}}{(b_1)_{nk}\cdots(b_q)_{nk}}\frac{w^k}{\Gamma(\alpha k+\beta)}\left(I_{0+}^{-\mu',-\mu,-\eta',-\eta,-\delta} t^{\sigma+\nu k-1}\right)(x)$$
$$= x^{\sigma+\mu+\mu'-\delta-1}\sum_{k=0}^{\infty}\frac{(a_1)_{mk}\cdots(a_p)_{mk}}{(b_1)_{nk}\cdots(b_q)_{nk}}\frac{(wx^{\nu})^k}{k!}$$
$$\times\,\frac{\Gamma(\sigma+\nu k)\,\Gamma(\sigma+\mu+\mu'+\eta'-\delta+\nu k)\,\Gamma(\sigma+\mu-\eta+\nu k)\,\Gamma(k+1)}{\Gamma(\alpha k+\beta)\,\Gamma(\sigma+\mu+\mu'-\delta+\nu k)\,\Gamma(\sigma+\mu+\eta'-\delta+\nu k)\,\Gamma(\sigma-\eta+\nu k)}. \tag{3.2}$$

Next, using (1.1), (1.5) and rearranging the terms, we get

$$\left\{D_{0+}^{\mu,\mu',\eta,\eta',\delta}\left[t^{\sigma-1} M_{p,q;m,n}^{\alpha,\beta}\left(wt^{\nu}\right)\right]\right\}(x) = x^{\sigma+\mu+\mu'-\delta-1}\,\frac{\prod_{j=1}^{q}\Gamma\left(b_j\right)}{\prod_{i=1}^{p}\Gamma\left(a_i\right)}$$

$$\times {}_{p+4}\psi_{q+4}\left[\begin{array}{l} (a_1,m),\ldots,(a_p,m),(1,1),(\sigma,\nu),(\sigma+\mu+\mu'+\eta'-\delta,\nu), \\ (b_1,n),\ldots,(b_q,n),(\beta,\alpha),(\sigma+\mu+\mu'-\delta,\nu),(\sigma+\mu+\eta'-\delta,\nu), \end{array}\right.$$

$$\left.\begin{array}{l}(\sigma+\mu-\eta,\nu) \\ (\sigma-\eta,\nu)\end{array} ;\ wx^{\nu}\right],$$

which completes the proof of Theorem 1. □

If we take $\mu = \mu + \eta, \mu' = \eta' = 0, \eta = -\delta$ and $\delta = \mu$ then Theorem 1 reduces to the following result

Corollary 3.1. *Let $\mu, \eta, \delta, \alpha, \beta, \sigma \in C, x > 0, \nu > 0$ and $m, n > 0$ be such that $\Re(\delta) > 0, \Re(\nu) > 0, \Re(\alpha) > 0$ and $pm \leq qn + \Re(\alpha)$, then we have*

$$\{D_{0+}^{\mu,\eta,\delta}[t^{\sigma-1}M_{p,q;m,n}^{\alpha,\beta}(wt^{\nu})]\}(x) = x^{\sigma+\eta-1}\frac{\prod_{j=1}^{q}\Gamma(b_j)}{\prod_{i=1}^{p}\Gamma(a_i)}$$

$$\times_{p+3}\,\psi_{q+3}\left[\begin{array}{l} (a_1,m),\ldots,(a_p,m),(1,1),(\sigma,\nu),(\sigma+\mu+\eta+\delta,\nu) \\ (b_1,n),\ldots,(b_q,n),(\beta,\alpha),(\sigma+\eta,\nu),(\sigma+\delta,\nu)\end{array} ;wx^{\nu}\right]. \quad (3.3)$$

If we put $\eta = -\mu$ in above Corollary 1.1, we can easily obtain the following result.

Corollary 3.2. *Let $\mu, \alpha, \beta, \sigma \in C, x > 0, \nu > 0$ and $m, n > 0$ be such that $\Re(\mu) > 0, \Re(\nu) > 0, \Re(\alpha) > 0$ and $pm \leq qn + \Re(\alpha)$, then there holds the formula*

$$\{D_{0+}^{\mu}[t^{\sigma-1}M_{p,q;m,n}^{\alpha,\beta}(wt^{\nu})]\}(x) = x^{\sigma-\mu-1}\frac{\prod_{j=1}^{q}\Gamma(b_j)}{\prod_{i=1}^{p}\Gamma(a_i)}$$

$$\times_{p+2}\,\psi_{q+2}\left[\begin{array}{l} (a_1,m),\ldots,(a_p,m),(1,1),(\sigma,\nu) \\ (b_1,n),\ldots,(b_q,n),(\beta,\alpha),(\sigma-\mu,\nu)\end{array} ;lwx^{\nu}\right]. \quad (3.4)$$

If we set $m = n = 1$ in (3.1), then $M_{p,q;m,n}^{\alpha,\beta}(z)$ reduces to the generalized M-series $M_{p,q}^{\alpha,\beta}(z)$ [11], we get the following Corollary.

Corollary 3.3. *Let $\mu, \mu', \eta, \eta', \delta, \alpha, \beta, \sigma \in C, x > 0, \nu > 0$ and $m, n > 0$ be such that $\Re(\delta) > 0, \Re(\nu) > 0, \Re(\alpha) > 0$ and $p \leq q + \Re(\alpha)$, then we have*

$$\{D_{0+}^{\mu,\mu',\eta,\eta',\delta}[t^{\sigma-1}M_{p,q}^{\alpha,\beta}(wt^{\nu})]\}(x) = x^{\sigma+\mu+\mu'-\delta-1}\frac{\prod_{j=1}^{q}\Gamma(b_j)}{\prod_{i=1}^{p}\Gamma(a_i)}$$

$$\times_{p+4}\,\psi_{q+4}\left[\begin{array}{l} (a_1,1),\ldots,(a_p,1),(1,1),(\sigma,\nu),(\sigma+\mu+\mu'+\eta'-\delta,\nu), \\ (b_1,1),\ldots,(b_q,1),(\beta,\alpha),(\sigma+\mu+\mu'-\delta,\nu),(\sigma+\mu+\eta'-\delta,\nu), \end{array}\right.$$

$$\left.\begin{array}{l}(\sigma+\mu-\eta,\nu) \\ (\sigma-\eta,\nu)\end{array} ;wx^{\nu}\right]. \quad (3.5)$$

If we take $\beta = 1$ in (3.5), then generalized M-series $M_{p,q}^{\alpha,\beta}(z)$ reduces to the M-series $M_{p,q}^{\alpha}(z)$ [10], we arrive at

Corollary 3.4. *Let $\mu,\mu',\eta,\eta',\delta,\alpha,\beta,\sigma \in C, x>0, \nu>0$ and $m,n>0$ be such that $\Re(\delta)>0, \Re(\nu)>0, \Re(\alpha)>0$, then there holds the formula*

$$\left\{D_{0+}^{\mu,\mu',\eta,\eta',\delta}\left[t^{\sigma-1}M_{p,q}^{\alpha}\left(wt^{\nu}\right)\right]\right\}(x) = x^{\sigma+\mu+\mu'-\delta-1}\frac{\prod_{j=1}^{q}\Gamma\left(b_j\right)}{\prod_{i=1}^{p}\Gamma\left(a_i\right)}$$
$$\times\ {}_{p+4}\psi_{q+4}\left[\begin{array}{l}(a_1,1),\ldots,(a_p,1),(1,1),(\sigma,\nu),(\sigma+\mu+\mu'+\eta'-\delta,\nu),\\ (b_1,1),\ldots,(b_q,1),(1,\alpha),(\sigma+\mu+\mu'-\delta,\nu),(\sigma+\mu+\eta'-\delta,\nu),\end{array}\right.$$
$$\left.\begin{array}{l}(\sigma+\mu-\eta,\nu)\\ (\sigma-\eta,\nu)\end{array};\ wx^{\nu}\right] \quad (3.6)$$

Again, if we put $p=q=1$ in (3.1), then $M_{p,q;m,n}^{\alpha,\beta}(z)$ reduces to the generalized Mittag-Leffler function $E_{\alpha,\beta,n}^{a_1,b_1,m}(z)$ [8], we obtain

Corollary 3.5. *Let $\mu,\mu',\eta,\eta',\delta,\alpha,\beta,\sigma \in C, x>0, \nu>0$ and $m,n>0$ be such that $\Re(\delta)>0, \Re(\nu)>0, \Re(\alpha)>0$ and $m\leq n+\Re(\alpha)$, then we have*

$$\{D_{0+}^{\mu,\mu',\eta,\eta',\delta}[t^{\sigma-1}E_{\alpha,\beta,n}^{a_1,b_1,m}(wt^{\nu})]\}(x) = x^{\sigma+\mu+\mu'-\delta-1}\frac{\Gamma(b_1)}{\Gamma(a_1)}$$
$$\times_5\psi_5\left[\begin{array}{l}(a_1,m),(1,1),(\sigma,\nu),(\sigma+\mu+\mu'+\eta'-\delta,\nu),(\sigma+\mu-\eta,\nu)\\ (b_1,n),(\beta,\alpha),(\sigma+\mu+\mu'-\delta,\nu),(\sigma+\mu+\eta'-\delta,\nu),(\sigma-\eta,\nu)\end{array};wx^{\nu}\right]. \quad (3.7)$$

Further, if we set $\alpha=\beta=1$ and $m=n=1$, then $M_{p,q;m,n}^{\alpha,\beta}(z)$ reduces to generalized hypergrometric function ${}_pF_q(.)$ ([4], [5], [6]), we have

Corollary 3.6. *Let $\mu,\mu',\eta,\eta',\delta,\alpha,\beta,\sigma \in C$, $x,\nu>0$ and $m,n>0$ be such that $\Re(\delta)>0, \Re(\nu)>0, \Re(\alpha)>0$, then we have*

$$\{D_{0+}^{\mu,\mu',\eta,\eta',\delta}[t^{\sigma-1}{}_pF_q(wt^{\nu})]\}(x) = x^{\sigma+\mu+\mu'-\delta-1}\frac{\prod_{j=1}^{q}\Gamma(b_j)}{\prod_{i=1}^{p}\Gamma(a_i)}$$
$$\times_{p+3}\psi_{q+3}\left[\begin{array}{l}(a_1,1),\ldots,(a_p,1),(\sigma,\nu),(\sigma+\mu+\mu'+\eta'-\delta,\nu),(\sigma+\mu-\eta,\nu)\\ (b_1,1),\ldots,(b_q,1),(\sigma+\mu+\mu'-\delta,\nu),(\sigma+\mu+\eta'-\delta,\nu),(\sigma-\eta,\nu)\end{array};\ wx^{\nu}\right]. \quad (3.8)$$

Theorem 3.2. *Let $\mu,\mu',\eta,\eta',\delta,\alpha,\beta,\sigma \in C$, $x,\nu>0$ and $m,n>0$ be such that $\Re(\delta)>0, \Re(\nu)>0, \Re(\alpha)>0$ and $pm\leq qn+\Re(\alpha)$, then there holds the formula*

$$\{D_{-}^{\mu,\mu',\eta,\eta',\delta}[t^{\delta-\sigma}M_{p,q;m,n}^{\alpha,\beta}(wt^{-\nu})]\}(x) = x^{\mu+\mu'-\sigma}\frac{\prod_{j=1}^{q}\Gamma\left(b_j\right)}{\prod_{i=1}^{p}\Gamma(a_i)}$$
$$\times_{p+4}\psi_{q+4}\left[\begin{array}{l}(a_1,m),\ldots,(a_p,m),(1,1),(\sigma-\mu-\mu',\nu),(\sigma-\mu'-\eta,\nu),\\ (b_1,n),\ldots,(b_q,n),(\beta,\alpha),(\sigma-\delta,\nu),(\sigma-\mu-\mu'-\eta,\nu),\end{array}\right.$$
$$\left.\begin{array}{l}(\sigma+\eta'-\delta,\nu)\\ (\sigma-\mu'+\eta'-\delta,\nu)\end{array};wx^{-\nu}\right]. \quad (3.9)$$

Proof. By using (1.4), (2.3) and (2.8), we get

$$\left\{D_-^{\mu,\mu',\eta,\eta',\delta}\left[t^{\delta-\sigma}M_{p,q;m,n}^{\alpha,\beta}\left(wt^{-\nu}\right)\right]\right\}(x)$$
$$=\left\{D_-^{\mu,\mu',\eta,\eta',\delta}\left[t^{\delta-\sigma}\sum_{k=0}^{\infty}\frac{(a_1)_{mk}\dots(a_p)_{mk}}{(b_1)_{nk}\dots(b_q)_{nk}}\frac{\left(wt^{-\nu}\right)^k}{\Gamma(\alpha k+\beta)}\right]\right\}(x),$$

by interchanging the order of differentiation and summation, we have

$$=\sum_{k=0}^{\infty}\frac{(a_1)_{mk}\dots(a_p)_{mk}}{(b_1)_{nk}\dots(b_q)_{nk}}\frac{w^k}{\Gamma(\alpha k+\beta)}(D_-^{\mu,\mu',\eta,\eta',\delta}t^{\delta-\sigma-\nu k})(x)$$
$$=\sum_{k=0}^{\infty}\frac{(a_1)_{mk}\dots(a_p)_{mk}}{(b_1)_{nk}\dots(b_q)_{nk}}\frac{w^k}{\Gamma(\alpha k+\beta)}(I_-^{-\mu',-\mu,-\eta',-\eta,-\delta}t^{(1+\delta-\sigma-\nu k)-1})(x)$$
$$=x^{\mu+\mu'-\sigma}\sum_{k=0}^{\infty}\frac{(a_1)_{mk}\dots(a_p)_{mk}}{(b_1)_{nk}\dots(b_q)_{nk}}\frac{(wx^{-\nu})^k}{k!}$$
$$\times\frac{\Gamma(k+1)\Gamma(\sigma-\mu-\mu'+\nu k)\Gamma(\sigma-\mu'-\eta+\nu k)\Gamma(\sigma+\eta'-\delta+\nu k)}{\Gamma(\alpha k+\beta)\Gamma(\sigma-\delta+\nu k)\Gamma(\sigma-\mu-\mu'-\eta+\nu k)\Gamma(\sigma-\mu'+\eta'-\delta+\nu k)}. \tag{3.10}$$

Next, using (1.1), (1.5) and rearranging the terms, we obtain

$$\{D_-^{\mu,\mu',\eta,\eta',\delta}[t^{\delta-\sigma}M_{p,q;m,n}^{\alpha,\beta}(wt^{-\nu})]\}(x)=x^{\mu+\mu'-\sigma}\quad\frac{\prod_{j=1}^{q}\Gamma(b_j)}{\prod_{i=1}^{p}\Gamma(a_i)}$$
$$\times\ {}_{p+4}\psi_{q+4}\left[\begin{array}{l}(a_1,m),\dots,(a_p,m),(1,1),(\sigma-\mu-\mu',\nu),(\sigma-\mu'-\eta,\nu),\\(b_1,n),\dots,(b_q,n),(\beta,\alpha),(\sigma-\delta,\nu),(\sigma-\mu-\mu'-\eta,\nu),\end{array}\right.$$
$$\left.\begin{array}{l}(\sigma+\eta'-\delta,\nu)\\(\sigma-\mu'+\eta'-\delta,\nu)\end{array};\ wx^{-\nu}\right],$$

which completes the proof of the Theorem 2. □

If we take $\mu=\mu+\eta,\mu'=\eta'=0,\eta=-\delta$ and $\delta=\mu$ then (3.9) reduces to the following result

Corollary 3.7. *Let $\mu,\eta,\delta,\alpha,\beta,\sigma\in C,x>0,\nu>0$ and $m,n>0$ be such that $\Re(\delta)>0,\Re(\nu)>0,\Re(\alpha)>0$ and $pm\leq qn+\Re(\alpha)$, then we have*

$$\left\{D_-^{\mu,\eta,\delta}\left[t^{\mu-\sigma}M_{p,q;m,n}^{\alpha,\beta}\left(wt^{-\nu}\right)\right]\right\}(x)=x^{\mu+\eta-\sigma}\quad\frac{\prod_{j=1}^{q}\Gamma(b_j)}{\prod_{i=1}^{p}\Gamma(a_i)}$$
$$\times\ {}_{p+3}\psi_{q+3}\left[\begin{array}{l}(a_1,m),\dots,(a_p,m),(1,1),(\sigma-\mu-\eta,\nu),(\sigma+\delta,\nu)\\(b_1,n),\dots,(b_q,n),(\beta,\alpha),(\sigma-\mu,\nu),(\sigma-\mu-\eta+\delta,\nu)\end{array};\ wx^{-\nu}\right] \tag{3.11}$$

If we put $\eta=-\mu$ in above Corollary 2.1, then we obtain

Corollary 3.8. *Let $\mu, \alpha, \beta, \sigma \in C, x > 0, \nu > 0$ and $m, n > 0$ be such that $\Re(\mu) > 0, \Re(\nu) > 0, \Re(\alpha) > 0$ and $pm \leq qn + \Re(\alpha)$, then there holds the formula*

$$\{D_-^{\mu}[t^{\mu-\sigma} M_{p,q;m,n}^{\alpha,\beta}(wt^{-\nu})]\}(x) \\ = x^{-\sigma} \frac{\prod_{j=1}^{q} \Gamma(b_j)}{\prod_{i=1}^{p} \Gamma(a_i)} {}_{p+2}\psi_{q+2}\left[\begin{array}{l} (a_1, m), \ldots, (a_p, m), (1,1), (\sigma, \nu) \\ (b_1, n), \ldots, (b_q, n), (\beta, \alpha), (\sigma - \mu, \nu) \end{array} ; \; wx^{-\nu} \right]. \quad (3.12)$$

If we set $m = n = 1$ in (3.9), then $M_{p,q;m,n}^{\alpha,\beta}(z)$ reduces to the generalized M-series $M_{p,q}^{\alpha,\beta}(z)$ [11], we get the following Corollary.

Corollary 3.9. *Let $\mu, \mu', \eta, \eta', \delta, \alpha, \beta, \sigma \in C, x > 0, \nu > 0$ and $m, n > 0$ be such that $\Re(\delta) > 0, \Re(\nu) > 0, \Re(\alpha) > 0$ and $p \leq q + \Re(\alpha)$, then we have*

$$\{D_-^{\mu,\mu',\eta,\eta',\delta}[t^{\delta-\sigma} M_{p,q}^{\alpha,\beta}(wt^{-\nu})]\}(x) = x^{\mu+\mu'-\sigma} \frac{\prod_{j=1}^{q} \Gamma(b_j)}{\prod_{i=1}^{p} \Gamma(a_i)} \\ \times {}_{p+4}\psi_{q+4}\left[\begin{array}{l} (a_1, 1), \ldots, (a_p, 1), (1,1), (\sigma - \mu - \mu', \nu), (\sigma - \mu' - \eta, \nu), \\ (b_1, 1), \ldots, (b_q, 1), (\beta, \alpha), (\sigma - \delta, \nu), (\sigma - \mu - \mu' - \eta, \nu), \end{array} \right. \\ \left. \begin{array}{l} (\sigma + \eta' - \delta, \nu) \\ (\sigma - \mu' + \eta' - \delta, \nu) \end{array} ; \; wx^{-\nu} \right]. \quad (3.13)$$

If we put $\beta = 1$ in above Corollary 2.3, then generalized M-series $M_{p,q}^{\alpha,\beta}(z)$ reduces to the M-series $M_{p,q}^{\alpha}(z)$ [10], we arrive at

Corollary 3.10. *Let $\mu, \mu', \eta, \eta', \delta, \alpha, \beta, \sigma \in C, x > 0, \nu > 0$ and $m, n > 0$ be such that $\Re(\delta) > 0, \Re(\nu) > 0, \Re(\alpha) > 0$, then there holds the formula*

$$\{D_-^{\mu,\mu',\eta,\eta',\delta}[t^{\delta-\sigma} M_{p,q}^{\alpha}(wt^{-\nu})]\}(x) = x^{\mu+\mu'-\sigma} \frac{\prod_{j=1}^{q} \Gamma(b_j)}{\prod_{i=1}^{p} \Gamma(a_i)} \\ \times {}_{p+4}\psi_{q+4}\left[\begin{array}{l} (a_1, 1), \ldots, (a_p, 1), (1,1), (\sigma - \mu - \mu', \nu), (\sigma - \mu' - \eta, \nu), \\ (b_1, 1), \ldots, (b_q, 1), (1, \alpha), (\sigma - \delta, \nu), (\sigma - \mu - \mu' - \eta, \nu), \end{array} \right. \\ \left. \begin{array}{l} (\sigma + \eta' - \delta, \nu) \\ (\sigma - \mu' + \eta' - \delta, \nu) \end{array} ; \; wx^{-\nu} \right]. \quad (3.14)$$

Again, if we take $p = q = 1$ in (3.9), then $M_{p,q;m,n}^{\alpha,\beta}(z)$ reduces to the generalized Mittag-Leffler function $E_{\alpha,\beta,n}^{a_1,b_1,m}(z)$ [8], we have

Corollary 3.11. *Let $\mu, \mu', \eta, \eta', \delta, \alpha, \beta, \sigma \in C, x > 0, \nu > 0$ and $m, n > 0$ be such that $\Re(\delta) > 0, \Re(\nu) > 0, \Re(\alpha) > 0$ and $m \leq n + \Re(\alpha)$, then we have*

$$\left\{D_-^{\mu,\mu',\eta,\eta',\delta}\left[t^{\delta-\sigma} E_{\alpha,\beta,n}^{a_1,b_1,m}\left(wt^{-\nu}\right)\right]\right\}(x) = x^{\mu+\mu'-\sigma} \frac{\Gamma(b_1)}{\Gamma(a_1)} \\ \times {}_5\psi_5\left[\begin{array}{l} (a_1, m), (1,1), (\sigma - \mu - \mu', \nu), (\sigma - \mu' - \eta, \nu), (\sigma + \eta' - \delta, \nu) \\ (b_1, n), (\beta, \alpha), (\sigma - \delta, \nu), (\sigma - \mu - \mu' - \eta, \nu), (\sigma - \mu' + \eta' - \delta, \nu) \end{array} ; \; wx^{-\nu} \right]. \\ (3.15)$$

Further, if we set $\alpha=\beta=1$ and $m=n=1$ with arbitrary p,q then $M^{\alpha,\beta}_{p,q;m,n}(z)$ reduces to the generalized hypergrometric function ${}_pF_q\,(.)$ ([4], [5], [6]), we get

Corollary 3.12. *Let $\mu,\mu',\eta,\eta',\delta,\alpha,\beta,\sigma\in C,\ x,\nu>0$ and $m,n>0$ be such that $\Re(\delta)>0,\Re(\nu)>0,\Re(\alpha)>0$,then there holds the formula*

$$\{D_{-}^{\mu,\mu',\eta,\eta',\delta}[t^{\delta-\sigma}{}_pF_q(wt^{-\nu})]\}(x)=x^{\mu+\mu'-\sigma}\frac{\prod_{j=1}^{q}\Gamma(b_j)}{\prod_{i=1}^{p}\Gamma(a_i)}$$
$$\times_{p+3}\psi_{q+3}\left[\begin{array}{l}(a_1,1),\dots,(a_p,1),(\sigma-\mu-\mu',\nu),(\sigma-\mu'-\eta,\nu),\\(b_1,1),\dots,(b_q,1),(\sigma-\delta,\nu),(\sigma-\mu-\mu'-\eta,\nu),\end{array}\right.$$
$$\left.\begin{array}{l}(\sigma+\eta'-\delta,\nu)\\(\sigma-\mu'+\eta'-\delta,\nu)\end{array};\ wx^{-\nu}\right]. \quad (3.16)$$

4 Conclusion

We conclude this investigation by remarking the results obtained here are useful in deriving various derivative formulas in the theory of the fractional calculus operators. We can also obtain the number of special functions as the special cases of our main results, which are related with generalized M-Series and generalized Wright hypergeometric function.

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