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# A STUDY OF CERTAIN MIXED FAMILY OF LINEAR POSITIVE OPERATORS IN APPROXIMATION THEORY

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**Abstract**

In the present paper, we study about the integral modification of linear positive operators namely Meyer-Konig and Zeller operator. We also find the rate of convergence for function of bounded variation for these operators.

**Keywords and phrases:** Simultaneous approximation, Rate of convergence, Bounded variation.
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## 1 Introduction

The Meyer-Konig and Zeller operators [13] defined on interval $[0,1]$ are given as follows

$$P_n f(x) = \sum_{v=0}^{\infty} m_{n,v}(x) f\left(\frac{v}{n+v}\right), \quad x \in [0,1], \quad n \in N \tag{1.1}$$

where

$$m_{n,v} = \begin{pmatrix} n+v-1 \\ v \end{pmatrix} x^v (1-x)^n.$$

Some integral properties of operators (1.1) were discussed in [[3], [6], [8], [15]]. Here we discuss the other generalization of the Meyer-Konig Zeller operators.
In order to approximate Lebesgue integrable function on the interval $[0,1]$, we define the integral modification of the operators (1.1) as follows

$$M_{n,\alpha}(f,x) = \sum_{v=0}^{\infty} P_{n,v}^{(\alpha)}(x) \int_0^1 b_{n,v}(t) f(t) dt, \quad x \in [0,1] \tag{1.2}$$

for $\alpha \geq 1$, we have

$$P_{n,v}^{(\alpha)} = \left(\sum_{j=v}^{\infty} m_{n,j}\right)^{\alpha} - \left(\sum_{j=v+1}^{\infty} m_{n,j}\right)^{\alpha}$$

and

$$b_{n,v}(t) = \frac{(n+v)!}{v!(n-1)!} t^v (1-t)^{n-1}.$$

Operators (1.2) represent a new type of integral modification of Meyer-Konig and Zeller operatoprs. Other integral modifications of these operators were considered in [1], [2], [5],

[10] and [14]. For similar type of operators some approximation properties were studied in [4], [7], [9], [11] and [12].
For $\alpha = 1$, the operators (1.2) reduce to

$$M_{n,1}(f,x) = \sum_{v=0}^{\infty} m_{n,v}(x) \int_0^1 b_{n,v}(t) f(t) dt. \tag{1.3}$$

In the present paper, we study approximation properties of operators (1.2) for functions of bounded variation and also give an estimate of the rate of convergence of $M_{n,\alpha}(f,x)$. Last section deals with the estimation of some generalized Meyer-Konig-Zeller operators studied by [3], [15] and [16] etc. These operators are defined as

$$\widetilde{Q}_{n,\alpha}(f,x) \;=\; \sum_{v=0}^{\infty} \left( \frac{P_{n,v}^{(\alpha)}(x)}{\int_{\frac{v}{n+v}}^{\frac{(v+1)}{n+v+1}} dt} \right) \int_{\frac{v}{(n+v)}}^{\frac{(v+1)}{n+v+1}} f(t) dt = \int_0^1 f(t) L_{n,\alpha,2}(x,t) dt,$$

where $\alpha \leq 1$, $L_{n,\alpha,2}(x,t) = \sum_{v=0}^{\infty} \frac{P_{n,v}^{(\alpha)} \phi_v(t)}{\int_I \phi_v(u) du}$,
and $\phi_v$ is the characteristic of interval $\left[\frac{v}{n+v}, \frac{v+1}{n+v+1}\right]$ with respect to $I = [0,1]$ and $P_{n,v}^{(\alpha)}(x)$ are given in (1.2).

## 2 Basic Results

In order to prove our main result, we need the following lemmas.

**Lemma 2.1.** [16]. *For all $v, n \in N$, $x \in [0,1)$, we have*

$$m_{n,v}(x) = \frac{1}{\sqrt{2e}} \cdot \frac{1}{\sqrt{nx}},$$

*where the constant $\frac{1}{\sqrt{2e}}$ is the best possible.*

**Lemma 2.2.** *For $r \in N^0$, where $N^0$ is the set of non-negative integers, we define*

$$M_{n,1}((t-x)^r, x) = \sum_{v=0}^{\infty} m_{n,v}(x) \int_0^1 b_{n,v}(t)(t-x)^r dt,$$

*then*

$$M_{n,1}((t-x)^2, x) =\leq \frac{4x}{n-1} + \frac{2(1-x)^2}{(n-1)(n-2)}.$$

*Proof.* Taking $e_r(x) = x^r$, $r = 0,1,2,...$, the moments of operators (1.3) are given by

$$M_{n,1}(t^r, x) = \sum_{v=0}^{\infty} m_{n,v}(x) \frac{(n+v)!}{v!(n-1)!} B(v+r+1, n),$$

where $B$ being the Beta function and $M_{n,1}e^0 = e^0 = 1$.

$$M_{n,1}(t,x) \;=\; (1-x)^n \sum_{v=0}^{\infty} \binom{n+v-1}{v} x^v \frac{(v+1)}{(n+v+1)}$$

$$
\begin{aligned}
&\geq (1-x)^n \sum_{v=0}^{\infty} \binom{n+v-2}{v-1} x^v \frac{(v+1)}{v} \cdot \frac{(n+v-1)}{(n+v-1)} \\
&\geq (1-x)^n \sum_{v=0}^{\infty} \binom{n+v-1}{v} x^{v+1} \left[1-\frac{2}{n+1}\right] = \left[1-\frac{2}{n+1}\right] x,
\end{aligned}
$$

and

$$
\begin{aligned}
M_{n,1}(t^2,x) &= (1-x)^n \sum_{v=0}^{\infty} \binom{n+v-1}{v} x^v \frac{v(v-1)+4v+2}{(n+v+1)(n+v+2)} \\
&\leq \frac{(1-x)^n}{(n-1)!} \sum_{v=2}^{\infty} \frac{(n+v-3)!}{(v-2)!} x^v + 4\sum_{v=1}^{\infty} \frac{(n+v-3)!}{(v-1)!} x^v + 2\sum_{v=0}^{\infty} \frac{(n+v-3)!}{(v)!} x^v \\
&\leq (1-x)^n \left[\sum_{v=0}^{\infty} \binom{n+v-1}{v} x^{v+2} + \frac{4}{(n-1)} \sum_{v=0}^{\infty} \binom{n+v-1}{v} x^{v+1} \right. \\
&\quad \left. + \frac{2}{(n-1)(n-2)} \binom{n+v-1}{v} x^v \right] \\
&= x^2 + \frac{4x(1-x)}{(n-1)} + \frac{2(1-x)^2}{(n-1)(n-2)}.
\end{aligned}
$$

Combining these estimates, we have

$$
\begin{aligned}
M_{n,1}((t-x)^2,x) &= M_{n,1}(t^2,x) - 2xM_{n,1}(t,x) \\
&\leq \frac{4x}{(n-1)} + \frac{2(1-x)^2}{(n-2)(n-1)}.
\end{aligned}
$$

Particularly, for given $\mu > 4$ and $x \in (0,1)$, there exists an integer $N(\mu,x)$ such that $\forall$ $n \geq N(\mu,x)$

$$
M_{n,1}((t-x)^2,x) \leq \frac{\mu x}{n}.
$$

□

**Lemma 2.3.** *For all $x \in (0,1]$ and $v$ be a natural number, there holds*

$$
P_{n,v}^{(\alpha)}(x) \leq \alpha m_{n,v}(x) < \frac{\alpha}{\sqrt{2}enx}.
$$

*Proof.* Since $|a^\alpha - b^\alpha| \leq \alpha|a-b|$, $(0 \leq a,b \leq 1, \alpha \geq 1)$. Then by Lemma 1, we have

$$
P_{n,v}^{(\alpha)}(x) \leq \alpha m_{n,v}(x) < \frac{\alpha}{\sqrt{2}enx}.
$$

□

**Lemma 2.4.** *Let $L_{n,\alpha}(x,t) = \sum_{v=0}^{\infty} P_{n,v}^{(\alpha)}(x) b_{n,v}(t)$ and $\mu > 4$, $n \geq N(\mu,x)$, then*

$$
(i)\mu_{n,\alpha}(x,y) = \int_0^y L_{n,\alpha}(x,t)dt \leq \frac{\alpha\mu x}{n(x-y)^2}, \quad 0 \leq y < x.
$$

$$
(ii)1-\mu_{n,\alpha}(x,y) = \int_z^1 L_{n,\alpha}(x,t)dt \leq \frac{\alpha\mu x}{n(z-x)^2}, \quad x < z \leq 1.
$$

*Proof.* To prove (i),

$$\begin{aligned}\int_0^y L_{n,\alpha}(x,t)dt &\leq \int_0^y L_{n,\alpha}(x,t)\frac{(x-t)^2}{(x-y)^2}dt\\ &\leq \frac{1}{(x-y)^2}M_{n,\alpha}((t-x)^2,x) \leq \frac{\alpha M_{n,1}(t-x)^2,x)}{(x-y)^2} \leq \frac{\alpha\mu x}{n(x-y)^2}.\end{aligned}$$

Proof of $(ii)$ is similar as Lemma 2. □

**Lemma 2.5.** [15] *For $x \in (0,1)$, we have*

$$\left|\sum_{\frac{nx}{1-x}<v} m_{n,v}(x) - \frac{1}{2}\right| \leq \frac{5}{2\sqrt{nx}}.$$

## 3 Main Results

Following section deals with the main theorems

**Theorem 3.1.** *Consider a function $f$ of bounded variation on $[0,1]$, $\alpha \geq 1$. Then for every $x \in [0,1)$ and $\mu > 4$ and $n \geq \max N(\mu,x), 3$, we have*

$$\begin{aligned}\left|M_{n,\alpha}(f,x) - \left[\frac{1}{\alpha+1}f(x+) - \frac{\alpha}{\alpha+1}f(x-)\right]\right| &\leq \frac{1}{2}\left[\frac{\alpha^2+\alpha-2}{\alpha+1} + \frac{\alpha}{\sqrt{2e\sqrt{nx}}}\right] + |f(x+)-f(x-)|\\ &+\frac{(2\mu\alpha+x)}{nx}\sum_{v=1}^{n} V_{x-\frac{x}{\sqrt{v}}}^{x+\frac{(1-x)}{\sqrt{v}}} h_x(t),\end{aligned}$$

*where*

$$h_x(t) = \begin{cases} f(t)-f(x-), & t \in [0,x) \\ 0, & t{=}x \\ f(t)-f(x+), & t \in (x,1] \end{cases}$$

*$V_a^b(h_x)$ being the total variation of $h_x$ on $[a,b]$.*

*Proof.* Since

$$\begin{aligned}\left|M_{n,\alpha}(f,x) - \left[\frac{1}{\alpha+1}f(x+) + \frac{\alpha}{\alpha+1}f(x-)\right]\right| &\leq \left|M_{n,\alpha}(sgn(t-x),x+\left(\frac{\alpha-1}{\alpha+1}\right)\right|\frac{|f(x+)-f(x-)|}{2}\\ &+|M_{n,\alpha}(h_x,x)|,\end{aligned} \tag{3.1}$$

we have,

$$\begin{aligned}M_{n,\alpha}(sgn(t-x),x) &= \sum_{v=0}^{\infty} P_{n,v}^{(\alpha)}(x)\left(\int_x^1 b_{n,v}(t)dt - \int_0^x b_{n,v}(t)dt\right)\\ &= \sum_{v=0}^{\infty} P_{n,v}^{(\alpha)}(x)\left(\int_0^1 b_{n,v}(t)dt - 2\int_0^x b_{n,v}(t)dt\right)\end{aligned}$$

$$= \quad 1 - 2\sum_{v=0}^{\infty} P_{n,v}^{(\alpha)}(x)\int_0^x b_{n,v}(t)dt.$$

Applying Lemma 1, Lemma 3 and the fact $\sum_{j=0}^{v} m_{n,j}(x) = \int_x^1 b_{n,v}(t)dt$, we have

$$\begin{aligned}
M_{n,\alpha}(sgn(t-x),x) &= 1 - 2\sum_{v=0}^{\infty} P_{n,v}^{(\alpha)}(x)\left(1 - 2\sum_{j=0}^{v} m_{n,j}(x)\right) \\
&= -1 + 2\sum_{v=0}^{\infty} P_{n,v}^{(\alpha)}(x)\left(\sum_{j=0}^{v} m_{n,j}(x)\right) \leq -1 + 2\alpha\sum_{v=0}^{\infty} m_{n,v}(x)\sum_{j=0}^{v} m_{n,j}(x) \\
&= -1 + \alpha + \alpha\left[\sum_{v=0}^{\infty} m_{n,v}(x)\sum_{j=0}^{v} m_{n,j}(x) - \sum_{v=0}^{\infty}(m_{n,v}(x))^2\right] \\
&\leq \alpha - 1 + \alpha m_{n,v}(x)\sum_{v=0}^{\infty} m_{n,v}(x) \leq \alpha - 1 + \frac{\alpha}{\sqrt{2e\sqrt{nx}}}.
\end{aligned}$$

Hence

$$\left|M_{n,\alpha}(sgn(t-x), x + \left(\frac{\alpha-1}{\alpha+1}\right)\right| \leq \frac{\alpha^2+\alpha-2}{\alpha+1} + \frac{\alpha}{\sqrt{2e\sqrt{nx}}}. \tag{3.2}$$

Now to estimate $M_{n,\alpha}(h_x, x)$, applying the Lebesgue-Stieltjes integral representation, we get

$$\begin{aligned}
M_{n,\alpha}(h_x,x) &= \int_0^1 L_{n,\alpha}(x,t)h_x(t)dt \\
&= \left(\int_0^{x-\frac{x}{\sqrt{n}}} L_{n,\alpha}(x,t)dth_x(t)dt + \int_{x-\frac{x}{\sqrt{n}}}^{x+\frac{(1-x)}{\sqrt{n}}} L_{n,\alpha}(x,t)dth_x(t)dt\right. \\
&\quad \left. + \int_{x+\frac{(1-x)}{\sqrt{n}}}^{1} L_{n,\alpha}(x,t)dth_x(t)dt\right) \\
&= : J_1 + J_2 + J_3.
\end{aligned}$$

To estimate $J_1$, putting $x - \frac{x}{\sqrt{n}} = y$ and using Lebesgue-Stieltjes integral with $\mu_{n,\alpha}(x,t) = \int_0^t L_{n,\alpha}(x,z)dz$, we get

$$J_1 = \int_0^y h_x(t)dt(\mu_{n,\alpha}(x,t)) = h_x(y+)\mu_{n,\alpha}(x,y) - \int_0^y h_x(t)dt(\mu_{n,\alpha}(x,t)).$$

Since $|h_x(y+)| \leq V_{y+}^x(h_x)$, we follow that

$$|J_1| \leq V_{y+}^x(h_x)\mu_{n,\alpha}(x,y) + \int_0^y \mu_{n,\alpha}(x,t)dt - (V_t^x(h_x)).$$

Applying $(i)$ of Lemma 4, we get

$$|J_1| \leq V_{y+}^x(h_x)\frac{\alpha\mu x}{n(x-y)^2} + \frac{\alpha\mu x}{n}\int_0^y \frac{1}{(x-t)^2}dt(-V_t^x(h_x)).$$

Integrating by parts and by simple computation, we get

$$|J_1| \leq \frac{\alpha\mu x}{n}\left[\frac{V_0^x(h_x)}{x^2} + 2\int_0^y \frac{V_t^x(h_x)}{(x-t)^3}dt\right].$$

Changing the variable $y$ as $\left(x - \frac{x}{\sqrt{n}}\right)$, we have

$$|J_1| \leq \frac{\alpha\mu}{nx}\left[V_0^x(h_x) + \sum_{v=1}^{n} V^x_{x-\frac{x}{\sqrt{v}}}\right] \leq \frac{2\alpha\mu}{nx} V^x_{x-\frac{x}{\sqrt{v}}}(h_x). \tag{3.3}$$

According to Lemma 4, we have

$$|J_3| \leq \frac{2\mu\alpha}{nx}\sum_{v=1}^{n} V_x^{x+\frac{(1-x)}{\sqrt{v}}}(h_x). \tag{3.4}$$

Now to estimate $J_2$, for $t \in \left[x - \frac{x}{\sqrt{n}}, x + \frac{(1-x)}{\sqrt{n}}\right]$, we have

$$|h_x(t)| = |h_x(t) - h_x(x)| \leq V_{x-\frac{x}{\sqrt{v}}}^{x+\frac{(1-x)}{\sqrt{v}}}(h_x),$$

hence

$$|J_2| \leq V_{x-\frac{x}{\sqrt{v}}}^{x+\frac{(1-x)}{\sqrt{v}}}(h_x)\int_{x-\frac{x}{\sqrt{v}}}^{x-\frac{(1-x)}{\sqrt{v}}} dt(\mu_{n,\alpha}(x,t).$$

Since $\int_a^b \mu_n(x,t)dt \leq 1$ for $(a,b) \subset [0,1]$, hence

$$|J_2| \leq V_{x-\frac{x}{\sqrt{v}}}^{x+\frac{(1-x)}{\sqrt{v}}}(h_x) \leq \frac{1}{n}\sum_{v=1}^{n} V_{x-\frac{x}{\sqrt{v}}}^{x+\frac{(1-x)}{\sqrt{v}}}(h_x). \tag{3.5}$$

Collecting the estimates from (3.2) to (3.5), we get

$$|M_{n,\alpha}(h_x, x)| \leq \frac{(2\mu\alpha + x)}{nx}\sum_{v=1}^{n} V_{x-\frac{x}{\sqrt{v}}}^{x+\frac{(1-x)}{\sqrt{v}}}(h_x). \tag{3.6}$$

Combining (3.1),(3.2) and (3.6), we get the proof of the theorem. □

**Theorem 3.2.** *Let* $n > 2$, $\omega_n = \frac{4}{(n-1)\delta^2}\left(1 + \frac{1-\delta}{12(n-2)\delta}\right)$, *where* $0 < \delta \leq x \leq 1-\delta$, *then*
*(i) For* $0 < y < x$, *there holds*

$$\int_0^y L_{n,\alpha,2}(x,t)dt \leq \frac{\alpha\omega_n x^3(1-x)}{(x-y)^2}. \tag{3.7}$$

*(ii) For* $x < y < 1$, *there holds*

$$\int_z^1 L_{n,\alpha,2}(x,t)dt \leq \frac{\alpha\omega_n x(1-x)^3}{(x-y)^2}. \tag{3.8}$$

*Proof.* According to [10] for $x \in (0,1)$, $n > 2$, we get

$$\widetilde{Q}_{n,1}((t-x)^2,x) \leq \frac{4x(1-x)}{(n-1)} + \frac{1}{3}\frac{(1-x)^2}{(n-1)(n-2)}. \tag{3.9}$$

Using (3.9)

$$\begin{aligned}
\int_0^y L_{n,\alpha,2}(x,t)dt &\leq \int_0^y L_{n,\alpha,2}(x,t)\frac{(x-t)^2}{(x-y)^2}dt \leq \frac{\alpha}{(x-y)^2}\widetilde{Q}_{n,\alpha}((t-x)^2,x) \\
&\leq \frac{\alpha}{(x-y)^2}\left[\frac{4x(1-x)}{(n-1)} + \frac{(1-x)^2}{3(n-1)(n-2)}\right] \\
&\leq \frac{4\alpha x^3(1-x)}{\delta^2(n-1)(x-y)^2}\left[1 + \frac{\frac{1}{\delta}-1}{12(n-2)}\right] \\
&\leq \frac{\alpha w_n x^3(1-x)}{(x-y)^2},
\end{aligned}$$

which is the proof of (3.7). Proof of (3.8) is similar. □

**Theorem 3.3.** *Suppose $f$ be a function of bounded variation on the interval $[0,1)$, then $\forall x \in (0,1)$, $n > 2$ and $0 < \delta \leq (1-x)$, we have*

$$\begin{aligned}
\left|\widetilde{Q}_{n,\alpha}(f,x) - \frac{1}{2^\alpha}f(x+) - \left(1-\frac{1}{2^\alpha}\right)f(x-)\right| &\leq \left(\frac{5}{2} + \frac{1}{\sqrt{2e}}\right)\frac{\alpha}{\sqrt{nx}}|f(x+) - f(x-)| \\
&\quad + \left(2\alpha\omega_n x(1-x) + \frac{1}{n-1}\right)\sum_{v=1}^{n} V_{x-\frac{x}{\sqrt{v}}}^{x+\frac{(1-x)}{\sqrt{v}}}(h_x).
\end{aligned}$$

*where $h_x(t)$ and $V_a^b(h_x)$ are defined in theorem 1 and $\omega_n$ is given in theorem 2.*

*Proof.* Now

$$\begin{aligned}
\left|\widetilde{Q}_{n,\alpha}(f,x) - \frac{1}{2^\alpha}f(x+) - \left(1-\frac{1}{2^\alpha}\right)f(x-)\right| &\leq \left|\widetilde{Q}_{n,\alpha}(h_x,x)\right| \\
&\quad + \left|\frac{f(x+)-f(x-)}{2^\alpha} + \widetilde{Q}_{n,\alpha}(sgn(t-x),x)\right. \\
&\quad \left. + \left[f(x+) - \frac{1}{2^\alpha}f(x+) - \left(1-\frac{1}{2^\alpha}\right)f(x-)\right]\widetilde{Q}_{n,\alpha}(\delta_x,x)\right|.
\end{aligned}$$

According to [15] $\widetilde{Q}_{n,\alpha}(\delta_x,x) = 0$, and

$$\widetilde{Q}_{n,\alpha}(sgn(t-x),x) \leq \alpha 2^\alpha\left|\sum_{\frac{nx}{1-x}<v} m_{n,v}(x) - \frac{1}{2}\right| + 2^\alpha P_{n,v'}^{(\alpha)}(x), \ \ where \ x \in \left[\frac{v'}{n+v'}, \frac{v'+1}{n+v'+1}\right]$$

By using Lemma 3 and Lemma 5, we have,

$$\left|\widetilde{Q}_{n,\alpha}(sgn(t-x),x)\right| \leq \left(\frac{5}{2} + \frac{1}{\sqrt{2e}}\right)\frac{2^\alpha\alpha}{\sqrt{nx}}.$$

Now to estimate $\widetilde{Q}_{n,\alpha}(h_x,x)$, we follow

$$\begin{aligned}\widetilde{Q}_{n,\alpha}(h_x,x) &= \int_0^1 L_{n,\alpha,2}(x,t)h_x(t)dt \\ &= \left(\int_0^{x-\frac{x}{\sqrt{n}}} + \int_{x-\frac{x}{\sqrt{n}}}^{x+\frac{(1-x)}{\sqrt{n}}} + \int_{x+\frac{(1-x)}{\sqrt{n}}}^1\right) L_{n,\alpha,2}(x,t)h_x(t)dt \\ &= : E_1 + E_2 + E_3.\end{aligned}$$

Proofs of $E_1$ $E_2$, $E_3$ are according to similar lines of [15]. We have

$$|E_2| \leq V_{x-\frac{x}{\sqrt{n}}}^{x+\frac{(1-x)}{\sqrt{n}}}(h_x) \leq \frac{1}{n-1}\sum_{v=1}^{n} V_{x-\frac{x}{\sqrt{v}}}^{x+\frac{(1-x)}{\sqrt{v}}}(h_x).$$

Taking $y = x - \frac{x}{\sqrt{v}}$ and using integration by parts with $\gamma_{n,\alpha}(x,t) = \int_0^t L_{n,\alpha,2}(x,u)du$, we have

$$\begin{aligned}E_1 &= \int_0^y h_x(t)dt(\gamma_{n,\alpha}(x,t)) = h_x(y+)\gamma_{n,\alpha}(x,y) - \int_0^y \gamma_{n,\alpha}(x,t)dt(h_x(t)) \\ &\leq V_{y+}^x(h_x)\gamma_{n,\alpha}(x,y) + \int_0^y \gamma_{n,\alpha}dt\left(-V_t^x(h_x)\right).\end{aligned}$$

According to (3.7) of theorem 2, we get

$$E_1 \leq V_{y+}^x(h_x)\frac{\alpha\omega_n x^3(1-x)}{(x-y)^2} + \alpha w_n x^3(1-x)\int_0^y \frac{1}{(x-t)^2}dt\left(-V_t^x(h_x)\right).$$

Integrating by parts, we get,

$$|E_1| \leq \alpha\omega_n x^3(1-x)\left[\frac{V_0^x(h_x)}{x^2} + 2\int_0^y \frac{-V_t^x(h_x)}{(x-t)^3}dt\right].$$

Putting $y = \left(x - \frac{x}{\sqrt{u}}\right)$ in the last integral, we have

$$\begin{aligned}|E_1| &\leq \frac{1}{x^2}\alpha\omega_n x^3(1-x)\left[V_0^x(h_x) + \sum_{v=1}^{n} V_{x-\frac{x}{\sqrt{v}}}^x\right] \\ &\leq 2\alpha\omega_n x(1-x)\sum_{v=1}^{n} V_{x-\frac{x}{\sqrt{v}}}^x(h_x).\end{aligned}$$

Similarly, we get

$$|E_3| \leq 2\alpha\omega_n x(1-x)\sum_{v=1}^{n} V_{x-\frac{x}{\sqrt{n}}}^x(h_x).$$

Combining the estimates of $E_1$, $E_2$, $E_3$, we get the required theorem. □

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