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# A Basic Analogue of the Bessel-Clifford Equation

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**Summary.** A basic analogue of the Bessel-Clifford equation is studied, and a fundamental system of series solutions, valid for non-exceptional values of the parameter, is obtained. Recurrence relations for these solutions and a generating function are given. An orthogonality property is established and an expansion formula is indicated.

**1. Introduction** The usual system of numbers was extended to the so-called basic numbers by Heine [3] and then developed by Jackson [4], [5] and by other investigators. The basic analogue of the number $a$ is given by

$$[a]=(1-q^a)/(1-q), \tag{1.1}$$

where $q$ is any number, real or complex, called the base. Jackson [4] also introduced the operative symbol $\triangle$ defined by

$$\triangle\{\psi(x)\}=\frac{\psi(qx)-\psi(x)}{x(q-1)}, \tag{1.2}$$

which becomes the same as ordinary differentiation in the limit as $q$ tends to unity. Similarly, he defines basic integration as the inverse of basic differentiation employing the symbol $\underset{a}{\overset{b}{S}}$, which reduces in the limit $q\to 1$ to $\int_a^b$. These operations correspond exactly, in every way, to differentiation and integration.

We have the following elementary results :

$$\triangle x^n=[n]\, x^{n-1} \text{ and } \triangle E_q\,(ax)=aE_q(ax)$$

where $$E_q(x)=\sum_{m=0}^{\infty}\frac{x^m}{[m]!},\ [m]!=[1]\,[2]...[m], \tag{1.3}$$

and which corresponds to the exponential function. The basic analogue of the differentiation of a product is used later and is

$$\Delta\{u(x)\,v(x)\}=v(qx)\Delta u(x)+u(x)\Delta v(x). \tag{1.4}$$

It is assumed in what follows that all quantities and functions are real and that $0<q\leqslant 1$ unless otherwise stated.

**2. A Basic Analogue of the Bessel-Clifford Equation.** Consider the $q$-difference equation

$$q^{a+1}x\Delta^2 y+[a+1]\ \Delta y+y(qx)=0. \tag{2.1}$$

It is clear that as $q\to 1$, this equation reduces to the Bessel-Clifford equation

$$xy''+(a+1)y'+y=0. \tag{2.2}$$

See Hayek [2] for a detailed study of this latter equation. We also note that a parameter $h$ may be introduced into (2.1) by replacing $x$ by $hx$, when we have

$$q^{a+1}x\Delta^2 y+[a+1]\ \Delta y+hy(qx)=0. \tag{2.3}$$

We attempt a series solution of (2.1) in the form

$$y=\sum_{r=0}^{\infty} b_r\, x^{r+e},\ \ b_0\neq 0. \tag{2.4}$$

The indicial equation is

$$\{q^{a+1}\,[e-1]+[a+1]\}\,[e]=0 \tag{2.5}$$

which may be re-written in the form

$$[a+e]\,[e]=0, \tag{2.6}$$

and so the exponents are $e=0$ and $e=-a$.

It is easily seen that the coefficients $b_r$ corresponding to $e=0$ are given by

$$b_{r-1}\,q^{r-1}=[r]\,[a+r]\,b_r,$$

and if $b_0=1$, the resulting solution of (2.1) is

$$y_1=\sum_{r=0}^{\infty}\frac{(-1)^r\,q^{\frac{1}{2}r(r-1)}x^r}{[a+1]_r\,[r]\,!}, \tag{2.7}$$

provided that $a$ is not a negative integer.

The symbol $[a]_r$ denotes the basic Pochhammer product

$$[a]\,[a+1]...[a+r-1]=\Gamma_q(a+r)/\Gamma_q(a), \tag{2.8}$$

see Jackson [4] for a discussion of the basic gamma function $\Gamma_q(x)$. We take the standard solution of the basic analogue of the Bessel-Clifford equation under discussion to be

$$C_a(q\,;x)=\frac{y_1}{\Gamma_q(a+1)}=\frac{1}{\Gamma_q(a+1)}\sum_{r=0}^{\infty}\frac{(-1)^r\,q^{\frac{1}{2}r(r-1)}x^r}{[a+1]_r\,[r]\,!}. \tag{2.9}$$

It follows that a second independent solution of (2.1) is

$$x^{-a}C_{-a}(q\,;q^{-a}\,x), \tag{2.10}$$

unless $a$ is an integer, positive, negative or zero.

**3. Recurrence Formulae for the Function $\mathbf{C}_a$ (q ; x).** We may write the series representation of $C_a(q\,;x)$ in the form

$$C_a(q\,;x)=\sum_{r=0}^{\infty}\frac{(-1)^r\,q^{\frac{1}{2}r(r-1)}x^r}{\Gamma_q(a+1+r)\,[r]\,!}, \tag{3.1}$$

and $q$-differentiation term-by-term gives

$$\begin{aligned}\triangle C_a(q\,;x)&=\sum_{r=1}^{\infty}\frac{(-1)^r\,q^{\frac{1}{2}r(r-1)}x^{r-1}}{\Gamma_q(a+1+r)\,[r-1]\,!}\\&=-\sum_{r=0}^{\infty}\frac{(-1)^r\,q^{\frac{1}{2}r(r-1)}\,(qx)^r}{\Gamma_q(a+2+r)\,[r]\,!}\\&=-C_{a+1}(q\,;qx).\end{aligned} \tag{3.2}$$

This operation may be repeated several times, and we have the result

$$\triangle^n C_a(q\,;x)=(-1)^n\,C_{a+n}(q\,;q^n x). \tag{3.3}$$

If we now utilise the original $q$-difference equation (2.1), we obtain the expression

$$q^{a+1}\,x\,C_{a+2}(q\,;q^2x)-[a+1]\,C_{a+1}(q\,;qx)+C_a(q\,;qx)=0, \tag{3.4}$$

or, on replacing $qx$ by $x$

$$q^a x C_{a+2}(q\,;qx)-[a+1]\,C_{a+1}(q\,;x)+C_a(q\,;x)=0. \tag{3.5}$$

Again starting from the series rep.esentation of $C_a(q\,;x)$, (3.1), we have

$$\begin{aligned}\triangle\{x^{a+1}C_{a+1}(q\,;x)\}&=\sum_{r=0}^{\infty}\frac{(-1)^r\,q^{\frac{1}{2}r(r-1)}x^{a+r}}{\Gamma_q(a+1+r)\,[r]\,!}\\&=x^aC_a(q\,;x),\end{aligned} \tag{3.6}$$

or, after several applications of this process

$$\triangle^n\,\{x^{a+n}C_{a+n}(q\,;x)\}=x^aC_a. \tag{3.7}$$

Also,

$$\triangle^{n+1}\,\{x^{a+1}C_{a+1}(q\,;x)\}=x^{a-n}C_{a-n}(q\,;x), \tag{3 8}$$

and if we write (3.5) in the form

$$C_{a-1}(q\,;\,x)-[a]\,C_a(q\,;\,x)+q^a x\,C_{a+1}(q\,;\,qx)=0, \tag{3.9}$$

we have

$$C_{a-1}(q\,;\,x)-[a]\,C_a(q\,;\,x)-q^a\triangle C_a(q\,;\,x)=0, \tag{3.10}$$

because

$$C_{a+1}(q\,;\,qx)=-\triangle C_a(q\,;\,x). \tag{3.11}$$

If we add and subtract the result (3.11) to and from (3.10) in turn, we have the formulae

$$\begin{aligned} C_{a-1}(q\,;\,x)-C_{a+1}(q\,;\,qx) \\ =q^a(x+1)\triangle C_a(q\,;\,x)+[a]\,C_a(q\,;\,x) \end{aligned} \tag{3.12}$$

and

$$\begin{aligned} C_{a-1}(q\,;\,x)+C_{a+1}(q\,;\,qx) \\ =q^a(x-1)\triangle C_a(q\,;\,x)+[a]\,C_a(q\,;\,x). \end{aligned} \tag{3.13}$$

**4. A Generating Function of the Basic Bessel-Clifford Coefficient.** If the base $q$ is inverted in the series for the basic exponential function (1.3), we have the alternative form

$$E_{\frac{1}{q}}(x)=\sum_{r=0}^{\infty}\frac{q^{\frac{1}{2}r(r-1)}x^r}{[r]!}, \tag{4.1}$$

see Jackson [4].

Consider the product

$$E_q(t)\,E_{\frac{1}{q}}\left(-\frac{x}{t}\right)=\sum_{r=0}^{\infty}\frac{(-1)^r q^{\frac{1}{2}r(r-1)}}{[r]!}\left(\frac{x}{t}\right)^r\sum_{j=0}^{\infty}\frac{t^j}{[j]!}. \tag{4.2}$$

This may be re-arranged as

$$\sum_{r=0}^{\infty}\frac{(-1)^r q^{\frac{1}{2}r(r-1)}}{[r]!}x^r\sum_{j=0}^{\infty}\frac{t^{j-r}}{[j]!}$$

$$=\sum_{r=0}^{\infty}\frac{(-1)^r q^{\frac{1}{2}r(r-1)}}{[r]!}x^r\sum_{n=-r}^{\infty}\frac{t^n}{[n+r]!}$$

$$=\sum_{n=-\infty}^{\infty}t^n\sum_{r=0}^{\infty}\frac{(-1)^r q^{\frac{1}{2}r(r-1)}x^r}{[r]!\,[n+r]!}$$

$$=\sum_{n=-\infty}^{\infty}t^n\,C_n(q\,;\,x). \tag{4.3}$$

The function $C_n(q\,;x)$ is referred to as the basic Bessel-Clifford coefficient. This generating function enables us to define the basic Bessel-Clifford coefficient of negative order :

$$C_{-n}(q\,;x)=\sum_{r=0}^{\infty}\frac{(-x)^{n+r}\,q^{-\frac{1}{2}r(r-1)}}{[n+r]\,!\,[n]\,!}, \tag{4.4}$$

and so

$$E_q(t)\,E_{\frac{1}{q}}\left(-\frac{x}{t}\right)=C_0(q\,;x)$$
$$+\sum_{n=1}^{\infty}\left\{t^n+(-x)^n\,q^{-n\,(n-1)}\right\}C_n(q\,;x). \tag{4.5}$$

## 5. The Orthogonality of the Basic Bessel-Clifford Function.

Before discussing the orthogonality of the solutions of the basic Bessel-Clifford equation, we first give the following lemma :

LEMMA. Suppose that the base $q$ is real and such that $0<q\leqslant 1$, and that the real functions $r(x)$, $l(x)$ and $w(x)$ possess the appropriate number of $q$ derivatives on the interval $a\leqslant x\leqslant b$, and let $y_m(qx)$ and $y_n(qx)$ be eigen-functions corresponding to distinct eigen-values $\lambda_m$, $\lambda_n$ of the boundary-value system

$$\begin{aligned} &\Delta\{r\Delta y(x)\}+(l+\lambda w)\,y(qx)=0,\\ &h_1 y+h_2\Delta y=0 \qquad \text{at } x=a\\ \text{and}\quad &k_1 y+k_2\Delta y=0 \qquad \text{at } x=b, \end{aligned} \tag{5.1}$$

$h_1$, $h_2$, $k_1$ and $k_2$ being constants.

Then $y_m(qx)$ and $y_n(qx)$ are $q$-orthognal on the interval $a\leqslant x\leqslant b$ with respect to the weight function $w(x)$, that is

$$\underset{a}{\overset{b}{S}}\,w(x)\,y_m(qx)\,y_n(qx)\,d(qx)=0, \qquad m\neq n. \tag{5.2}$$

*Proof.* The functions $y_m(x)$ and $y_n(x)$ satisfy the equations

$$\Delta\{r\Delta y_m\}+(l+\lambda_m w)\,y_m(qx)=0 \tag{5.3}$$

and

$$\Delta\{r\Delta y_n\}+(l+\lambda_n w)\,y_n(qx)=0 \tag{5.4}$$

respectively. Multiply (2.3) and (2.4) by $y_n(qx)$ and $-y_m(qx)$ and add :

$$(\lambda_m-\lambda_n)\,w(x)\,y_m(qx)\,y_n(qx)$$
$$=y_m(qx)\Delta\{r\Delta y_n\}-y_n(qx)\Delta\{r\Delta y_m\}. \tag{5.5}$$

Consider the expression

$$\triangle\{r\triangle(y_n)y_m - r\triangle(y_m)y_n\}, \tag{5.6}$$

which on expansion by means of (1.4) becomes

$$\begin{aligned} &y_m(qx)\triangle(r\triangle y_n\} + r\triangle(y_n)\triangle(y_m) \\ &-y_n(qx)\triangle\{r\triangle y_m\} - r\triangle(y_m)\triangle(y_n). \end{aligned} \tag{5.7}$$

This is identical with the right-hand member of (5.5), so that

$$(\lambda_m - \lambda_n)w(x)y_m(qx)y_n(qx) = \triangle\{r\triangle(y_n)y_m - r\triangle(y_m)y_n\}. \tag{5.8}$$

If we $q$-integrate with respect to $x$ between the limits $a$ and $b$, the result

$$(\lambda_m - \lambda_n) \underset{a}{\overset{b}{S}} w(x)\ y_m(qx)\ y_n(qx)\ d(qx)$$

$$= \Big| r\triangle(y_n)y_m - r\triangle(y_m)y_n \Big|_a^b \tag{5.9}$$

is obtained.

The right-hand member of (5.9) is interpreted in the form

$$\begin{aligned} &r(a)\ \{\triangle y_n\}_a\ y_m(a) - r(a)\ \{\triangle y_m\}_a\ y_n(a) \\ &-r(b)\ \{\triangle y_n\}_b\ y_m(b) + r(b)\ \{\triangle y_m\}_b\ y_n(b), \end{aligned} \tag{5.10}$$

which clearly vanishes as a consequence of the boundary conditions. The lemma is thus established.

If $r(a)=0$ or $r(b)=0$, then either the first or the second boundary condition may be dropped. In particular, if $r(a)$ and $r(b)$ both vanish, we have the interesting case that the property of $q$-orthogonality holds without the imposition of any extrinsic boundary conditions. See [1], for example.

This lemma is now applied to the basic Bessel-Clifford equation in the form (2.3), where the parameter $h$ has been introduced. On multiplication by $x^a$, this equation may be written in the form

$$\triangle\{x^{a+1}\triangle y\} + hx^a\ y(qx) = 0, \tag{5.11}$$

and if the solution is $y = C_a\ (q\ ;\ hx)$, we have

$$\triangle\{x^{a+1}\triangle C_a\ (q\ ;\ hx)\} + hx^a C_a\ (q\ ;\ hqx) = 0. \tag{5.12}$$

Equation (5.12) is of the same form as that discussed in the above lemma, so that, if $h_1, h_2, h_3, \ldots$ are the positive roots of the equation $C_a\ (q\ ;\ hc) = 0$, for $a > -1$, then, since the function $x^{a+1}$ vanishes for $x = 0$, we have

$$\underset{0}{\overset{c}{S}}\ x^a\ C_a\ (q\ ;\ h_i qx)\ C_a\ (q\ ;\ h_j qx)\ d(qx) = 0, \tag{5.13}$$

provided that $i \neq j$. The required basic orthogonality property is thus established.

If it is assumed that an arbitrary function $f(x)$ may be expanded as a series of basic Bessel-Clifford functions in the form

$$f(x)=\sum_{i=0}^{\infty} A_i\, C_a\,(q\,;\,h_i qx), \tag{5.14}$$

it follows that $A_i$ is given by

$$A_i=\frac{\underset{0}{\overset{c}{S}}\, x^a\, C_a\,(q\,;\,h_i qx)\, f(x)\, d(qx)}{\underset{0}{\overset{c}{S}}\, x^a\, \{C_a\,(q\,;\,h_i qx)\}^2\, d(qx)}. \tag{5.15}$$

In order to investigate the basic integral

$$\underset{0}{\overset{c}{S}}\, x^a\, \{C_a\,(q\,;\,h_i qx)\}^2\, d(qx), \tag{5.16}$$

we consider the quantity

$$\frac{C^{a+1}}{h_i-h_j}\{-h_j C_a\,(q\,;\,h_i c)\, C_{a+1}\,(q\,;\,h_j c) + h_i C_a\,(q\,;\,h_j c)\, C_{a+1}\,(q\,;\,h_j c)\} \tag{5.17}$$

obtained from the right-hand member of (5.9) divided by $h_i-h_j$. Take the limit as $h_j \to h_i$ using de l' Hospital's rule. If we differentiate the numerator and denominator with respect to $h_j$, the denominator becomes equal to $-1$, and the numerator takes the form

$$C^{a+1}\,\{-C_a\,(q\,;\,h_i c)\, C_{a+1}\,(q\,;\,h_j c) - h_j\, C_a\,(q\,;\,h_i c)\, \frac{d}{dh_j}\, C_{a+1}\,(q\,;\,h_j c) + h_i\, C_{a+1}\,(q\,;\,h_i c)\, \frac{d}{dh_j}\, C_a\,(q\,;\,h_j c)\} \tag{5.18}$$

The required limit may be written as

$$\underset{0}{\overset{c}{S}}\, x^a\, \{C_a\,(q\,;\,h_i qx)\}^2\, d(qx) = -c^{a+1}\, h_i\, C_{a+1}\,(q\,;\,h_i c)\, \frac{d}{dh_i}\, C_a\,(q\,;\,h_i c) \tag{5.19}$$

noting that $C_a\,(q\,;\,h_i c)=0$.

All the results in this paper relating to the basic analogue of the Bessel-Clifford equation reduce to formulae obtained by Hayek [2] when $q \to 1$.

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