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An Integral Relation involving Spheroidal Functions and the H-function of Several Complex Variables

by

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ABSTRACT

In the present note we first evaluate an integral required to establish an integral relation involving the spheroidal function and the H-function of several complex variables (introduced by H. M. Srivastava and R. Panda [10. p. 271]) in the subsequent section. By suitably specializing the various parameters involved, this integral relation would yield the corresponding integral relations involving the H-function of two complex variables [8, p. 117], the Bessel function, the H-function of C. Fox [6, p. 408] and their several specializations available in the literature. However, we have not recorded them here for lack of space.

It is shown by an example how the main result (3.1) can be utilized to yield a large number of interesting integrals.

1. Introduction and Notations

The H function of several variables. Recently, Srivastava and Panda (cf. [10], p. 271, Eq. (4.1) et seq. ; see also [11] ; p. 130. Eq. (1.3)) have extended the H-function of the variables of Mittal and Gupta [8, p. 117] to several complex variables which, for ready reference, is defined and represented by the multiple integral :

$$H_{A,C:\,[B',D'];\,\ldots;\,[B^{(n)},D^{(n)}]}^{0,\lambda:\,(u',v');\,\ldots;\,(u^{(n)},v^{(n)})}\left(\begin{matrix}[(a):\theta',\ldots,\theta^{(n)}]:[(b'):\phi'];\ldots;[(b^{(n)}):\phi^{(n)}];\\ [(c):\psi',\ldots.\psi^{(n)}]:[(d'):\delta'];\ldots;[(d^{(n)}):\delta^{(n)}];\end{matrix}\; x_1,\ldots,x_n\right)$$

$$=\frac{1}{(2\pi i)^n}\int_{L_1}\cdots\int_{L_n}\prod_{k=1}^{n}[\phi_k(s_k)\,x^{s_k}]\,\psi(s_1,\ldots,s_n)\,ds_1\ldots ds_n, \quad \ldots(1.1)$$

where

$$\phi_k(s_k)=\frac{\prod_{j=1}^{u^{(k)}}\Gamma\left[d_j^{(k)}-\delta_j^{(k)}s_k\right]\prod_{j=1}^{v^{(k)}}\Gamma\left[1-b_j^{(k)}+\phi_j^{(k)}s_k\right]}{\prod_{j=u^{(k)}+1}^{D^{(k)}}\Gamma\left[1-d_j^{(k)}+\delta_j^{(k)}s_k\right]\prod_{j=v^{(k)}+1}^{B^{(k)}}\Gamma\left[b_j^{(k)}-\phi_j^{(k)}s_k\right]} \quad \ldots(1.2)$$

$$k=1, \ldots, n;$$

$$\psi(s_1, \ldots, s_n)=\frac{\prod_{j=1}^{\lambda}\Gamma\left[1-a_j+\sum_{k=1}^{n}\theta_j^{(k)}s_k\right]}{\prod_{j=\lambda+1}^{A}\Gamma\left[a_j-\sum_{k=1}^{n}\theta_j^{(k)}s_k\right]\prod_{j=1}^{C}\Gamma\left[1-c_j+\sum_{k=1}^{n}\psi_j^{(k)}s_k\right]}, \quad \ldots(1.3)$$

where L_k $(k=1, \ldots, n)$ are suitable contours.

Following Srivastava and Panda (see [10] and [11]), we have used the abbreviation (a) to denote the sequence of A parameters $a_1, \ldots, a_A$; for each $k=1, \ldots, n$, $(b^{(k)})$ abbreviates the sequence of $B^{(k)}$ parameters $b_j^{(k)}$, $j=1, \ldots, B^{(k)}$, with similar interpretations for (c), $(d^{(k)})$, etc., $k=1, \ldots, n$, it being understood, for example, that $b^{(1)}=b'$, $b^{(2)}=b''$, and so on.

The spheroidal functions. The spheroidal functions as defined and investigated by Stratton [12] and later by Chu and Stratton [13] are those solutions of the integral equation [12(27)] :

$$V_{\alpha n}(c)\,\psi_{\alpha n}(c, \eta)=\int_{-1}^{1} e^{ic\eta t}(1-t^2)\,\psi_{\alpha n}(c, t)\,dt, \quad -1\leqslant\eta\leqslant 1, \quad \ldots(1.4)$$

that remain finite at the singular points $\eta=\pm 1$, with eigenvalues $V_{\alpha n}(c)$, valid for $\alpha>-1$.

The spheroidal function can be expanded as [9(14)] :

$$\psi_{\alpha n}(c, \eta)=\frac{i^n\sqrt{2\pi}}{V_{\alpha n}(c)}\sum_{k=0,\text{ or }1}^{\infty}{}^{*}\; a_k(c\mid\alpha n)\,(c\eta)^{-(\alpha+\frac{1}{2})}J_{k+\alpha+\frac{1}{2}}(c\eta) \quad \ldots(1.5)$$

which represents the function uniformly on $(-\infty, \infty)$, where the coefficients $a_k(c\mid\alpha n)$ satisfy the recursion formula [14, Eq. (67)] and the asterisk over the summation sign indicates that the sum is taken over only even or odd values of k according as n is even or odd. As $c\to 0$, $a_k(c\mid\alpha n)\to 0$, $k\neq n$.

If we take $c \to 0$, $\eta \to \infty$ such that $c\eta$ remains finite and the normalization is chosen to be such that

$$\frac{\sqrt{2\pi}}{V_{\alpha n}(c)} \sum_{k=0,\,1}^{\infty}{}^{*} \; i^k a_k (c \mid \alpha n) = 1,\ k = n,$$

then

$$\psi_{\alpha n}(c, \eta) \text{ reduces to } (c\eta)^{-(\alpha+\frac{1}{2})} J_{n+\alpha+\frac{1}{2}}(c\eta).$$

The known results [5, p. 140], [2, p. 4], [4, p. 292], required in the sequel, may be recalled as follows :

(i) $$\int_0^{\pi/2} \frac{\sin^{2u-1}\theta \cos^{2v-1}\theta}{(a\sin^2\theta + b\cos^2\theta)^{u+v}}\, d\theta = \frac{\Gamma(u)\,\Gamma(v)}{2a^u b^v\, \Gamma(u+v)}, \qquad \ldots(1.6)$$

Re $(u, v) > 0$;

(ii) $$\Gamma(2z) = 2^{2z-\frac{1}{2}} (2\pi)^{-\frac{1}{2}}\, \Gamma(z)\, \Gamma(z+\tfrac{1}{2}) \; ; \qquad \ldots(1.7)$$

(iii) $$\int_0^{\infty} z^{\rho} e^{-z} L_k^{\sigma}(z)\, dz = \frac{\Gamma(\sigma-\rho+k)\,\Gamma(\rho+1)}{k!\,\Gamma(\sigma-\rho)},\ \text{Re}(\rho) > -1; \qquad \ldots(1.8)$$

$$= \frac{(-1)^k\, \Gamma(\rho-\sigma+1)\, \Gamma(\rho+1)}{k!\, \Gamma(\rho-\sigma-k+1)}\; ; \qquad \ldots(1.9)$$

(which is derivable from [2, p. 3 and 4]).

2. An Integral Required. We shall use the following integral for establishing the desired integral relation.

$$\int_0^{\pi/2} \frac{\sin^{2u-1}\theta \cos^{2v-1}\theta}{(a\sin^2\theta+b\cos^2\theta)^{u+v}}\, \psi_{\alpha n}\left(c^{\sigma}, \frac{2z^{\sigma/2}(\sin\theta\cos\theta)^{\sigma}}{(a\sin^2\theta+b\cos^2\theta)^{\sigma}}\right)$$

$$H\left[\frac{\beta_1 z^{s_1}(\sin\theta\cos\theta)^{2s_1}}{(a\sin^2\theta+b\cos^2\theta)^{2s_1}}, \ldots, \frac{\beta_r z^{s_r}(\sin\theta\cos\theta)^{2s_r}}{(a\sin^2\theta+b\cos^2\theta)^{2s_r}}\right] d\theta$$

$$= \frac{i^n \pi\, 2^{-(u+v+\alpha)}}{V_{\alpha n}(c^{\sigma})\, a^u\, b^v} \sum_{k=0,\,1}^{\infty}{}^{*} \sum_{m=0}^{\infty} \frac{(-1)^m a_k(c^{\sigma} \mid \alpha n)}{m!\, \Gamma(m+k+\alpha+\frac{3}{2})} \left(\frac{c}{2\sqrt{ab}}\right)^{\sigma(2m+k)}$$

$$.z^{\sigma(2m+k)/2}\, H^{0,\lambda+2}_{A+2,\, C+2\,:\,[B',\, D'];\, \ldots;\, [B^{(r)},\, D^{(r)}]} {}^{:\,(u',\, v')\,;\, \ldots\,;\, (u^{(r)},\, v^{(r)})}$$

$$\left(\begin{matrix}(\ldots),\, P,\, Q : (\ldots)\, ;\, \ldots\, ;\, (\ldots)\, ; \\ (\ldots),\, R,\, S : (\ldots)\, ;\, \ldots\, ;\, (\ldots)\, ;\end{matrix}\; \frac{\beta_1 z^{s_1}}{(4ab)^{s_1}}, \ldots, \frac{\beta_r z^{s_r}}{(4ab)^{s_r}}\right), \qquad \ldots(2.1)$$

where

(i) (...) stands everywhere for the same parameters as given in the definition (1.1) at corresponding places ;

(ii) P and Q stand for the quantities :

$$\left(1-u-\frac{\sigma}{2}(2m+k)\,;\,s_1,\,\ldots,\,s_r\right)$$

and

$$\left(1-v-\frac{\sigma}{2}(2m+k)\,;\,s_1,\,\ldots,\,s_r\right),$$

respectively ;

(iii) R and S stand for the quantities :

$$\left(1-\frac{u}{2}-\frac{v}{2}-\frac{\sigma}{2}(2m+k)\,;\,s_1,\,\ldots,\,s_r\right)$$

and

$$\left(\frac{1}{2}-\frac{u}{2}-\frac{v}{2}-\frac{\sigma}{2}(2m+k);\,s_1,\,\ldots,\,s_r\right),$$

respectively.

The sufficient conditions for the validity of the integral (2.1) are given below :

(*a*) $\sigma>0$, $s_1>0$, ..., $s_r>0$; a, b, c and z are non-negative numbers,

$$\operatorname{Re}\left[u+v+\sum_{i=1}^{r} s_i\left(d_j^{(i)}\Big/\delta_j^{(i)}\right)\right]>0,\ j=1,\ \ldots,\ u^{(i)}$$

(*b*) The conditions (1.6) and (1.7) in [11] are also satisfied by the H-function occurring in the integrand.

(*c*) The double series occurring on the right-hand side of (2.1) is absolutely and uniformly convergent by M-Test [1, p. 246 and 266].

Proof. In order to prove (2.1), we express the spheroidal function involved in the integrand in its expansion form with the help of (1.5) and the well-known Bessel series [3, p. 4], and change the order of integration and summation which is permissible under the conditions stated). We then express the H-function of r variables in terms of its multiple integral of Barnes type by virtue of (1.1), and change the order of the resulting integrals. Now, evaluating the inner integral thus obtained by virtue of known result (1.6) and then interpreting the result thus obtained with the help of (1.1) after using (1.7), we arrive at the desired result.

The change of order of integration in each case is justified [1, p. 504] due to the absolute convergence of the integrals involved in the process.

3. Integral Relation. We now proceed to establish the following integral relation :

$$\int_0^\infty \int_0^\infty \frac{x^{2v-1}y^{2u-1}}{(bx^2+ay^2)^{u+v}} \psi_{\alpha n}\left(c^\sigma, \frac{2x^\sigma y^\sigma (x^2+y^2)^{\sigma/2}}{(bx^2+ay^2)^\sigma}\right)$$

$$\cdot H\left[\frac{\beta_1 (xy)^{2s_1}(x^2+y^2)^{s_1}}{(bx^2+ay^2)^{2s_1}}, \ldots, \frac{\beta_r (xy)^{2s_r}(x^2+y^2)^{s_r}}{(bx^2+ay^2)^{2s_r}}\right]$$

$$\cdot f(x^2+y^2)dx\, dy$$

$$=\frac{i^n \pi 2^{-(u+v+\alpha+1)}}{V_{\alpha n}(c^\sigma) a^u b^v} \sum_{k=0,1}^{\infty}{}^{*} \sum_{m=0}^{\infty} \frac{(-1)^m a_k(c^\sigma \mid \alpha n)}{m!\,\Gamma(m+k+\alpha+\frac{3}{2})}$$

$$\cdot\left(\frac{c}{2\sqrt{ab}}\right)^{\sigma(2m+k)} \int_0^\infty z^{(\sigma/2)(2m+k)} f(z)$$

$$\cdot H^{0,\lambda+2}_{A+2,C+2} \begin{matrix} : (u', v') ; \ldots ; (u^{(r)}, v^{(r)}) \\ : [B', D'] ; \ldots ; [B^{(r)}, D^{(r)}] \end{matrix}$$

$$\left(\begin{matrix} (\ldots), P, Q : (\ldots) ; \ldots ; (\ldots) ; \\ (\ldots), R, S : (\ldots) ; \ldots ; (\ldots) ; \end{matrix} \frac{\beta_1 z^{s_1}}{(4ab)^{s_1}}, \ldots, \frac{B_r z^{s_r}}{(4ab)^{s_r}}\right) dz, \quad \ldots(3.1)$$

where the symbols have the same meaning as defined in Section 2, and $f(z)=0(z^{-\mu})$ for large z and $f(z)=o(z^\varepsilon)$ for small z, $\mu>0$, $\varepsilon>0$, and $\mathrm{Re}\left[2\sum_{i=1}^{r} s_i\left(d_j^{(i)}/\delta_j^{(i)}\right)+\varepsilon\right]>0$ $(j=1, \ldots, u^{(i)})$.

Remaining conditions of its validity are the same as given for (2.1).

Proof. To prove (3.1), we first set $z=r^2$ in (2.1), then multiply both sides by $rf(r^2)$ and integrate with respect to 'r' between the limits $(0, \infty)$. Then, changing the order of integration and double summation (which is justified under the conditions stated), substituting $x=r\cos\theta$, $y=r\sin\theta$ and making a slight change of variable on the right side of the resulting equation, we get the desired integral relation.

The importance of the relation thus established can be visualized by evaluating a large number of integrals by choosing $f(z)$ in convenient forms. To illustrate this fact, we present the following example.

Example.

Let us choose $f(z)=z^{\rho} e^{-z} L_p^q(z)$, where $L_p^q(z)$ is the Laguerre polynomial. Substituting this value of $f(z)$ in 3.1 and then evaluating the resulting integral on the right-hand side on similar lines of (2.1), by using the result (1.9) instead of (1.6), we obtain the following interesting integral :

$$\int_0^\infty \int_0^\infty \frac{x^{2v-1} y^{2u-1} (x^2+y^2)^{\rho+1}}{(bx^2+ay^2)^{u+v}} \psi_{\alpha n}\left(c^{\sigma}, \frac{2(xy)^{\sigma}(x^2+y^2)^{\sigma/2}}{(bx^2+ay^2)^{\sigma}}\right)$$

$$\cdot e^{-(x^2+y^2)} L_p^q(x^2+y^2)$$

$$\cdot H\left[\frac{\beta_1 (xy)^{2s_1}(x^2+y^2)^{s_1}}{(bx^2+ay^2)^{2s_1}}, \ldots, \frac{\beta_r (xy)^{2s_r}(x^2+y^2)^{s_r}}{(bx^2+ay^2)^{2s_r}}\right] dx\, dy$$

$$=\frac{(-1)^p\ i^n\ \pi 2^{-(u+v+\alpha+1)}}{p\,!\, V_{\alpha n}(c^{\sigma})\, a^u\ b^v} \sum_{k=0,1}^{\infty}{}^{*} \sum_{m=0}^{\infty} \frac{(-1)^m a_k (c^{\sigma} \mid \alpha n)}{m\,!\,\Gamma(m+k+\alpha+\frac{3}{2})}$$

$$\left(\frac{c}{2\sqrt{ab}}\right)^{\sigma(2m+k)} H^{0,\,\lambda+4\ :\ (u',\, v')\, ;\, \ldots\, ;\, (u^{(r)},\, v^{(r)})}_{A+4,\, C+3\, :\, [B',\, D']\, ;\, \ldots\, ;\, [B^{(r)},\, D^{(r)}]}$$

$$\left(\begin{matrix}(\ldots), P, Q, T, U : (\ldots), \ldots, (\ldots)\, ; \\ (\ldots), R, S, W\ : (\ldots), \ldots, (\ldots)\, ;\end{matrix}\ \frac{\beta_1}{(4ab)^{s_1}}, \ldots, \frac{\beta_r}{(4ab)^{s_r}}\right), \quad \ldots(3.2)$$

where, in addition to the symbols already stated in (3.1), we have used the following symbols :

(i) T and U stand for the quantities: $(q-\rho-\sigma(2m+k)\,;\, s_1, \ldots, s_r)$ and $(-\rho-\sigma(2m+k)\,;\, s_1, \ldots, s_r)$, respectively ;

(ii) W stands for the quantity

$$(p+q-\rho-\sigma(2m+k)\,;\, s_1, \ldots, s_r).$$

The sufficient conditions under which the integral (3.2) is valid are given below :

(*a*) $\operatorname{Re}\left[2\rho+1+2\sum_{i=1}^{r}\left(s_i\, d_j^{(i)} \big/ \delta_j^{(i)}\right)\right]>0,\ j=1, \ldots, u^{(i)}.$

(*b*) The double series on the right hand side of (3.2) is absolutely and uniformly convergent.

Remark 1. Regarding the convergence of the series occurring on the right-hand side of our result, it would be worth mentioning that the ratio of the gamma functions involved in these results is bounded for large values of k (even or odd) [2, p. 47, Eq. (4)], and the ratio a_{k+2}/a_k is $-c^2/4k^2$. Hence the series are absolutely and uniformly convergent by M-Test.

Remark 2. On account of the general nature of the functions involved in the integrals of our results, a number of new and intresting results can be deduced by specializing the various parameters involved therein, but we shall not record them for lack of space.

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