Jñānabha, Vol. 8, 1978.

On a Class of Functions of Bounded Variation

by

Prem Chandra

School of Studies in Mathematics & Statistics, Vikram University, Ujjain–456 001, M.P., India

(Received : October 6, 1977)

1. Introduction. In this note, we prove some interesting results concerning the Lebesgue-integrable functions ([2], Chapter 10) of a class of functions of bounded variation ([2], Chapter 11). We write $f \in L(0, c)$, whenever the functions f are Lebesgue-integrable over $(0, c)$, and $f \in BV(0, c)$ for the functions f of the class of all functions of bounded variation over $(0, c)$. For the convenience in working, we further use the following notations :

(1.1) $$A(t)=\int_0^t g(u)\,du.$$

(1.2) $$f_g(t)=\frac{1}{A(t)}\int_0^t g(u)\,f(u)\,du.$$

(1.3) $$F(t)=f(t)-f_g(t).$$

2. We prove the following theorems :

Theorem 1. *Let* $c>0$ *and let* $g(t)>0$ *be such that* $A(t)>0$ *for* $t>0$. *Then*

$$\left.\begin{array}{ll} (2.1) & F(t)\in BV(0, c) \\ \textit{and} & \\ (2.2) & \dfrac{g(t)\,F(t)}{A(t)}\in L(0, c) \end{array}\right\} \Leftrightarrow$$

(2.3) $$f(t)\in BV(0, c).$$

Theorem 2. *Let constant* $k>c>0$ *and* $b\geqslant 0$. *Then*

(2.4) $$\left\{\frac{1}{t}\int_0^t f(u)\left(\log\frac{k}{u}\right)^b du\right\}\in BV(0, c) \Leftrightarrow$$

(2.5) $$f_1(t)\left(\log\frac{k}{t}\right)^b \in BV(0, c).$$

Theorem 3. *Let constant $k>c>0$ and $b\geqslant 0$. Then*

$$\left.\begin{array}{ll}(2.6) & f(+0)=0 \\ \text{and} & \\ (2.7) & \left\{\frac{1}{t}\int_0^t\left(\log\frac{k}{u}\right)^b df(u)\right\}\in BV(0,c)\end{array}\right\}\Leftrightarrow$$

$$(2.8)\qquad \left\{\frac{f(t)\left(\log\frac{k}{t}\right)^b}{t}\right\}\in BV(0,c).$$

Theorem 4. *Let constant $k>c>0$ and $b\geqslant 0$ and let $0\leqslant g(t)$ be such that $A(t)>0$ for $t>0$. Then*

$$\left.\begin{array}{ll}(2.9) & \int_0^c\left(\log\frac{k}{t}\right)^b \mid dF(t)\mid<\infty \\ \text{and} & \\ (2.10) & \left\{\frac{g(t)\,F(t)}{A(t)}\left(\log\frac{k}{t}\right)^b\right\}\in L(0,c)\end{array}\right\}\Leftrightarrow$$

$$(2.11)\qquad \int_0^c\left(\log\frac{k}{t}\right)^b \mid df(t)\mid<\infty,$$

whenever, for $0<u<c$,

$$(2.12)\qquad \int_u^c\frac{g(t)}{A^2(t)}\left(\log\frac{k}{t}\right)^b dt=O\left\{\frac{\left(\log\frac{k}{u}\right)^b}{A(u)}\right\}.$$

3. Proof of the Theorems. The proof of Theorem 2, for $b=0$, is trivial. And the proofs of Theorems 3 and 4, for $b=0$, follow from Theorem 1. For the proofs of the theorems, we require the following lemma :

Lemma. *Let $0<c$ be finite or infinite and let $g(t)\geqslant 0$ be such that $A(t)>0$, for $t>0$. Then*

$$f(t)\in BV(0,c)\Rightarrow f_g(t)\in BV(0,c).$$

Proof. Since $A(t)=\int_0^t g(u)\,du$, therefore $A(+0)=0$ and $A'(t)=g(t)$. Now

$$\begin{aligned} d(f_g(t))&=\frac{g(t)f(t)}{A(t)}-\frac{g(t)}{A^2(t)}\int_0^t g(u)f(u)\,du\\ &=\frac{g(t)}{A(t)}\left[f(t)-\frac{1}{A(t)}\left\{A(t)\,f(t)-\int_0^t A(u)\,df(u)\right\}\right]\\ &=\frac{g(t)}{A^2(t)}\int_0^t A(u)\,df(u).\end{aligned}$$

Therefore

$$\int_0^c |d(f_g(t))| \leqslant \int_0^c \frac{g(t)}{A^2(t)}\,dt \int_0^t A(u)\,|df(u)|$$
$$=\int_0^c A(u)\,|df(u)|\int_u^c \frac{d}{dt}\left(\frac{-1}{A(t)}\right)dt$$
$$\leqslant \int_0^c |df(u)|,$$

which is convergent by the hypothesis.

This terminates the proof of the lemma.

COROLLARY 1. Let $0<c$ be finite or infinite and $g=1$ in the above lemma. Then $f(t)\in BV(0, c) \Rightarrow f_1(t) \in BV(0, c)$.

3.1. Proof of Theorem 1. By the lemma, we follow that (2.3) implies (2.1). We now prove that (2.3) implies (2.2) also.

By (2.3), we have

$$F(t)=\frac{1}{A(t)}\int_0^t A(u)\,df(u).$$

Then, proceeding as in the proof of the lemma and using (2.3), we have

$$\int_0^c \frac{g(t)\,|F(t)|}{A(t)}\,dt \leqslant \int_0^c \frac{g(t)}{A^2(t)}\,dt \int_0^t A(u)\,|df(u)|$$
$$< \infty.$$

Finally, we prove that (2.1) and (2.2) implies (2.3). By (1.3), we have

$$\int_0^c |df(t)| \leqslant \int_0^c |dF(t)| + \int_0^c |df_g(t)|$$
$$=\int_0^c |dF(t)| + \int_0^c \frac{g(t)\,|F(t)|}{A(t)}\,dt,$$

which is convergent, by (2.1) and (2.2).

This terminates the proof of Theorem 1.

Let g be a non-zero constant function. Then we have the following corollaries :

COROLLARY 2. Let $c>0$. Then $F(t) \in BV(0, c)$ and
$$t^{-1}F(t) \in L(0, c) \Leftrightarrow f(t) \in BV(0, c).$$

COROLLARY 3. Let constant $k>c>0$ and $b>1$. Then

$$F(t)\left(\log\frac{k}{t}\right)^b \in BV(0, c) \Rightarrow f(t)\in BV(0, c).$$

3.2. Proof of Theorem 2. Integrating by parts, we have

$$\frac{1}{t}\int_0^t f(u)\left(\log\frac{k}{u}\right)^b du = f_1(t)\left(\log\frac{k}{t}\right)^b + \frac{b}{t}\int_0^t f_1(u)\left(\log\frac{k}{u}\right)^{b-1} du.$$

Let (2.5) hold. Then $\left\{\frac{1}{t}\int_0^t f_1(u)\left(\log\frac{k}{u}\right)^{b-1} du\right\} \in BV(0, c)$, by Corollary 1, since $f_1(t)\left(\log\frac{k}{t}\right)^{b-1} \in BV(0, c)$. This proves that (2.4) holds.

Now we prove that (2.4) implies (2.5).

Let $$h_1(t) = \frac{1}{t}\int_0^t f(u)\left(\log\frac{k}{u}\right)^b du.$$

Then

$$f_1(t) = \frac{1}{t}\int_0^t \left(\log\frac{k}{u}\right)^{-b} d(uh_1(u)).$$

Thus, integrating by parts,

$$(3.2.1)\quad f_1(t)\left(\log\frac{k}{t}\right)^b = h_1(t) - \frac{b\left(\log\frac{k}{t}\right)^b}{t}\int_0^t \frac{h_1(u)}{\left(\log\frac{k}{u}\right)^{1+b}} du.$$

Since $h_1(t) \in BV(0, c)$, to prove that (2.4) implies (2.5), we only require to show that

$$(3.2.2)\quad J(t) = \left\{\frac{\left(\log\frac{k}{t}\right)^b}{t}\int_0^t \frac{h_1(u)}{\left(\log\frac{k}{u}\right)^{1+b}} du\right\} \in BV(0, c).$$

Again, integrating by parts,

$$\begin{aligned} J(t) &= h_1(t)\frac{\left(\log\frac{k}{t}\right)^b}{t}\int_0^t \left(\log\frac{k}{u}\right)^{-1-b} du \\ &\quad - \frac{\left(\log\frac{k}{t}\right)^b}{t}\int_0^t \left\{\int_0^u \left(\log\frac{k}{x}\right)^{-1-b} dx\right\} dh_1(u) \\ &= J_1(t) + J_2(t), \text{ say.} \end{aligned}$$

Now,

$$\begin{aligned} \int_0^c |dJ_2(t)| &\leqslant \int_0^c |dh_1(t)|\, t^{-1}\left(\log\frac{k}{t}\right)^b \int_0^t \left(\log\frac{k}{u}\right)^{-1-b} du \\ &\quad + \int_0^c \left| d\left\{t^{-1}\left(\log\frac{k}{t}\right)^b\right\}\right| \int_0^t |dh_1(u)| \int_0^u \left(\log\frac{k}{x}\right)^{-1-b} dx \end{aligned}$$

$$\leqslant \int_0^c \left(\log \frac{k}{t}\right)^{-1} | dh_1(t) |$$

$$+\int_0^c | dh_1(t) | \int_0^u \left(\log \frac{k}{x}\right)^{-1-b} dx \int_u^c \left| d\left\{t^{-1}\left(\log \frac{k}{t}\right)^b\right\}\right|$$

$$\leqslant \left(\log \frac{k}{c}\right)^{-1} \int_0^c | dh_1(t) |$$

$$+\int_0^c u^{-1}\left(\log \frac{k}{u}\right)^b | dh_1(u) | \int_0^u \left(\log \frac{k}{x}\right)^{-1-b} dx$$

$$\leqslant 2\left(\log \frac{k}{c}\right)^{-1} \int_0^c | dh_1(t) | < \infty,$$

by (2.4). Hence $J_2(t) \in BV(0, c)$. Now, to prove (3.2.2), it is sufficient to show that $J_1(t) \in BV(0, c)$, which follows, whenever (2.4) holds and $I(t) \in BV(0, c)$, where

$$I(t) = t^{-1}\left(\log \frac{k}{t}\right)^b \int_0^t \left(\log \frac{k}{u}\right)^{-1-b} du.$$

Integrating by parts,

$$I(t) = \left(\log \frac{k}{t}\right)^{-1} - (1+b)\left(\log \frac{k}{t}\right)^b \frac{1}{t} \int_0^t \left(\log \frac{k}{u}\right)^{-b-2} du$$

$$= I_1(t) - (1+b) I_2(t), \text{ say.}$$

Since $I_1(t) \in BV(0, c)$ therefore we only require to prove that $I_2(t) \in BV(0, c)$. Now,

$$\int_0^c | dI_2(t) | \leqslant \int_0^c t^{-1}\left(\log \frac{k}{t}\right)^{-2} dt$$

$$+\int_0^c \left| d\left\{t^{-1}\left(\log \frac{k}{t}\right)^b\right\}\right| \int_0^t \left(\log \frac{k}{u}\right)^{-b-2} du$$

$$\leqslant 2\int_0^c t^{-1}\left(\log \frac{k}{t}\right)^{-2} dt$$

$$< \infty.$$

This terminates the proof.

COROLLARY 4. $f(t)\left(\log \frac{k}{t}\right)^b \in BV(0, c)$, where $k > c > 0$ and $b \geqslant 0$, implies that $F(t)\left(\log \frac{k}{t}\right)^b \in BV(0, c)$, whenever g be a non-zero constant function.

The proof follows by using Corollary 1 and Theorem 2.

3.3. Proof of Theorem 3. It is clear that (2.8) implies (2.6). Let (2.8) hold then integrating by parts, we have

$$\frac{1}{t}\int_0^t \left(\log\frac{k}{u}\right)^b df(u)=\frac{f(t)\left(\log\frac{k}{t}\right)^b}{t}+\frac{b}{t}\int_0^t \frac{f(u)\left(\log\frac{k}{u}\right)^{b-1}}{u}\,du.$$

Since $(2.8)\Rightarrow\left\{\frac{f(u)}{u}\left(\log\frac{k}{u}\right)^{b-1}\right\}\in BV(0,c)$, therefore, by Cor. 1,

$$\left\{\frac{1}{t}\int_0^t \frac{g(u)}{u}\left(\log\frac{k}{u}\right)^{b-1} du\right\}\in BV(0,c).$$

Thus (2.8) implies (2.6).

Now we prove that (2.6) and (2.7) implies (2.8). Let

$$H_1(t)=\frac{1}{t}\int_0^t \left(\log\frac{k}{u}\right)^b df(u).$$

Then, since $f(+0)=0$,

$$f(t)=\int_0^t df(u)=\int_0^t \left(\log\frac{k}{u}\right)^{-b} d(uH_1(u)).$$

Thus

$$\frac{f(t)\left(\log\frac{k}{t}\right)^b}{t}=\left(\log\frac{k}{t}\right)^b\frac{1}{t}\int_0^t \left(\log\frac{k}{u}\right)^{-b} d\{uH_1(u)\}.$$

Arguing as in the proof of Theorem 2 with $H_1(u)$ in place of $h_1(u)$, we follow the proof.

3.4. Proof of Theorem 4. By (1.3), we have

$$\begin{aligned}\int_0^c \left(\log\frac{k}{t}\right)^b |\,df(t)\,| &\leqslant\int_0^c \left(\log\frac{k}{t}\right)^b |\,dF(t)\,|+\int_0^c \left(\log\frac{k}{t}\right)^b |\,df_g(t)\,|\\ &=\int_0^c \left(\log\frac{k}{t}\right)^b |\,dF(t)\,|+\int_0^c \frac{|\,g(t)F(t)\,|}{A(t)}\left(\log\frac{k}{t}\right)^b dt\\ &<\infty,\end{aligned}$$

by (2.9) and (2.10). Thus (2.11) holds.

We now prove that (2.11) implies (2.9) and (2.10). Since

$$\left(\log\frac{k}{c}\right)^c\int_0^c |\,df(t)\,|\leqslant\int_0^c \left(\log\frac{k}{t}\right)^b |\,df(t)\,|$$
$$<\infty,$$

by (2.11), therefore, $f(t)\in BV(0,c)$ which further implies, by the lemma, that $f_g(t)\in BV(0,c)$. Also, we have

$$F(t)=\frac{1}{A(t)}\int_0^t A(u)\,df(u).$$

So that

$$\int_0^c \frac{|F(t)|\,g(t)}{A(t)}\left(\log\frac{k}{t}\right)^b dt \leqslant \int_0^c \frac{g(t)}{A^2(t)}\left(\log\frac{k}{t}\right)^b dt \int_0^t A(u)\,|\,df(u)\,|$$

$$=\int_0^c A(u)\,|\,df(u)\,|\,\int_u^c \left(\log\frac{k}{t}\right)^b \frac{d}{dt}\left(\frac{-1}{A(t)}\right) dt$$

$$=O\left\{\int_0^c \left(\log\frac{k}{u}\right)^b |\,df(u)\,|\right\} \quad \text{by (2.12)}$$

$$=O(1),$$

by (2.11), which proves that (2.11) implies (2.10). Further, we observe that

$$\int_0^c \left(\log\frac{k}{t}\right)^b |\,df_g(t)\,| = \int_0^c \frac{g(t)\,|\,F(t)\,|}{A(t)}\left(\log\frac{k}{t}\right)^b dt.$$

Therefore, (2.11) implies (2.10) which, in turn, implies that

$$\text{(3.4.1)} \qquad \int_0^c \left(\log\frac{k}{t}\right)^b |\,df_g(t)\,| < \infty.$$

Finally, to prove that (2.11) implies (2.9), we have, by using (1.3),

$$\int_0^c \left(\log\frac{k}{t}\right)^b |dF(t)| \leqslant \int_0^c \left(\log\frac{k}{t}\right)^b |\,df(t)\,| + \int_0^c \left(\log\frac{k}{t}\right)^b |\,df_g(t)\,|$$

$$<\infty,$$

by (2.11) and (3.4.1).

This terminates the proof of Theorem 4*.

Remark 1. Let $k>c>0$. Then, for finite b, (2.9) and (2.10) imply (2.11).

COROLLARY 5. Let $k>c>0$ and $b\geqslant 0$ and let g be a non-zero constant function. Then

$$\left.\begin{array}{l}\text{(i)}\ \displaystyle\int_0^c \left(\log\frac{k}{t}\right)^b |\,dF(t)\,| < \infty \\[2ex] \text{(ii)}\ \displaystyle\frac{F(t)}{t}\left(\log\frac{k}{t}\right)^b \in L(0,c)\end{array}\right\} \Leftrightarrow \text{(iii)}\ \int_0 \left(\log\frac{k}{t}\right)^b |\,df(t)\,| < \infty.$$

Remark 2. By the application of Lemma 7 of Chandra [1], the conditions (i) and (ii) of Corollary 5 are equivalent to

$$\text{(iv)} \qquad F(t)\left(\log\frac{k}{t}\right)^b \in BV(0,c)$$

and (ii). Therefore, Corollary 5 can be put in the following form :

---

* In Theorems 2, 3 and 4, $\left(\log\frac{k}{t}\right)^b$ $(b\geqslant 0)$ may be replaced by $\left(\log_h \frac{k}{t}\right)^b$ $(b\geqslant 0)$, where $\log_h = \log\log_{h-1}$ $(h>1)$ and $\log_1 = \log$.

COROLLARY 5A. Let $k>c>0$ and $b\geqslant 0$ and g be 1. Then

$$\text{(ii) and (iv)} \Leftrightarrow \text{(iii)}.$$

REFERENCES

[1] P. Chandra, On the absolute Riesz summability of Fourier series, its factored conjugate series and their derived series, *Rend. Mat.* (6), 3 (1970), pp. 291-311.

[2] E. C. Titchmarsh, *The Theory of Functions*, Oxford, 1961.