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On Sound Waves in a Pipe of variable cross-section

by

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Summary. An analytical solution in terms of confluent hypergeometric functions is obtained of the problem of determining the periods of the normal modes of sound waves in a pipe whose cross-sectional area is proportional to an exponential function where the exponent is quadratic in the longitudinal coordinate. Numerical results are given in the case where both ends of the pipe are open, and an application to brass instruments is indicated.

1. Introduction. It is well known (see [1], page 96, for example) that sound waves in a pipe of slowly varying cross-sectional area $A(x)$ are governed by the partial differential equation

$$\frac{\partial^2 u}{\partial t^2}=v^2 \frac{\partial}{\partial x}\left\{\frac{1}{A} \frac{\partial}{\partial x} (Au)\right\} \tag{1.1}$$

where v is the velocity of the waves. It is assumed that the displacement is small, and that any effects due to viscosity and friction are negligable.

For normal modes, let

$$u(x, t)=F(x) \cos (pt+\varepsilon) \tag{1.2}$$

when (1.1) takes the form

$$F''+\frac{A'}{A} F'+\left(\frac{A'}{A}\right)' F=-\frac{p^2}{v^2} F \tag{1.3}$$

primes denoting differentiations with respect to the independent variable x.

A case which has been widely discussed is that of the exponential horn, where the cross-sectional area is given by

$$A(x)=A_0 e^{bx}. \tag{1.4}$$

In this instance, the periods are the same as those of a cylindrical pipe.

An obvious generalisation of (1.4) is to let the exponent be quadratic in x, that is

$$A(x)=A_0\, e^{bx+cx^2} \tag{1.5}$$

It is then evident that equation (1.3) may now be written in the form

$$F''+(b+2cx)\,F'+\left(2c+\frac{p^2}{v^2}\right)F=0 \tag{1.6}$$

2. The solution of the equation (1.6). If we take a new independent variable in (1.6) given by

$$z=-\frac{1}{4c}\,(b+2cx)^2 \tag{2.1}$$

we have

$$z\ddot{F}+\left(\frac{1}{2}-z\right)\dot{F}-\left(\frac{1}{2}+\frac{p^2}{4cv^2}\right)F=0 \tag{2.2}$$

where dots denote differentiations with respect to z.

The problem under consideration is now seen to be soluble in terms of confluent hypergeometric functions, because (2.2) is a special case of the confluent hypergeometric equation

$$z\ddot{F}+(c-z)\,\dot{F}-aF=0 \tag{2.3}$$

{See [2], page 2.}

Two independent solutions of (2.3) are

$$F_1={}_1F_1\,(a\,;\,c\,;\,z) \tag{2.4}$$

and
$$F_2=z^{1-c}\,{}_1F_1\,(1+a-c\,;\,2-c\,;\,z) \tag{2.5}$$

The confluent hypergeometric function is defined by the relation

$${}_1F_1\,(a\,;\,c\,;\,x)=\sum_{r=0}^{\infty}\frac{(a)_r\,x^r}{(c)_r\,r\,!}, \tag{2.6}$$

where the Pochhammer symbol $(a)_r$ is given by

$$(a)_r=a\,(a+1)\ldots(a+r-1)$$

$$(a)_0=1$$

and
$$(a)_{-r}=(-1)^r/(1-a)_r \tag{2.7}$$

For both of the solutions (2.4) and (2.5) to be defined, it i sufficient that the parameter c is not an integer.

The solutions of (2.2) are thus :

$$F_1 = {}_1F_1\left(\frac{1}{2}+\frac{p^2}{4cv^2}\,;\,\frac{1}{2}\,;\,-\frac{1}{4c}\,(b+2cx)^2\right) \tag{2.8}$$

and

$$F_2 = \frac{i}{2c^{1/2}}\,(b+2cx)\,{}_1F_1\left(1+\frac{p^2}{4cv^2}\,;\,\frac{3}{2}\,;-\frac{1}{4c}(b+2cx)^2\right) \tag{2.9}$$

3. Discussion of the Normal Modes for the Pipe with Open Ends. At an open end of the pipe, dF/dx vanishes, so that, if we write

$$F = AF_1 + BF_2, \tag{3.1}$$

where A and B are arbitrary constants, then, for a pipe open at its ends x_1 and x_2, we have

$$AF_1'(x_1)+BF_2'(x_1)=0$$

and

$$AF_1'(x_2)+BF_2'(x_2)=0, \tag{3.2}$$

and so the eigenvalue equation is

$$\begin{vmatrix} F_1'(x_1) & F_2'(x_1) \\ F_1'(x_2) & F_2'(x_2) \end{vmatrix} = 0$$

or

$$F_1'(x_1)\,F_2'(x_2) - F_1'(x_2)\,F_2'(x_1) = 0. \tag{3.3}$$

For convenience, we let x_1 and x_2 be respectively $-b/2c$ and $1-b/2c$, so that the overall length of the pipe is 1.

At the end $x=-b/2c$, $F_1'=0$ and $F_2'=1$, so that the eigenvalues are determined by the equation

$${}_1F_1\left(\frac{3}{2}+\frac{p^2}{4cv^2}\,;\,\frac{3}{2};\,-cl^2\right)=0. \tag{3.4}$$

Numerical solutions of this equation were obtained without difficulty using a micro computer, and these are listed below for $l=1$.

$c=\cdot 1$		$c=\cdot 2$		$c=\cdot 5$	
$p_1=1027$		$p_1=1011$		$p_1=964$	
	1051		1059		1084
$p_2=2078$		$p_2=2070$		$p_2=2048$	
	1045		1048		1055
$p_3=3123$		$p_3=3118$		$p_3=3103$	
	1045		1046		1050
$p_4=4168$		$p_4=4164$		$p_4=4153$	
	1043		1044		1047
$p_5=5211$		$p_5=5208$		$p_5=5200$	
	1044		1044		1046
$p_6=6255$		$p_6=6252$		$p_6=6246$	

$c=1$		$c=2$	
$p_1=887$		$p_1=741$	
	1126		1216
$p_2=2013$		$p_2=1957$	
	1068		1088
$p_3=3081$		$p_3=3045$	
	1055		1064
$p_4=4136$		$p_4=4109$	
	1050		1056
$p_5=5186$		$p_5=5165$	
	1046		1050
$p_6=6232$		$p_6=6215$	

From these results, by considering the first differences of the eigenvalues, it will be seen that, in particular for small values of c, the lowest eigenvalues are smaller in value than what would be expected in the case of a cylindrical pipe, whereas the higher-order eigenvalues become progressively less affected. As would be expected, the parameter b does not enter into these results.

4. An Application to Brass Instruments. It is sometimes observed that the fundamental tone of a brass instrument such as a french horn is slightly flat as compared with the other harmonics produced. This phenomenon may readily be accounted for if the exponent of the rate of flare of the tube involves a small quadratic term.

Corresponding to a horn pitched in B-flat alto, the fundamental should be of the order of 56 vibrations per second, and it is expected that the higher harmonics are integral multiples of this number. In practice, the author has noticed on his own instrument that the fundamental note is about a quarter of a tone flat, without correction by 'lipping'. The length of the vibrating column of air is approximately 2·955 metres, and if the parameter c is of the order of 0·028, the effect mentioned above is easily accounted for, as is shown below:

$c=0{\cdot}028$	$l=2{\cdot}955$	
$p_1=339{\cdot}9$	$f_1=54{\cdot}1$	
		57·2
$p_2=699{\cdot}6$	$f_2=111{\cdot}3$	
		56·5
$p_3=1054{\cdot}6$	$f_3=167{\cdot}8$	
		56·3
$p_4=1408{\cdot}6$	$f_4=224{\cdot}1$	
		56·4
$p_5=1762{\cdot}2$	$f_5=280{\cdot}5$	
		56·3
$p_6=2116{\cdot}0$	$f_6=336{\cdot}8$	

$f_1, \ldots, f_6$ are the frequencies of the harmonics in cycles per second.

The overall results are slightly sharper than those obtained in practice because the effects of damping have not been taken into account.

REFERENCES

[1] C. A. Coulson, *Waves*, University Mathematical Texts, Oliver and Boyd, Edinburgh and London, 1952.

[2] L. J. Slater, *Confluent Hypergeometric Functions*, Cambridge University Press, 1960.