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A note on q-fractional differentiation and basic hypergeometric transformations

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1. Introduction. Recently, M. Upadhyay [4] used the q-fractional derivative operator D_q^{α}, defined by Agarwal [1] and Al-Salam [2], to derive certain interesting basic hypergeometric transformations. The object of this note is to give a further application of this q-fractional derivative operator in obtaining some more basic hypergeometric expansions and transformations.

2. Definitions and notations. For $|q|<1$, let

$$\begin{cases}[q^a]_n=[a]_n=(1-q^a)(1-q^{a+1})\ldots(1-q^{a+n-1})\\ [a]_0=[q^a]_0=1.\end{cases} \tag{2.1}$$

The generalized basic hypergeometric series is defined as

$$ {}_A\Phi_B\begin{bmatrix} q^{(a)} ; & \\ & x \\ q^{(b)} ; & \end{bmatrix}={}_A\Phi_B\begin{bmatrix} (a) ; & \\ & x \\ (b) ; & \end{bmatrix}=\sum_{n=0}^{\infty}\frac{[(a)]_n\, x^n}{[q]_n\,[(b)]_n},\ |x|<1. \tag{2.2}$$

As usual, (a_N) stands for the sequence of N parameters a_1, a_2, ..., a_N ; when $N=A$, we shall simply write (a) instead of (a_A). We shall also use the following function :

$$[x-y]_{\mu}=x^{\mu}\sum_{k=0}^{\infty}\frac{[-\mu]_k}{[q]_k}\left\{\frac{q^{\mu}y}{x}\right\}^k, \tag{2.3}$$

and the well-known basic double hypergeometric function of higher order (cf. [5], p. 141), which we conveniently denote here as

$$(2.4)\qquad \Phi\left[\begin{matrix}(a):(b);(c);\\ (d):(e);(g);\end{matrix}\; x, y\right]$$

$$= \sum_{m,n=0}^{\infty} \frac{[(a)]_{m+n}\,[(b)]_m\,[(c)]_n}{[(d)]_{m+n}\,[(e)]_m\,[(g)]_n} \frac{x^m}{(q)_m} \frac{y^n}{(q)_n},\ |x|<1,\ |y|<1.$$

For the various known special cases of the double q-hypergeometric functions (2.4), see [4], p. 110.

3. Chaturvedi [3] has proved the following result :

$$(3.1)\qquad {}_4\Phi_3\left[\begin{matrix}\eta,\ 1+\alpha-\gamma-\delta,\ 1+\alpha-\beta-\delta,\ 1+\alpha-\beta-\gamma\ ;\\ 1+\alpha-\beta,\ 1+\alpha-\gamma,\ 1+\alpha-\delta\qquad ;\end{matrix}\; x\right]$$

$$= \sum_{r=0}^{\infty} \frac{(\alpha)_r\,(\beta)_r\,(\gamma)_r\,(\delta)_r\,(\eta)_{2r}\, q^{(\alpha-\beta-\gamma-\delta)r}\, x^r}{(q)_r\,(1+\alpha-\beta)_r\,(1+\alpha-\eta)_r\,(1+\alpha-\delta)_r\,(\alpha)_{2r}}\; {}_2\Phi_1\left[\begin{matrix}\eta+2r,\ \eta-\alpha\ ;\\ 1+\alpha+2r\ ;\end{matrix}\; x\right]$$

By the Euler's identity

$${}_2\Phi_1\left[\begin{matrix}\alpha,\ \beta\ ;\\ \gamma\ ;\end{matrix}\; x\right] = {}_1\Phi_0\,[\alpha+\beta-\gamma\ ;\ -\ ;\ x]\; {}_2\Phi_1\left[\begin{matrix}\gamma-\alpha,\ \gamma-\beta\\ \gamma\qquad ;\end{matrix}\; xq^{\alpha+\beta-\gamma}\right]$$

we have

$$(3.2)\qquad {}_2\Phi_1\left[\begin{matrix}\eta+2r,\ \eta-\alpha\ ;\\ 1+\alpha+2r\ ;\end{matrix}\; x\right]$$

$$= {}_1\Phi_0\,[1+2\eta-2\alpha\ ;\ -;\ x]\; {}_2\Phi_1\left[\begin{matrix}1+\alpha-\eta,\ 2\alpha-\eta-1-2r\ ;\\ 1+\alpha+2r\qquad ;\end{matrix}\; xq^{2(\eta-\alpha-\frac{1}{2})}\right]$$

Throughout this paper, we shall use

$D_{q,\,x}^{\alpha,\ \mu-1}\,[f(x)]$ to denote (see [4 ; p. 109, Eq. (1.2)]) $D_q^{\alpha}\,[x^{\mu-1} f(x)]$

Substituting (3.2) in (3.1), and applying $D_{q,\,x}^{\theta-\psi,\ \theta-1}$, we have the expansion

$$(3.3)\qquad {}_5\Phi_4\left[\begin{matrix}\theta,\ \eta,\ 1+\alpha-\gamma-\delta,\ 1+\alpha-\beta-\delta,\ 1+\alpha-\beta-\gamma\ ;\\ \psi,\ 1+\alpha-\beta,\ 1+\alpha-\gamma,\ 1+\alpha-\delta\qquad ;\end{matrix}\; x\right]$$

$$= \sum_{r=0}^{\infty} \frac{(\alpha)_r\,(\beta)_r\,(\gamma)_r\,(\delta)_r\,(\eta)_{2r}\, q^{(\alpha-\beta-\gamma-\delta)r}\, x^r}{(q)_r\,(1+\alpha-\beta)_r\,(1+\alpha-\eta)_r\,(1+\alpha-\delta)_r\,(\alpha)_{2r}\,(\psi)_r}$$

$$\cdot\,\Phi\left[\begin{matrix} x \\ xq^{2\eta-2\alpha-1} \\ \\ \end{matrix}\ \middle|\ \begin{matrix}\theta+r\\ 1+2\eta-2\alpha;\,(1+\alpha-\eta),\,(2\alpha-\eta-1-2r)\\ \psi+r\\ -\ ;\qquad 1+\alpha+2r\end{matrix}\right]$$

provided $|x|<1$ and $|xq^{2\eta-2\alpha-1}|<1$.

Now multiplying both sides of (3.1) by the series equivalent of $[1-y]_{-\mu}$, given by (2.3), putting xu and yu for x and y, respectively, and then applying

$$\prod_{i=1}^{G} D_{q,u}^{g_i\ \ h_i-g_i-1},\ \prod_{i=1}^{L} D_{q,x}^{l_i-j_i,\ l_i-1} \text{ and } \prod_{i=1}^{F} D_{q,y}^{f_i-s_i,\ f_i-1}$$

successively, we get another general expansion as

(3.4)

$$\Phi\left[\begin{array}{c|c} x & \begin{array}{c}(g)\\(l),\ 1+\alpha-\gamma-\delta,\ 1+\alpha-\beta-\delta,\ 1+\alpha-\beta-\gamma,\ \eta\ ;\ (f),\ \beta\end{array}\\ yuq^{-b} & \begin{array}{c}(h_G)\\(j_L),\ 1+\alpha-\beta,\ 1+\alpha-\gamma,\ 1+\alpha-\delta\ ;\ (s_F)\end{array}\end{array}\right]$$

$$=\sum_{r=0}^{\infty}\frac{(\alpha)_r\,(\beta)_r\,(\gamma)_r\,(\delta)_r\,(\eta)_{2r}\,(xu)^r\,q^{(\alpha-\beta-\gamma-\delta)r}}{(q)_r\,(1+\alpha-\beta)_r\,(1+\alpha-\eta)_r\,(1+\alpha-\delta)_r\,(\alpha)_{2r}}$$

$$\cdot\Phi\left[\begin{array}{c|c} xu & \begin{array}{c}(g)+r\\(l)+r,\ \eta+2r,\ \eta-\alpha\ ;\ (f)+r,\ \beta\end{array}\\ yuq^{-b} & \begin{array}{c}(h_G)+r\\(j_L)+r,\ 1+\alpha+2r\ ;\ (s_F)+r\end{array}\end{array}\right]$$

provided $|xu|<1$, and $|yuq^{-c}|<1$.

We conclude by remarking that several interesting special cases of our main expansion formula (3.4) can be readily given in terms, for example, of the basic analogues of Appell functions (cf. [4], p. 110).

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REFERENCES

[1] R. P. Agarwal, Certain fractional q-integrals and q-derivatives, Proc. Cambridge Philos. Soc. 66 (1969), pp. 365-370.

[2] W. A. Al-Salam, Some fractional q-integrals and q-derivatives, Proc. Edinburgh Math. Soc. (2) **15** (1969), pp. 135-145.

[3] K. Chaturvedi, *A Study of Basic Hypergeometric Functions*, Ph.D. Thesis University of Indore, India. 1974.

[4] M. Upadhyay, q-fractional differentiation and basic hypergeometric transformations, Ann. Polon. Math. 25 (1971), pp. 109-124.

[5] M. Upadhyay, Certain fractional q-integral operators, Pure Appl. Math. Sci. 4 (1976), pp. 141-149.