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The (R, e^{W^α}, r) (C, 1)–Summability of a Fourier Series

by

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1. Introduction. Let Σa_n be a given infinite series with the sequence of partial sums $\{s_n\}$. Then the $(C, 1)$— mean of $\{s_n\}$ is given by

$$T_n = \left(\frac{1}{n+1}\right) \sum_{k=0}^{n} s_k.$$

Let $\lambda(w)$ be continuous and monotonic increasing in $(0, \infty)$ and tend to infinity as $w \to \infty$. We write,

$$A_\lambda^r(w) = \sum_{n \leqslant w} \{\lambda(w) - \lambda(n)\}^r a_n, \qquad \text{where } r > 0$$

$$= r \int_0^w \{\lambda(w) - \lambda(x)\}^{r-1} \lambda'(x) A_\lambda(x)\, dx,$$

where $\qquad A_\lambda(x) = A_n = a_0 + a_1 + \ldots + a_n = \sum_{n \leqslant x} a_n.$

Also, we write

$$C_\lambda^r(w) = \{\lambda(w)\}^{-r} A_\lambda^r(w)$$

$$= r\{\lambda(w)\}^{-r} \int_0^w \{\lambda(w) - \lambda(x)\}^{r-1} \lambda'(x) A_\lambda(x)\, dx.$$

$C_\lambda^r(w)$ is known as the $(R, \lambda(w), r\}$— mean of $\{A_n\}$. If $C_\lambda^r(w)$ tends to s as $w \to \infty$, we say that Σa_n is summable $(R, \lambda(w), r)$ to the sum s (see [4]). Symbolically we write

$$\Sigma a_n \in s\ (R, \lambda(w), r).$$

We define the $(R, e^{W^\alpha}, r)(C, 1)$– mean of $\{s_n\}$ as the (R, e^{W^α}, r)– mean of $\{T_n\}$. Thus, writing $I(w)$ for the $(R, e^{W^\alpha}, r)(C, 1)$– mean of $\{s_n\}$, we obtain

$$I(w)=\alpha r e^{-rW^\alpha} \int_0^w (e^{W^\alpha}-e^{x^\alpha})^{r-1}\ e^{x^\alpha}\ x^{\alpha-1} T_x\, dx.$$

If $I(w) \to s$ as $w \to \infty$, we say that Σa_n is summable $(R, e^{W^\alpha}, r)(C, 1)$ to the sum s. Symbolically we write

$$\Sigma\, a_n \in s\ (R, e^{W^\alpha}, r)(C, 1).$$

Let f be a 2π-periodic function and L-integrable over $(-\pi, \pi)$. Let the Fourier series of f be

$$\tfrac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) \equiv \sum_{n=0}^{\infty} A_n(t) \tag{1.1}$$

We use the following notations throughout this paper, where s and x are fixed real numbers :

$$\phi(t)=\tfrac{1}{2}\{f(x+t)+f(x-t)-2s\} \tag{1.2}$$

$$\phi_1(t)=\left(\frac{1}{t}\right)\int_0^t \phi(u)\, du \tag{1.3}$$

$$\Phi_1(t)=\int_0^t |\ \phi_1(u)\ |\ du \tag{1.4}$$

$$e(w)=e^{W^\alpha}\quad (0<\alpha<1) \tag{1.5}$$

$$e^1(w)=\frac{d}{dw}\, e(w) \tag{1.6}$$

$$E(w, t)=\int_0^w \{e(w)-e(x)\}^{r-1} e(x)\, x^{\alpha-1} \sin t(x+1)\, dx \tag{1.7}$$

$$K(w, t)=\int_0^w \{e(w)-e(x)\}^{r-1}\ e(x)\ \frac{x^{\alpha-1}}{(x+1)} \sin^2 \tfrac{1}{2}t\ (x+1)\, dx. \tag{1.8}$$

In 1942, Wang [2] studied Riesz summability of type $e(w)$ and order $r>0$, of the Fourier series of f, when

$$\int_0^t |\ \phi(u)\ |\ du = o\left\{\frac{t}{\log(1/t)}\right\}\quad (t \to 0).$$

The object of this paper is to prove the following theorem concerning the $(R, e(w), r)$ $(C, 1)$—summability of the Fourier series of f at a point x.

Theorem. *Let*

$$\Phi_1(t)=o\left\{\frac{t}{\log(1/t)}\right\} \tag{1.9}$$

as $t\to 0$. *Then* $\Sigma A_n(x)\in s(R, e(w), r)(C, 1)$ *where* $r>0$.

2. We shall require the following order estimates, which hold good for $0<r<1$ and $w\to\infty$.

$$E(w, t)=O\{we^r(w)t\} \tag{2.1}$$
$$E(w, t)=O\{e^r(w)\} \tag{2.2}$$
$$E(w, t)=O\{e^r(w)w^{r(\alpha-1)}t^{-r}\} \tag{2.3}$$
$$E(w, t)=O\{e^r(w)w^{\alpha-1}(\log w)^{(1-r)/r}t^{-1}\} + O\{e^r(w)(\log w)^{-1}\} \tag{2.4}$$
$$K(w, t)=O\{we^r(w)t^2\} \tag{2.5}$$
$$K(w, t)=O\{e^r(w)t\} \tag{2.6}$$
$$K(w, t)=O\{e^r(w)w^{r(\alpha-1)}t^{1-r}\} \tag{2.7}$$
$$K(w, t)=O\{e^r(w)(\log w)^{-1}t\} + O\{e^r(w)(\log w)^{(1-r)/r}w^{\alpha-1}\} \tag{2.8}$$

Proofs of (2.1), (2.2), (2.5) and (2.6), being simple, are omitted and (2.3) and (2.7) may be compared with Chandra ([5], proof of (3.2)). The proof of (2.8) runs parallel to the proof of (2.4). Therefore we prove (2.4) only.

Proof of (2.4). For all sufficiently large w, there is a unique number w_1 such that

$$e(w)-e(w_1)=e(w)(\log w)^{-1/r} \tag{2.9}$$

Now,

$$E(w, t)=\left(\int_0^A+\int_A^{w_1}+\int_{w_1}^w\right)\{e(w)-e(x)\}^{r-1}e(x)x^{\alpha-1}\sin t(x+1)\,dx$$
$$=I_1+I_2+I_3, \text{ say.}$$

Then, for large w,

$$I_1=O\{e^{r-1}(w)\}.$$

And, by the second mean value theorem,

$$I_2=\{e(w)-e(w_1)\}^{r-1}e(w_1)w_1^{\alpha-1}\int_{w'}^{w_1}\sin t(x+1)\,dx, \qquad (A<w'<w_1)$$
$$=O\{e^r(w)(\log w)^{-(r-1)/r}w^{\alpha-1}t^{-1}\}, \text{ by (2.9)}$$

Finally,

$$| I_3 | \leqslant \int_{w_1}^{w} \{e(w)-e(x)\}^{r-1}\, e(x)\, x^{\alpha-1}\, dx$$
$$= O\{e^r(w)(\log w)^{-1}\}.$$

Thus collecting the results, we obtain the required result.

3. For the proof of the theorem we require the following lemmas:

Lemma 1. (Hardy and Riesz [3], Theorem 16, p. 29). Let $r'>r>0$. Then $\Sigma a_n \in s(R, \lambda(w), r)$ implies that $\Sigma a_n \in s(R, \lambda(w), r')$.

Lemma 2. The integral

$$J=\int_0^{\pi} t\phi_1(t)\, \frac{\cos(\frac{1}{2}t)}{\sin^3(\frac{1}{2}t)}\, K(w, t)\, dt$$
$$=o\{e^r(w)\},\ w\rightarrow\infty.$$

Proof. $$J=\left(\int_0^{w^{-1}}+\int_{w^{-1}}^{w^{\alpha-1}}+\int_{w^{\alpha-1}}^{\xi}+\int_{\xi}^{\pi}\right) t\phi_1(t)\frac{\cos(\frac{1}{2}t)}{\sin^3(\frac{1}{2}t)}\, K(w, t)\, dt$$
$$=J_1+J_2+J_3+J_4, \text{ say,}$$

where

$$\xi=w^{\alpha-1}(\log w)^{(1-r)/r}.$$

Now, by using (2.5), (2.6) and (2.7) respectively in J_1, J_2 and J_3 and using (1.9), we obtain that

$$J_i=o\{e^r(w)\} \quad (i=1, 2 \text{ and } 3)$$

as $w\rightarrow\infty$. Finally, by using (2.8), we obtain that

$$J_4=o\{e^r(w)\}, \quad w\rightarrow\infty.$$

Thus collecting the results, we obtain that

$$J=o\{e^r(w)\}, \quad w\rightarrow\infty.$$

This completes the proof of Lemma 2.

4. Proof of the Theorem. In view of Lemma 1, it is enough to show that the theorem holds for $0<r<1$.

Writing $T_n(x)$ for the $(C, 1)$— mean of (1.1), we have (see [1], p. 412).

$$T_n(x)=\frac{1}{\pi(n+1)}\int_0^{\pi} \phi(t)\ \frac{\sin^2 \frac{1}{2}(n+1)t}{\sin^2(\frac{1}{2}t)}\ dt+s.$$

Now, if the $(R, e(w), r)\,(C, 1)$— mean of $\{s_n(x)\}$ is denoted by $I(w)$, then

$$I(w)-s=\frac{\alpha r}{e^r(w)}\int_0^w \{e(w)-e(x)\}^{r-1}\, e(x)\, x^{\alpha-1}.$$

$$\left\{\frac{1}{\pi(x+1)}\int_0^\pi \phi(t)\,\frac{\sin^2 \frac{1}{2}(x+1)t}{\sin^2(\frac{1}{2}t)}\,dt\right\}dx$$

$$=\frac{\alpha r\,\phi_1(\pi)}{e^r(w)}\int_0^w \{e(w)-e(x)\}^{r-1}\, e(x)\,\frac{x^{\alpha-1}}{x+1}\cos^2(\tfrac{1}{2}\pi x)\,dx$$

$$+\frac{\alpha r}{\pi\, e^r(w)}\int_0^\pi t\,\phi_1(t)\,\frac{\cos(\frac{1}{2}t)}{\sin^3(\frac{1}{2}t)}\,K(w,t)\,dt$$

$$-\frac{\alpha r}{2\pi\, e^r(w)}\int_0^\pi \frac{t\,\phi_1(t)}{\sin^2(\frac{1}{2}t)}\,E(w,t)\,dt$$

(on integrating by parts)

$$=R_1+R_2-R_3, \text{ say.} \tag{4.1}$$

By Lemmas 1 and 2 respectively, it follows that

$$R_i=o\,(1) \qquad (i=1 \text{ and } 2) \tag{4.2}$$

as $w\to\infty$. And

$$R_3=\frac{\alpha r}{2\pi e^r(w)}\left(\int_0^{w^{-1}}+\int_{w^{-1}}^{w^{\alpha-1}}+\int_{w^{\alpha-1}}^{\xi}+\int_{\xi}^{\pi}\right)\frac{t\,\phi_1(t)}{\sin^2(\frac{1}{2}t)}\,E(w,t)\,dt$$

$$=\frac{\alpha r}{2\pi e^r(w)}\,(R_{3,1}+R_{3,2}+R_{3,3}+R_{3,4}), \text{ say.}$$

Now using (2.1), (2.2) and (2.3), respectively, in $R_{3,1}$, $R_{3,2}$ and $R_{3,3}$ and using (1.9), it follows that

$$R_{3,i}=o\,\{e^r(w)\} \qquad (i=1, 2 \text{ and } 3)$$

as $w\to\infty$. Finally, by using (2.4), we obtain that

$$R_{3,4}=o\,\{e^r(w)\}$$

as $w\to\infty$.

Thus collecting the results, we obtain that

$$R_3=o\,(1) \tag{4.3}$$

as $w\to\infty$ and it follows from (4.1) – (4.3) that

$$I(w)\to s$$

as $w\to\infty$.

This completes the proof of the theorem.

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