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ON THE GENERALIZED FUNCTIONS OF SLOW GROWTH

By

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ABSTRACT

The theory of generalized functions, discussed by Lighthill and others, includes functions which, although not integrable over $(-\infty,\infty)$, are such that $|f(x)|/(1+x^2)^N$ is integrable for some $N > 0$. While the theory of distributions, given by L. Schwartz, does not allow us to include these many functions, the approach in this paper is to include some more functions which can generate the generalized functions of slow growth in the sense of L. Schwartz.

1. INTRODUCTION.

The sequential approach to the theory of generalized functions has been discussed by many authors. Notable among them are Temple ([1] and [2], Lighthill [3] and Jones [4]. These authors introduced the generalized functions via sequences of good functions. Their definition provides a very wide range of generalized functions. It includes all the integrable functions and measures. It also includes

(*) the functions $f(x)$ which, although not integrable over $(-\infty,\infty)$, are such that $|f(x)|/(1+x^2)^N$ is integrable for some $N > 0$, i.e., such that

$$\int_{-\infty}^{\infty} (1+x^2)^{-N} \; |f(x)| \; dx < \infty .$$

On the other hand, Schwartz ([5] and [6]) used a different approach based on the theory of linear topological spaces to define generalized functions. Subsequently, Gel′ fand and Shilov [7] presented a systematic development of the theory in which the generalized functions are defined as follows :

A continuous linear functional $\langle f, \phi \rangle$, defined on some fundamental space ϕ, is said to be a generalized function. For our purpose we take the space S of good functions as the fundamental space. In this case, the generalized function defined is called a generalized functions of slow growth. A regular generalized function $\langle f, \phi \rangle$ is given by

$$\langle f, \phi \rangle = \int_{R^n} f(x)\, \phi(x)\, dx \tag{1}$$

where $f(x)$ is a fixed function (absolutely) integrable in every finite domain (we shall call such functions locally integrable) and the r.h.s. of (1) converges absolutely for every good function $\phi(x)$ belonging to S.

This definition, according to Gel' fand and Shilov ([8], Ch. II, (1.5)) includes only those locally integrable functions $f(x)$ which have growth not higher than a power type at infinity, i. e.,

$$|f(x)| \leqslant C(1+|x|)^k \tag{2}$$

for some $k \geqslant 0$.

The condition (2) for absolute convergence of (1) has been stated as the necessary and sufficient condition by Gel' fand and Shilov [8] and does not allow to include as many functions to generate the generalized functions as Temple's and Lighthill's approach provides. The present paper shows that the condition (2) is not true as such, and as a matter of fact, we can include some more functions to generate Schwartz generalized functions.

2. The statement regarding the condition (2) as given by Gel' fand and Shilov is not true completely. In fact, it is only a sufficient condition and not a necessary one. A counter example is provided as under :

Example. Consider a sequence of intervals $\{\delta_n\}$, where

$$\delta_n = \left(n - 2^{-(n+1)} e^{-n},\ n + 2^{-(n+1)} e^{-n}\right)$$

on R^1. We define the function f(x) by

$$f(x) = \begin{cases} e^n, & x \in \delta_n, \ n = 1,2,3,\ldots \\ 0, & x \notin \delta_n \end{cases}$$

Clearly, f(x) is locally integrable on R^1 and allows r.h.s. of (1) to converge absolutely for every good function ϕ belonging to S, while it does not satisfy the condition (2). In this example, the function f(x) itself implies the existence of regular generalized function.

We now consider a function of the type (*), that is, a function f(x) in the ordinary sense of classical analysis which although not integrable over $(-\infty, \infty)$, is such that $|f(x)|/(1+x^2)^N$ is integrable for some $N > 0$. Then since for any ϕ in S (R^n) and for some constant $A < \infty$. we have

$$|(1+x^2)^N \phi(x)| < A,$$

we can write

$$f(x)\phi(x) = h(x)\ (1+x^2)^N\ \phi(x);$$

where $h(x) = f(x)/(1+x^2)^N$. Thus $f(x)\phi(x)$ is integrable and allows (2) to converge absolutely.

The functional so obtained is continuous, too. For

$$\lim_{\alpha\to\infty} \phi_\alpha(x) = 0 \text{ in } S(R^n)$$

implies

$$\lim_{\alpha\to\infty} \langle f, \phi_\alpha \rangle = \lim_{\alpha\to\infty} \int_{R^n} h(x)\,(1+x^2)^N\, \phi_\alpha(x)\, dx = 0.$$

The above discussion can be summarized as the following theorem :

Theorem 1. *If* f(x) *is a function of x (in the ordinary sense) not integrable in* $(-\infty, \infty)$ *and is such that there exists a positive number* N *such that*

$$(3) \qquad \int_{R^n} (1+x^2)^{-N}\ |f(x)|\ dx < \infty$$

then the function f(x) *defined by*

$$\langle f, \phi \rangle = \int_{R^n} f(x)\phi(x)\ dx,$$

for all $\phi(x)$ *in* $S(R^n)$, *is a regular generalized function of slow growth in the sense of L. Schwartz.*

By this precedure, we can increase the number of generalized functions available to us, by using not only the ordinary functions such that (3) is finite but also the new generalized functions which can be obtained by differentiation in accordance with the rule

$$\int_{-\infty}^{\infty} f^{(n)}(x)\,\phi(x)\,dx = (-1)^n \int_{-\infty}^{\infty} f(x)\phi^{(n)}(x)\,dx,$$

$\phi(x)$ belongs to $S(R^n)$.

It is now a simple matter to prove.

Theorem 2. *If* $f(x)$ *is an ordinary differentiable function such that both* $f(x)$ *and its derivative* $f'(x)$ *satisfy* (3), *then the derivative of the generalized function generated by* $f(x)$ *is identical with the generalized function generated by* $f'(x)$.

REMARK. Theorem 1 gives a sufficient condition for an ordinary function to define a generalized function. If f(x) is a locally integrable function, then condition (2) itself implies the assumption of Theorem 1, but the converse is not true.

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REFERENCES

[1] G. Temple, Theory and application of Generalized Functions, J. London Math. Soc. 28 (1953), 134—148.

[2] G. Temple, The theory of Generalized functions. Proc. Roy. Soc. Ser. A, 228(1955), 175—190.

[3] M. J. Lighthill, An Introduction to Fourier Analysis and generalized functions, Cambridge University, London, 1958.

[4] D. S. Jones , Generalized Functions, McGraw-Hill, New York, 1966.

[5] L. Schwartz, Theorie des Distributions, Vol. 1, Hermann, Paris, 1950.

[6] L. Schwartz, Theorie des Distributions, Vol. 2, Hermann, Paris, 1951.

[7] I.M. Gel'fand and G.E. Shilov, Generalized Functions, Vol. 1, Academic Press, 1964.

[8] I.M. Gel'fand and G. E. Shilov, Generalized Functions, Vol. 2, Academic Press, 1968.