Jñānabha, Vol. 7, 1977

Dedicated to Professor Charles Fox (1897–1977)

DIFFERENT TYPES OF CONVEXITIES IN METRIC LINEAR SPACE

By

T. D. NARANG

Department of Mathematics,
University of Delhi, Delhi--**11007**, *India*

(Received : August 24, 1976)

In the literature, different types of convexities of the norms are available in normed linear spaces—uniform convexity [3], uniform convexity in every direction [4], locally uniformly convex [5], locally uniformly convex near a point [5], weakly locally uniformly convex [5] and strict convexity [2]. In [1], the notions of uniform convexity and strict convexity weie extended to metric linear spaces and some relationships between these two types of convexities were established. In this paper, the notions of some other types of convexities are extended to metric linear spaces and some relationships between them have been established. Also some questions concerning the relationships between these convexities have been raised. We start with the following definition :

Definition 1. A metric linear space (X, d) is said to be *strictly convex* [1] if

$$d(x, 0) \leqslant r,\ d(y, 0) \leqslant r \text{ imply } d\left(\frac{x+y}{2},\ 0\right) < r$$

unless $x=y$; $x, y \in X$ and r is any positive real number.

Definition 2. A metric linear space (X, d) is said to be *uniformly convex* [1] if to each pair of positive number (ε , r) there corresponds a positive number δ such that if x and y lie in X with $d(x, y) \geqslant \varepsilon$, $d(x, 0) < r+\delta$, $d(y, 0) < r+\delta$ then $d\left(\frac{x+y}{2}, 0\right) < r$.

The following theorem was proved in [1].

Theorem 1. (a) Every uniformly convex metric linear space is strictly convex.

(b) Every totally complete strictly convex metric linear space is uniformly convex (A metric linear space is said to be totally complete [1] if its metrically—bounded closed sets are compact).

The concept of uniform convexity in a metric linear space is based on the geometric condition that if two members of the sphere are far apart then their mid-point is well inside the sphere. We consider here a generalization of this concept whose geometric significance is that the collection of all chords of the sphere that are parallel to a fixed direction and whose lengths are bounded below by a positive number has the property that the mid-points of the chords lie uniformly deep inside the sphere.

Definitions 3. A metric linear space (X, d) is said to be *uniformly convex in every direction* (UCED) if, for every non-zero member z of X and $\varepsilon > 0$, there exists $\delta > 0$ such that $|\alpha| < \varepsilon$ if $d(x, 0) < r+\delta$, $d(y, 0) < r+\delta$, $x-y=\alpha z$ and $d\left(\frac{x+y}{2}, 0\right) \geqslant r$.

Does UCED implies strict convexity ? Of course converse is not true. For example, the space C[0, 1] of all continuous functions on [0, 1] with the metric

$$d(f, 0) = \sup \{ |f(t)| + (\int_0^1 |f(t)^2| \, dt)^{1/2} \}$$

is strictly convex, but not UCED.

Next, we consider another weaker form of uniform convexity, called local uniform convexity. Geometrically, this differs from uniform convexity in that it is required that one end point of the variable chord remain fixed.

Definition 4. A metric linear space (X, d) is said to be *locally uniformly convex* if given $\varepsilon > 0$ and an element x with $d(x, 0) \leqslant r$, there exists $\delta(\varepsilon, x) > 0$ such that

$$d\left(\frac{x+y}{2}, 0\right) < r \text{ whenever } d(x, y) \geqslant \varepsilon \text{ and } d(y, 0) < r+\delta.$$

The following theorem shows that uniform convexity implies local uniform convexity and local uniform convexity implies strict convexity.

Theorem 2. (a) Every uniformly convex metric linear space is locally uniformly convex.

(b) Every locally uniformly convex metric linear space is strictly convex.

Proof. (a) is an immediate consequence of the two definitions.

(b) Let (X, d) be a locally uniformly convex metric linear space and $x, y \in X$ be such that $d(x, 0) \leqslant r$, $d(y, 0) \leqslant r$.

Let $d\left(\frac{x+y}{2}, 0\right) \geqslant r$. Then for any positive ε, the definition of local uniform convexity implies $d(x, y) < \varepsilon$ and hence $x=y$.

To illustrate that local uniform convexity is stronger than strict convexity, consider the following example :

Let C[0, 1] denotes the space of all real continuous functions on [0, 1] with

$$d(f, 0) = \max_{0 \leqslant t \leqslant 1} |f(t)|$$

Let $<t_n>$ be dense sequence of points in [0, 1] which does not include 0. Define a new metric d_1 on [0, 1] as

$$d_1(f, 0) = \left[\{d(f, 0)\}^2 + \sum_{n=1}^{\infty} \frac{1}{2^n} \; |f(t_n)|^2 \right]^{\frac{1}{2}}.$$

Then it can be seen that d_1 is strictly convex but not locally uniformly convex.

We raise the following problem :

Problem 1. Under what conditions strict convexity will imply local uniform convexity ? Of course, Theorem 1 (b) gives one such set of conditions, but can we improve upon Theorem 1 (b) for this case.

We next give another weaker form of uniform convexity.

Definition 5. A metric linear space (X, d) is said to be *locally uniformly convex near a point* x_0 if there is a sphere about x_0 in which the oondition for uniform convexity holds.

Geometrically, this differs from uniform convexity in that the variable chord of the sphere is contained in a sphere about the point x_0, whereas local uniform convexity requires only that one end point of the chord remained fixed.

Problem 2. How is this concept of local uniform convexity near a point related to other types of convexities ?

REFERENCES

[1] G. C. Ahuja, T. D. Narang and Swaran Trehan, Best approximation on convex sets in metric linear spaces, *Math. Nachr.* (to appear).

[2] E. W. Cheney, *Introduction to Approximation Theory,* McGraw-Hill, New York, 1966.

[3] J. A. Clarkson, Uniformly convex spaces, *Trans. Amer. Math. Soc.* **40** (1936), 396-414.

[4] M.M.Day, R.C. James and S. Swaminathan, Normed linear spaces that are uniformly convex in every direction, *Canad. J. Math.* **23** (1971), 1051-1059.

[5] A. R. Lovaglia, Locally uniformly convex banach spaces, *Trans. Amer. Math. Soc.* **78** (1955), 225-238.