ON SUMMATION OF LAGUERRE SERIES BY RIESZ LOGARITHMIC MEANS

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In the present paper we prove the strong logarithmic summability of Laguerre series at the point \( x=0 \).

1. Introduction

A series \( \sum_{n=0}^{\infty} a_n \) with its partial sum \( S_\nu \) is said to be summable by Riesz logarithmic means or strongly summable by logarithmic means with index one denoted as \( (R, \log n, 1) \) summable to \( S \) if

\[
\frac{1}{\log n} \sum_{\nu=0}^{n} \frac{|S_\nu - S|}{\nu + 1} = o(1), \quad \text{as } n \to \infty.
\]

The Laguerre series associated with a Lebesgue measurable function \( f(x) \) in the interval \( [0, \infty) \) is given by

\[
f(x) = \sum_{n=0}^{\infty} a_n L_n^{(\alpha)}(x), \quad \alpha > -1
\]

where

\[
a_n = \frac{\Gamma(\alpha + 1)}{\Gamma(n + \alpha)} \left( \frac{n + \alpha}{n} \right)^{-1} \int_{0}^{\infty} e^{-y} y^\alpha f(y) L_n^{(\alpha)}(y) \, dy
\]

and \( L_n^{(\alpha)}(x) \) are Laguerre polynomials of order \( \alpha > -1 \), defined by the generating function

\[
(1) (1-w)^{\alpha} \exp \left( \frac{-xw}{1-w} \right) = \sum_{n=0}^{\infty} w^n L_n^{(\alpha)}(x); \quad |w| < 1.
\]
existence of the integral (1.2) is being assumed.

We write throughout the paper

\begin{equation}
\phi(y) = \{\Gamma(a+1)\}^{-1} e^{-y} y^a [f(y)-A], A \text{ is constant.}
\end{equation}

Cesaro summability of (1.1) has been studied by Szego [2, p. 245] and Pandey [1]. The following theorem is due to Pandey [1]:

**THEOREM A.** If \( a > -1, \beta > 0 \),

\begin{equation}
\Phi(t) \equiv \int_0^t |\phi(y)| \, dy = o \left( t^{a+1} \right), \text{ as } t \to 0,
\end{equation}

\begin{equation}
\int_{3n}^{3n+1} e^{-y/2} y^{-a-\beta/2-1} |\phi(y)| \, dy = o \left( n^{\beta/2} \right)
\end{equation}

and

\begin{equation}
\int_{3n}^{3n+1} e^{-y/2} y^{-a-\beta-5/6} \, dy = o \left( \log n \right), \text{ as } n \to \infty.
\end{equation}

then the series (1.1) is summable \((C, a+\beta+\frac{1}{2})\) at \( x=0 \), to the sum \( S \).

2. The object of the present paper is to study the series (1.1) at \( x=0 \) with respect to the Riesz logarithmic summability of order one. We establish the following

**Theorem.** If for \(-1 < a < -\frac{1}{2}\)

\begin{equation}
\Phi(t) \equiv \int_t^w \frac{|\phi(y)|}{y^{a/2+3/4}} \, dy = o \left( \log \frac{1}{t} \right), \text{ as } t \to 0
\end{equation}

\begin{equation}
\int_n^{n+1} e^{y/2} y^{-a/2-3/4} \, dy = o \left( \log n \right)
\end{equation}

and

\begin{equation}
\int_n^{n+1} e^{y/2} y^{-1/3} \, dy = o \left( \log n \right), \text{ as } n \to \infty
\end{equation}
then the series (1.1) is summable \((R, \log n, 1)\) to the sum \(S\), at \(x = 0\).

3. To prove our theorem we need the following lemmas.

**Lemma 1.** [2, p. 175]. If \(a\) be arbitrary and real, \(c\) and \(w\) are fixed positive constants, \(n \to \infty\), then

\[
L_n^{(a)}(x) = \begin{cases} 
2^{a/2-1/4} O\left(n^{a/2-1/4}\right); c/n \leq x \leq w, \\
O\left(n^a\right); 0 \leq x \leq c/n.
\end{cases}
\]

**Lemma 2.** [2, p. 239]. If \(\alpha\) and \(\lambda\) be arbitrary and real, \(\lambda > 0, 0 < \eta < 4\), then for \(n \to \infty\), we have

\[
\max \ e^{-x/2} x^\lambda | L_n^{(a)}(x) | \sim n^Q
\]

where

\[
Q = \begin{cases} 
\max \{-(\lambda-1/2), (\alpha/2-1/4); w \leq x \leq (4-\eta)n, \\
\max \{-(\lambda-1/3), (\alpha/2-1/4); x \geq w.\}
\end{cases}
\]

4. **Proof of the Theorem.** we have [2, p. 269],

\[
S_\nu(0) = \sum_{m=0}^{\nu} a_m L_m^{(a)}(0) = \{\Gamma(\alpha+1)\}^{-1} \sum_{m=0}^{\nu} e^{-y} y^\alpha \int_{\mathbb{R}} L_m^{(a)}(y) dy
\]

\[
= \{\Gamma(\alpha+1)\}^{-1} \int_{0}^{\infty} e^{-y} y^\alpha \left[ f(y) - S \right] L_v^{(\alpha+1)}(y) dy.
\]

By using orthogonal property of Laguerre polynomials we obtain

\[
S_\nu(0) - S = \{\Gamma(\alpha+1)\}^{-1} \int_{0}^{\infty} e^{-y} y^\alpha \left[ f(y) - S \right] L_v^{(\alpha+1)}(y) dy
\]

\[
= \int_{0}^{\infty} \phi(y) L_v^{(\alpha+1)}(y) dy
\]

Therefore
\[
\frac{1}{\log n} \sum_{\nu = 0}^{n} \frac{|S_\nu \rightarrow S_\nu|}{\nu + 1} = \frac{1}{\log n} \int_0^\infty \phi(y) \sum_{\nu = 0}^{m} \frac{L_\nu^{(a+1)}(y)}{\nu + 1} \, dy
\]

\[
= \int_0^{c/n} + \int_{c/n}^w + \int_{n}^w + \int_n^\infty
\]

(4.1) \quad = I_1 + I_2 + I_3 + I_4, \text{ say.}

Using second condition of lemma 1, we have

\[
I_1 = O \left( \frac{1}{\log n} \right) \int_0^{c/n} |\phi(y)| \left\{ \sum_{\nu = 0}^{n} \frac{\nu \alpha + 1}{\nu + 1} \right\} \, dy
\]

\[
= O \left( \frac{n^{\alpha + 1}}{\log n} \right) \phi(y) \left( n^{-a/2 - 3/4} \log n \right) \quad \text{[by (1.2)]}
\]

(4.2) \quad = o(1), \text{ as } n \to \infty.

Next using the first condition of Lemma 1, we have

\[
I_2 = O \left( \frac{1}{\log n} \right) \int_{c/n}^w |\phi(y)| y^{-a/2 - 3/4} \left\{ \sum_{\nu = 0}^{n} \frac{\nu \alpha/2 + 1/4}{\nu + 1} \right\} \, dy
\]

\[
= O \left( \frac{1}{\log n} \right) \left[ \int_{c/n}^w \frac{|\phi(y)|}{y^{a/2 + 3/4}} \, dy \right] \quad \text{[by (2.1)]}
\]

(4.3) \quad = o(1), \text{ as } n \to \infty.

Now
Using the first condition of Lemma 2, we have

$$I_3 = \mathcal{O}\left( \frac{1}{\log n} \right) \int^n e^{y/2} y^{-\alpha/2-3/4} |\phi(y)| dy$$

$$= \mathcal{O}\left( \frac{1}{\log n} \right) \mathcal{O}(1) \cdot o\left( \log n \right)$$

(by 2.2)

(4.4) $= o(1)$, as $n \to \infty$.

Finally

$$I_4 = \mathcal{O}\left( \frac{1}{\log n} \right) \int^n e^{y/2} y^{-\alpha/2-5/6} |\phi(y)| dy$$

$$= \mathcal{O}\left( \frac{1}{\log n} \right) \mathcal{O}(1) \cdot o\left( \log n \right)$$

(by 2.3)

(4.5) $= o(1)$, as $n \to \infty$.

Thus collecting the results (4.1) to (4.5), we see that
\[
\frac{1}{\log n} \sum_{v=0}^{n} \frac{|S_v - S|}{v+1} = o(1), \text{ as } n \to \infty.
\]

Hence the Theorem is proved.

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REFERENCES
