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CHARLES FOX : A MATHEMATICIAN I HAVE KNOWN *

By

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Charles Fox was born on March 17, 1897, in London, England. His parents were Morris and Fanny Fox. He began his education at Coopers Company School in Bow (London), and at the City of London School, where he was a scholar. In 1915 he won a Mathematical Scholarship to Cambridge University, and he went into residence at Sidney Sussex College. He took Part I of the celebrated Mathematical Tripos in 1916 with First Class Honours.

He secured a First Class Honours also in Part II of the Mathematical Tripos, in the year 1917. That same year he joined the British Expeditionary Forces in France, during World War I, and was wounded in action in 1918. He then returned to Cambridge and completed his B. A. His mathematical studies were carried on under such eminent mathematicians as Professor G. H. Hardy.

Fox began his professional career in 1919 as a Demonstrator and Lecturer in Mathematics at the Imperial College of Science in London. The following year he joined Birkbeck College of the University of London as a Lecturer in Mathematics, where he remained untill 1948. During his long association with Birkbeck College for twenty-eight years, a number of important developments took place in Fox's professional as well as personal life. In 1922 he obtained his Cambringe M. A. degree, in 1923 he was elected a Fellow of the Cambridge Philosophical Society, and in 1925 he published his first paper in the *Proceedings of the London Mathematical Society*. Three years later, in 1928, he was awarded the D. Sc. degree by the University of London.

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In 1932 he married Eileen Kaye in London, and in 1949 they moved to Montreal, Canada, where he joined the faculty of McGill University as an Associate Professor of Mathematics. He was very happy in his family life. Two children were born; a son, Edward, and a daughter, Frances, and there are seven grandchildren.

Within a year after his arrival at McGill University, Fox finished his only book : *An Introduction to the Calculus of Variations*, which was published by Oxford University Press. His motivation for writing this book is fairly well reflected in the second paragraph of its preface, which reads : "*...During my many years of teaching at London University I felt that none of the existing texts covered the subject as I would like to teach it and so I undertook the task of writing one of my own...*,"

Fox was promoted to the rank of Professor of Mathematics at McGill University in 1956. Five years later he was elected a Fellow of the Royal Society of Canada in 1961. Indeed, he was a member of the London Mathematical Society, the Canadian Mathematical Congress, and the American Mathematical Society.

Upon his retirement from McGill University in 1967, Fox accepted a visiting professorship at Sir George Williams University (now Sir George Williams Campus of Concordia University), also in Montreal. Thus he remained an active teacher of mathematics until 1975, when he reluctantly gave up his lectureship at Concordia University. The following year he was awarded an honorary LL.D. degree by Concordia University.

He passed away (in Montreal) on April 30, 1977 of a cardiac arrest. He had had a heart condition for quite a long time, which slowed him down considerably but did not stop his activities altogether till his last year or so.

Fox's publications span over a period of half a century, his first paper having appeared in 1925, while the last one was published a couple of years before his death. His papers may be divided into six main (not mutually exclusive) groups :

(i) Theory of null series and null integrals (*cf.*[1] and [2]);

(ii) Hypergeometric functions and their generalizations (*cf.* [3], [5], [7] and [32]);

(iii) Integral transforms and integral equations (*cf.* [4], [8], [12], [14], [16], [18], [22] through [25], [27] through [29], [31] through [39], [41], and [42]);

(iv) Mathematics of navigation † (*cf.* [20] and [43]);
(v) Theory of statistical distributions (*cf.* [29] and [40]);
(vi) Notes on miscellaneous topics (*cf.* [6], [9], through [11], [13], [15], [17], [19], [21], [26], [28] and [30]).

It seems worthwhile to remark that Fox's publication record shows a trend comparable with that of many other mathematicians. His book and some of his most significant papers were written in the prime of his career, while both the number and importance of his research papers decreased towards the end of his career. He did indeed continue his research activities after his retirement from McGill University and, as indicated earlier, was producing interesting papers until a couple of years before his death.

One of Fox's major contributions to the theory of hypergeometric functions is his systematic study of the asymptotic expansion of the generalized hypergeometric function defined by (*cf.* [5], p. 389 *et seq.*)

$$(1)\quad {}_p\overset{*}{F}_q\,[(\alpha_1, A_1), \ldots,(\alpha_p, A_p); (\beta_1, B_1),\ldots,(\beta_q, B_q);z] = \sum_{n=0}^{\infty} \frac{f(n)z^n}{n!},$$

where, for convenience,

$$(2)\quad f(t)=\left\{\prod_{j=1}^{p}\Gamma(\alpha_j + A_j t)\right\}\left\{\prod_{j=1}^{q}\Gamma(\beta_j + B_j t)\right\}^{-1},$$

and the coefficients $A_1,\ldots,A_p$, $B_1,\ldots,B_q$ are positive real numbers such that

$$(3)\quad \omega = 1 + \sum_{j=1}^{q} B_j - \sum_{j=1}^{p} A_j > 0$$

Although a substantially more general integral function than (1) was studied earlier by G.N. Watson in 1913, Fox's methods were an improvement over that of E. W. Barnes (who, in 1907, had discussed the asymptotic expansions of the generalized hypergeometric function (1) in the familiar special case when $A_j=1$, $j=1, .., p$, and $B_j=1$, $j=1,\ldots,q$) and

† These references [20] and [43] have just been reproduced from Professor Fox's list of publications, which was kindly supplied to the author by Mrs. Eileen Fox.

differed from those of Watson mainly in that no appeai was made to the properties of certain inverse factorial series. It should be mentioned that Frx's methods were further generalized by E. M. Wright in 1935 (and again in 1940) in order to cover the case of the integral function (1) when

(4) $$|\arg(-z)| \leqslant \pi - \tfrac{1}{2}\pi\varepsilon,$$

where $0 < \varepsilon \leqslant 2$ and ε need not be rational; in the subsequent literature, therefore, the integral function (1) is quite often refesred to as Wright's generalized hypergeometric function !

The contribution to the theory of special functions by which Fox will always be remembered by workers in these areas of applicable analysis is, beyond any manner of doubt, his paper [32] in which he formally introduced the *H*-function defined by [*op. cit.*, p. 408].

(5) $$H^{m,n}_{p,q}\left[z \,\middle|\, \begin{matrix} (\alpha_1,A_1), \ldots,(\alpha_p, A_p) \\ (\beta_1,B_1),\ldots,(\beta_q, B_q) \end{matrix} \right] = \frac{1}{2\pi i}\int_L \theta(\zeta)\, z^{\zeta}\, d\zeta,$$

where

(6) $$\theta(\zeta) = \frac{\prod_{j=1}^{m}\Gamma(\beta_j \quad B_j\,\zeta)\ \prod_{j=1}^{n}\Gamma(1-\alpha_j + A_j\,\zeta)}{\prod_{j=m+1}^{q}\Gamma(1-\beta_j+B_j\zeta)\ \prod_{j=n+1}^{p}\Gamma(\alpha_j - A_j\,\zeta)},$$

$$0 \leqq m \leqq q,\ 0 \leqq n \leqq p.$$

and L is a suitable contour of the Mellin-Barnes type (in the complex ζ-plane) which separates the poles of one product from those of the other. If the positive coefficients $A_1,\ldots,A_p$ and $B_1,\ldots,B_q$ are constrained by the inequality

(7) $$\Omega = \sum_{j=1}^{n} A_j - \sum_{j=n=1}^{p} A_j + \sum_{j=1}^{m} B_j - \sum_{j=m+1}^{q} B_j > 0,$$

then, under certain additional conditions, the integral in (5) is absolutely convergent and defines the *H*-function, analytic in the sector

(8) $$|\arg(z)| < \tfrac{1}{2}\,\Omega\,\pi.$$

the point $z=0$ being tacitly excluded.

The *H*-function may be looked upon as an appropriate further extension of the generalized hypergeometric function defined by (1); it also provides an elegant generalization of T. M. MacRobert's *E*-funetion and C.S. Meijer's *G*-function, both of which evidently correspond to the special case of (5) when $A_j=1, j=1,\ldots,p$. and $B_j=1, j=1,\ldots,q$. It may be remarked in passing that a study of one form or the other of the H-function, which was initiated as long ago as 1888 by S. Pincherle, appeared in the works of E. W. Barnes in 1908, H. Mellin in 1910, A. L. Dixon and W. L. Ferrar in 1936, S. Bochner in 1958, and several others. Nonetheless, a first systematic presentation of the properties of the H-function as a symmetrical Fourier kernel was made in the aforementioned 1961 paper by Fox whose name has naturally been associated with this function in the literature ever since.

Fox did not pursue his H-function beyond the invaluable discovery of its properties (as a symmetrical Fourier kernel) incorporated in his paper [32]; instead, he turned to the solution of certain classes of integral equations by operational techniques involving integral transformations. Nevertheless, a large number of research wotkers have since been engaged in the investigation of tke H-function and its natural extensions in two and more complex variables; untll his death Fox encouraged and was appreciative of some of these developments specially in his correspondence and long discussions (on different occasions) with the present author.

And so passes yet anotherable mathematician who, over a remarkably long and exceptionally active life, served his discipline with devotion and distinction. He has left behind him the memory of a quiet family man, one who was an effective teacher of mathematics, one who made significant contributions to his field of expertise, and one who inspired many of his colleagues and students to carry out independent researches for themselves. Charles Fox will indeed be remembered not only for his great intellectual gifts and research contributions, not only for his courtesy and kindliness, but, beyond everything else, for his extraordinary qualities of honesty and integrity.

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