Jñānabha ज्ञानाभ

Volume 6

1976

Published by :
THE VIJNANA PARISHAD OF INDIA
DAYANAND VEDIC POSTGRADUATE COLLEGE
BUNDELKHAND UNIVERSITY
ORAI, U. P., INDIA

JÑĀNABHA

Composed by : Friends Composing House, Agra–282 007, India
Printed at : Modern Printers, Bagh Muzaffar Khan, Agra–282 002, India

Jñānabha Sect. A, Vol. 6, 1976

CLAST PETROGRAPHY : BOULDER CONGLOMERATE STAGE OF UPPER SIWALIK AROUND UDHAMPUR

By **Yash P. Sharda and V. K. Verma**

Department of Geology, University of Delhi,

Delhi–110007, *India*

(Received : June 2, 1976)

Abstract. Murree and Siwalik sediments are well exposed around Udhampur, Jammu and Kashmir State. Some clasts collected at random were studied under petrological microscope. Schistose clasts were found to be dominating followed by sandstones, siltotones and clay and a limestone clast. These petrographic studies directly indicate the nature of distributive province.

Introduction. Murree and Siwalik sediments are well exposed around Udhampur, Jammu and Kashmir State.

The clasts from Boulder Conglomerate Stage of Upper Siwalik ; thirty in number were selected at random and their petrography was studied under microscope. It could throw some light on the source area of Upper Siwalik sediments. Out of the thirty clast samples, seventeen were of different types of schists such as quartz-sericite schist and chlorite schist, eight of sandstones, three of siltstones and shales and one of limestone. The petrography of different rock types is as follows :

Schists. Three different types of schists were encountered during the study of petrography of schistose clasts. These are quartz sericite schist, sericite schist, chlorite schist. The dominant minerals present in these schistose clasts are quartz, sericite, chlorite, plagioclase, mica, tourmaline and some opaque minerals. The schists in which quartz dominates, prefix quartz has been used in that such as quartz-sericite schist. In sericite quartz schist, the sericite dominates over quartz. The porphyroblasts of quartz are dominantly composed of two generations and chlorite gives the schistose structure (Figs. 1, 3). In one of the clasts some carbonaceous material is disseminated through the rock and minor amounts of sphene, calcite and micro-needles of rutile are present. The rocks show typical schistose structure defined by chlorite, muscovite, biotite and sericite in different clasts. In some cases these are at times marked by fine grained carbonaceous matter. The schistosity is further defined by stretched quartz grains showing strain effect.

The major part of schistose rocks in which quartz is dominating is occupied by xenoblastic grains varying in size from 0·002 to 0·03 mm. These quartz grains are flattened in the direction of schistosity. At times the schistosity is crenulated, which might be a later effect. Some times the alignment of quartz grains is oblique to the direction of schistosity. They look as if placed on fine grained sericitic mass, are sometimes augen shaped and when observed closely have sutured contacts indicating recrystallization. Sometimes very fine quartz grains are observed between coarser ones as irregular aggregates which could have recrystallized to coarser quartz grains with the increase in degree of metamorphism. The boundaries of majority of quartz grains are serrated (Fig. 3) and very commonly fine mosaic of quartz occurs at the margins within the coarse quartz grains along the fractures, sometimes as pockets between the coarse quartz grains. It indicates that probably the coarse quartz now present has also developed from finer siliceous material with increase in degree of metamorphism. The strain effect is indicated by the undulose extinction and presence of shadow zones (Figs. 1, 3). The two generations of quartz can be clearly distinguished from each other, the first having some opaque inclusions whereas second is inclusion free. The sutured contacts, stylolitic contacts defined by very fine chlorite needless indicate the role of pressure solutiong during deformation episode (Fig. 4).

A few plagioclase grains having sharp twinning, occur sometimes as flattened parallel to schistosity and sometimes oblique to it. Their size varies from 0·002 to 0·03 mm.

A few grains of tourmaline, sub-idoblastic to xenoblastic, showing pale brown to greenish brown pleochroism have been observed in some of the schistose clasts. The tourmaline xenoblasts show random orientation. In some of the clasts the tourmaline has been replaced by silica, chlorite and sericite. Generally the schistosity wraps around the porphyroblasts of quartz, plagioclase felspar, tourmaline etc. and at places the schistosity abuts against the porphyroblasts of quartz indicating its subsequent recrystallization.

Minor amouts of opaque minerals show the xenoblastic outlines and unoriented nature with respect to schistosity.

The crystallization of sericite to muscovite as also stressfree growth of chlorite (Fig. 2) have also been noticed in some of the clasts.

In one of the clasts, a quartz vein has been observed traversing the rock making a small angle with schistosity. The vein quartz is much coarser then the porphyroblasts present in the rock. The quartz grains present in the vein are highly strained, show irregular and sutured boundaries among themselves. Sometimes fine crystallized quartz occur along the periphery of larger ones showing marginal granulation. The quartz grains present in the vein are flattened at an oblique angle to schistosity exhibiting translation lamellae. A few grains of zircon have also been observed in the quartz vein.

Similar schists have been described by Pascoe (1949) in Salkhala series of Kashmir, which could be probable source of these schist-clasts present in Upper Siwalik Boulder conglomerate in the area.

Sandstones. Most of the sandstone clasts are very similar to each other in petrographical characters. Quartz grains form the major part of framework of grains. These are medium to fine sized, moderately sorted, subangular to subrounded. The framework quartz grains have straight and slightly undulose extinction. The overgrowth on the quartz grains can be observed by the presence of dustline, development of crystalline edges over subrounded quartz grains and occasionally slight variation in optical continuity. The siliceous cement is present in the rock in large quantities. The grains show sutured contacts with each other due to siliceous cements. Some of the quartz grains show etched surfaces and Bohem's lamellae. Most of the quartz grains show sutured contacts with the clay and chloritic matrix present in the rock (Fig. 5). The other cement present is ferruginous. It can be observed between the quartz grains and replacing the matrix. The calcic cement is altogether absent in these clasts. The chert transforming to quartz and replacing clay matrix can be clearly observed indicating that it is later addition to the rock.

A few plagioclase grains have been also observed. The plagioclase grains are fresh and show little alteration to sericite. Some of the plagioclase grains show overgrowth also.

The matrix consists of clay minerals, sericite and chlorite. The chlorite seems to be forming the secondary matrix produced during diagenesis by the reaction of clay and iron-oxides when favourable conditions existed (Sharda, 1975). The clay and chlorite darticles are observed eating into quartz grains in almost all the clasts (Fig. 6). The chlorite matrix replacing the clay and sericite matrix has been noticed.

Detrital muscovite, biotite and chlorite have been observed in the rock Some of these detrital mica and chlorite flakes are bent around the quartz grains showing undulatory extinction and are broken accross the cleavage (Fig. 5). These are bent due to overburden and some of them are bent due to pressure from overgrowing quartz grains. Few biotite grains record different stages of alteration. Zircon and epidote have also been observed.

Authigenic mica flakes developing along the contacts of quartz grains and chlorite in the matrix have been noted.

The rock fragments of schists, phyllites, quartzites, siltstones and some chert fragments have been noticed.

The sandstone clasts show all the stages of diagenesis such as redoxomorphic, locomorphic, and phyllomorphic (Dapples, 1962) as evidenced from the petrography.

The detailed petrography shows that these sandstone clasts are similar to Murree sandstones to a great extent indicating that Murrees have also supplied detritus to Upper Siwalik of this area.

Siltstones and Shales. Some of the clasts can be classed as siltstones and shales on the basis of their petrography. In these quartz grains of every small size form about twenty percent of the bulk. These have been observed floating in the matrix of clay, chlorite and sericite and ferruginous cement. This ferruginous cement is more dominating in the shales where the quartz grains are very small. The quartz grains are rounded and show irregular contacts with the clay and chlorite matrix. Some micaflakes (very tiny) follow a crude parallelism (bedding ?). Small chlorite flakes are also oriented in this direction.

Limestone Clast. In this the clastic grains and cement consist of same minerals. The clastic grains are rounded and turbid. The cement consists of well crystallized calcite showing twinning lamellae and is of micrite type. Some of the cement shows grumus texture also indicating that it partially crystaillized. Presumably these limestones were porous and the cement could have been deposited later during diagenesis shortly after deposition. Some of the detrital grains are in contact with each other whereas others are floating in the calcic cement (Fig. 7). The grains which are in contact with each other show that the cement has been reduced to minimum indicating that pressure and differential solution along the grain boundaries have produced a close packing of original grains before complete cementation (William *et al.*, 1969). Some of the detrital quartz grains showing straight and slightly undulatory extinction have been observed in calcic cement. A few quartz grains are cemented to clastic calcite/dolomite grains. They show highly sutured contacts with calcic cement. In some of the grains the replacement by calcic cement in advanced stages has been observed. Some quartz grains also show the surface etched by calcic cement. But percentage of quartz grains is very low. Cherty bands are also present in these clasts. The quartz in these bands is cryptocrystalline to microcrystalline and replaces calcite/dolomite crystals which are rhomb-shaped. Some quartz veins have been observed in these limestone clasts. The quartz in these veins is also cryptocrystalline. Some of these veins join some vugs present in these limestone clasts. These veins contain prismatic quartz crystals. Some coarsely crystalline carbonate (calcite/dolomite) grains in fine matrix have also been observed. These grains show irregular boundaries indicating replacement by surrounding cryptocrystalline mass.

Petrographically these limestone clasts are quite similar to Jammu limestones (Wadia, 1932) and Sirban Limestones (Sharma and Krishnaswamy, 1969) lying in north of Udhampur area which could have acted as one of the source rocks for Upper Siwalik of Udhampur.

Conclusions. The clast petrographic studies indicate that older rocks such as Murree sediments, Salkhala series of Kashmir and Sirban or Jammu Limestones exposed in north of Udhampur have also supplied detritus during Upper-Siwalik sedimentation in Udhampur area.

REFERENCES

1. Dappies, E C. (1962), Stages of Diagenesis in the development of sandstones, **Bull. Geol. Soc. Amer. 73**, 913-934
2. Pascoe, E. H., (1949). A Mannual of Geology of India and Burma, 3 Vols., Govt. of India Publ , 2130
3. Sharda, Yash P. (1975), Diagenesis in Siwalik Sandstones around Udhampur, Jammu and Kashmir Symp. Sediment, Sedimentation and Sedimentary Envint., Delhi University, 22-30
4. Sharma, M. R. and Krishnaswamy (1969), Petrography of Sirban Limestones of Riasi, Jammu and Kashmir State, India, **J. Sed. Petrol**, **39**, 1023-1031.
5. Wadia, D N. (1932), Tertiary geosyncline of North West Punjab, **Qart. J. Geol. Min. Met. Soc. India, 4**, 70-96
6. William, H., F. J. Turner and C. M. Gilbert (1954), Petrography.

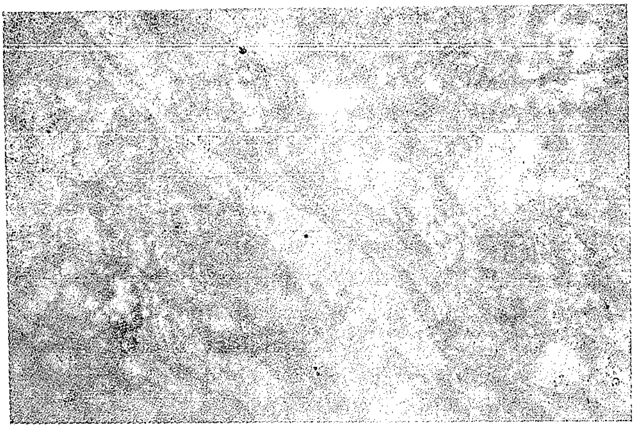

Fig. 1. Photomicrograph showing quartz grain forming pressure shadow zone wrapped up by chlorite altering to biotite. The ground mass is constituted of micro-crystalline quartz grains and sericite in schistose clast (x nicols) (10×8)

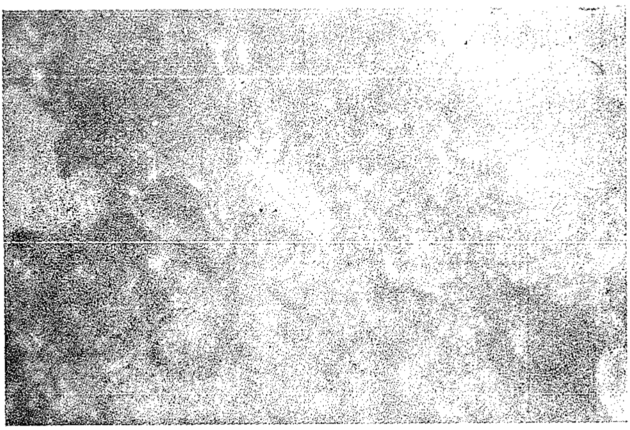

Fig. 2. Photomicrograph showing stressfree growth of chlorite and sericite into radiating fibres. Detrital felspar also shows partial replacement due to pressure solution phenomena. The groundmass consists of microcrystalline quartz in schistose clast (x nicols) (10×8)

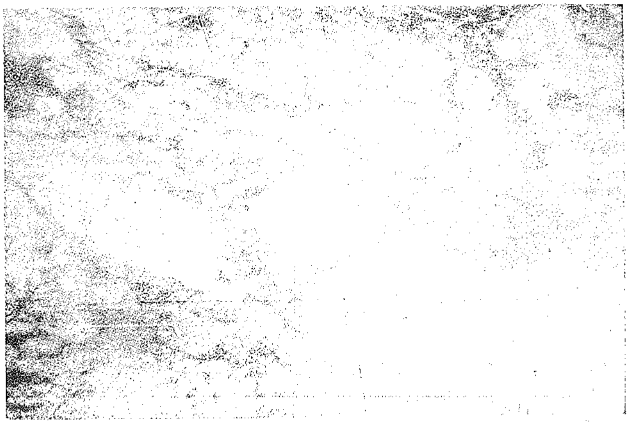

Fig. 3. Photomicrograph showing the schistosity defined by chlorite and biotite flakes wrapping around angular quartz grains being replaced by chlorite along two ends forming pressure shadow zones in schistose clast (x nicols) (10×8)

Fig. 4. Photomicrograph showing sub-angular quartz grains having sutured margins developed due to siliceous cement. The matrix is composed of chlorite and sericite in sandstone clast(x nicols) (10×8)

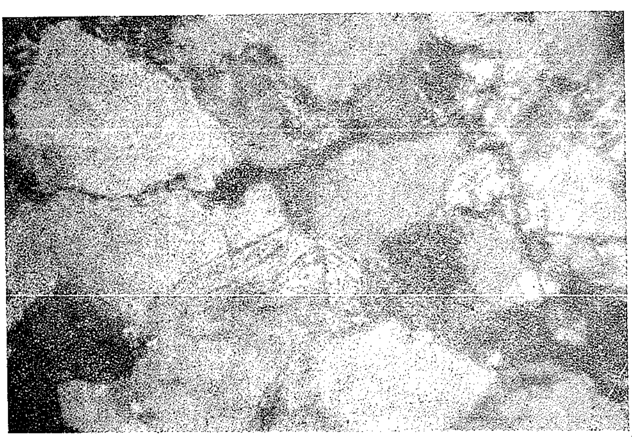

Fig. 5. Photomicrograph showing detrital chlorite flake in the matrix bent due to overgrowing quartz grain in sandstone clast (x nicols) (10×8

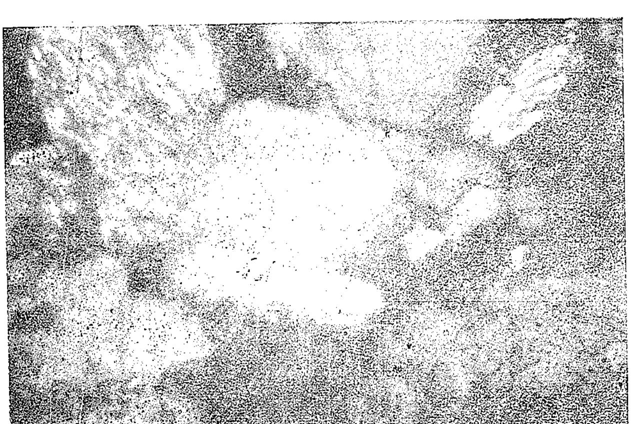

Fig. 6. Photomicrograph showing the chlorite particles eatinginto detrital quartz grain giving rise to embayed boundaries in sandstone clast (x nicols) (25×8)

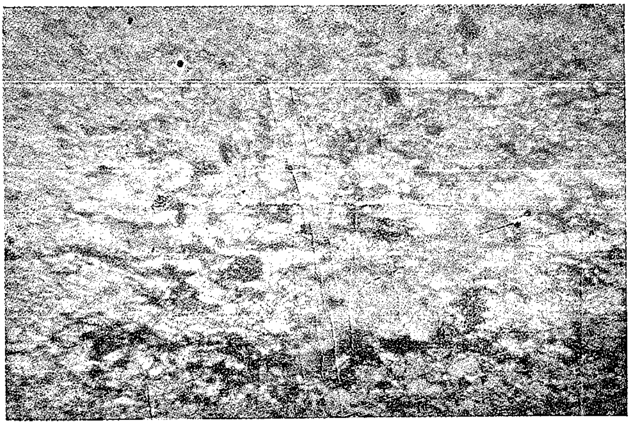

Fig. 7. Photomicrograph showing coarse grained sparitic calcite surrounded by fine grained micrite. The sparry calcite shows some stretching due to deformation in limestone clast (x nicols) (10×8)

Jñānabha Sect. A, Vol. 6, 1976.

ON THE H-FUNCTION

By **V. B. L. Chaurasia**
Deparment of Mathematics,
Malaviya Regional Engineering College,
Jaipur–302004, *India.*

(Received : March 17, 1975 ; in revised form : August 27, 1975)

Abstract. This paper presents some finite integrals involving the H-function. Since the G-function and several other functions are special cases of the H-function, the corresponding known or new integrals for these functions can be obtained from the results presented here.

1. Introduction. The H-function introduced by Fox [5, p. 408] will be represented and defined as follows :

$$(1.1)\qquad H_{p,q}^{m,n}\left[z\left|\begin{matrix}(a_p, e_p)\\(b_q, f_q)\end{matrix}\right.\right]$$

$$=\frac{1}{2\pi i}\int_L \frac{\prod_{i=1}^{m}\Gamma(b_i-f_i s)\prod_{i=1}^{n}\Gamma(1-a_i s+e_i s)z^s}{\prod_{i=m+1}^{q}\Gamma(1-b_i+f_i s)\prod_{i=n+1}^{p}\Gamma(a_i-e_i s)}\,ds,$$

where an empty product is interpreted as unity, $0\leqslant m\leqslant q$, $0\leqslant n\leqslant p$; e's and f's are positive, L is a suitable contour of Barnes type such that the poles of $\Gamma(b_i-f_i s)$, $i=1,\dots, m$ lie on the right-hand side of the contour and those of $\Gamma(1-a_i+e_i s)$, $i=1, \dots, n$ lie on the left-hand side.

Asymptotic expansion and analytic continuation of the H-function have been discussed by Braaksma [2].

For the sake of brevity,

$$\sum_1^n e_i-\sum_{n+1}^p e_i+\sum_1^m f_i-\sum_{m+1}^q f_i\equiv T$$

(a_p, e_p) or $(a_i, e_i)_{1,\,p}$ stands for the sequence

$$(a_1, e_1), \ldots, (a_p, e_p).$$

If one or more of e_i, f_i be assigned zero values, the definition integral may still make sense and the corresponding transformation formulae may be obtained. Thus we have, for example

$$(1.2)\qquad H_{p,\,q}^{m,\,n}\left[z\,\middle|\,\begin{matrix}(A,0),\,(A_i,e_i)_{2,\,p}\\(B_i,f_i)_{1,\,q}\end{matrix}\right]$$

$$=\Gamma(1-A)\,H_{p-1,\,q}^{m,\,n-1}\left[z\,\middle|\,\begin{matrix}(A_i,e_i)_{2,\,p}\\(B_i,f_i)_{1,\,q}\end{matrix}\right],$$

$$p\geqslant n\geqslant 1,\ R(1-A)>0.$$

$$(1.3)\qquad H_{p,\,q}^{m,\,n}\left[z\,\middle|\,\begin{matrix}(A_i,e_i)_{1,\,p-1},\,(A,0)\\(B_i,f_i)_{1,\,q}\end{matrix}\right]$$

$$=\frac{1}{\Gamma(A)}\,H_{p-1,\,q}^{m,\,n}\left[z\,\middle|\,\begin{matrix}(A_i,e_i)_{1,\,p-1}\\(B_i,f_i)_{1,\,q}\end{matrix}\right],$$

$$p-1\geqslant n\geqslant 0,\ R(A)>0.$$

$$(1.4)\qquad H_{p,\,q}^{m,\,n}\left[z\,\middle|\,\begin{matrix}(A_i,e_i)_{1,\,p}\\(B,0),\,(B_i,f_i)_{2,\,q}\end{matrix}\right]$$

$$=\Gamma(B)\,H_{p,\,q-1}^{m-1,\,n}\left[z\,\middle|\,\begin{matrix}(A_i,e_i)_{1,\,p}\\(B_i,f_i)_{2,\,q}\end{matrix}\right],$$

$$q\geqslant m\geqslant 1,\ R(B)>0.$$

$$(1.5)\qquad H_{p,\,q}^{m,\,n}\left[z\,\middle|\,\begin{matrix}(A_i,e_i)_{1,\,p}\\(B_i,f_i)_{1,\,q-1},\,(B,0)\end{matrix}\right]$$

$$=\frac{1}{\Gamma(1-B)}\,H_{p,\,q-1}^{m,\,n}\left[z\,\middle|\,\begin{matrix}(A_i,e_i)_{1,\,p}\\(B_i,f_i)_{1,\,q-1}\end{matrix}\right],$$

$$q-1\geqslant m\geqslant 0,\ R(1-B)>0.$$

We shall require the following results [1]

$$(1.6)\qquad F_4\,[p, s\,;\,1+c,\,1+d\,;\,-xt,\,(1-x)\,t]$$

$$=\sum_{r=0}^{\infty}\frac{(p,r)(s,r)\,t^r}{(1+c,r)(1+d,r)}\,P_r^{(c,\,d)}(1-2x),$$

$$0<x<1,\ |\,t\,|<1.$$

$$(1.7)\qquad \int_0^1 t^{w-1}\,(1-t)^v\,P_r^{(u,\,v)}\,(1-2t)\,H_{p,\,q}^{m,\,n}\left[z\,t^h(1-t)^k\,\middle|\,\begin{matrix}(a_p,e_p)\\(b_q,f_q)\end{matrix}\right]dt$$

$$=\frac{1}{r\,!}\,H_{p+3,\,q+2}^{m+1,\,n+2}$$

$$\left[z\,\middle|\,\begin{matrix}(1-w,h),\,(-v-r,k),\,(a_p,e_p),\,(1+u-w,h)\\(1+u+r-w,h),\,(b_q,f_q),\,(-v-w-r,h+k)\end{matrix}\right],$$

where $R(w)>0$, $R(v)>-1$, $R(w+h\, b_i/f_i)>0$, $R(v+kb_i/f_i)>-1$, $i=1,\dots,m$, $T>0$, $|\arg z|<\frac{1}{2}T\pi$.

To establish (1.7), we express the H-function on left hand side in contour integral as in (1.1), change the order of integration and evaluate the integral thus obtained with the help of a known result [4, p. 284(2)]. Finally, interpreting with the help of (1.1), we arrive at (1.7).

2. The Integrals to be established are :

(2.1) $$\int_0^1 x^{w-1}(1-x)^d\, F_4\,[u, v\,;\, 1+c,\, 1+d\,;\, -xt,\, (1-x)t]$$

$$\times\, H^{m,\, n}_{p,\, q}\left[zx^h(1-x)^k \,\middle|\, \begin{matrix}(a_p,\, e_p)\\(b_q,\, f_q)\end{matrix}\right] dx = \sum_{r=0}^{\infty} \frac{(u,\, r)(v,\, r)\, t^r}{(1+c,\, r)(1+d,\, r)\, r\,!}$$

$$\times\, H^{m+1,\, n+2}_{p+3,\, q+2}\left[z\,\middle|\, \begin{matrix}(1-w,\, h),\, (-d-r,\, k),\, (a_p,\, e_p)(c-w+1,\, h)\\(c+r+1-w,\, h),\, (b_q, f_q),\, (-w-d-r,\, h+k)\end{matrix}\right],$$

where $h,\ k>0$, $R(w)>0$, $R(d)>-1$, $|t|<1$, $R(w+h\, b_i/f_i)>0$, $R(d+k\, b_i/f_i)>-1$, $i=1,\dots,m$, $T>0$, $|\arg z|<\frac{1}{2}T\pi$.

(2.2) $$\int_0^1 x^{w-1}(1-x)^d\, F_4\,[u, v\,;\, 1+c,\, 1+d;\, -xt,\, (1-x)t]$$

$$\times\, H^{m,\, n}_{p,\, q}\left[zx^{-h}(1-x)^{-k} \,\middle|\, \begin{matrix}(a_p,\, e_p)\\(b_q,\, f_q)\end{matrix}\right] dx = \sum_{r=0}^{\infty} \frac{(u,\, r)(v,\, r)t^r}{(1+c,\, r)(1+d,\, r)\, r\,!}$$

$$\times\, H^{m+2,\, n+1}_{p+2,\, q+3}\left[z\,\middle|\, \begin{matrix}(w-c-r,\, h),\, (a_p,\, e_p),\, (1+w+d+r,\, h+k)\\(w,\, h),\, (1+d+r,\, k),\, (b_q, f_q),\, (w-c,\, h)\end{matrix}\right],$$

where $h, k>0$, $R(w)>0$, $R(d)>-1$, $|t|<1$, $R\{w-h(a_i-1)/e_i\}>0$, $R\{d-k(a_i-1)/e_i\}>-1$, $i=1,\,\dots,\,n$, $T>0$, $|\arg z|<\frac{1}{2}T\pi$.

(2.3) $$\int_0^1 x^{w-1}(1-x)^d\, F_4\,[u, v\,;\, 1+c,\, 1+d\,;\, -xt,\, (1-x)t]$$

$$\times\, H^{m,\, n}_{p,\, q}\left[zx^{-h}\,(1-x)^k \,\middle|\, \begin{matrix}(a_p,\, e_p)\\(b_q,\, f_q)\end{matrix}\right] dx = \sum_{r=0}^{\infty} \frac{(u,\, r)(v,\, r)t^r}{(1+c,\, r)(1+d,\, r)\, r!}$$

$$\times\, H^{m+1,\, n+2}_{p+3,\, q+2}$$

$$\left[z\,\middle|\, \begin{matrix}(-d-r,\, k),\, (w-c-r,\, h),\, (a_p,\, e_p),\, (d+w+r+1,\, h-k)\\(w,\, h),\, (b_q, f_q),\, (w-c,\, h)\end{matrix}\right],$$

where $h, k>0$, $R(w)>0$, $R(d)>-1$, $|t|<1$, $R\{w-h(a_i-1)/e_i\}>0$, $R(d+k\, b_i'/f_i')>-1$, $i=1,\dots,n$; $i=1,\dots,m$, $T>0$, $|\arg z|<\frac{1}{2}T\pi$.

$$(2.4)\quad \int_0^1 x^{w-1}(1-x)^d F_4[u, v; 1+c, 1+d; -xt, (1-x)t]$$

$$\times H_{p,q}^{m,n}\left[zx^h(1-x)^{-k}\left|\begin{matrix}(a_p, e_p)\\(b_q, f_q)\end{matrix}\right.\right]dx = \sum_{r=0}^{\infty}\frac{(u,r)(v,r)t^r}{(1+c,r)(1+d,r)\,r!}$$

$$\times H_{p+2,\ q+3}^{m+2,\ n+1}$$

$$\left[z\left|\begin{matrix}(1-w,h),(a_p,e_p),(1+c-w,h)\\(1+c-w+r,h),(1+d+r,k),(a_p,e_p),(-d-w-r,k-h)\end{matrix}\right.\right].$$

where $h, k>0$, $R(w)>0$, $R(d)>-1$, $R(w+h\,b_i/f_i)>0$,
$R\{d-k(a_{i'}-1)/e_{i'}\}>-1$, $i=1,\ldots,m$; $i'=1,\ldots,n$, $T>0$, $|\arg z|<\frac{1}{2}T\pi$

3. Proof. To establish (2.1), multiplying both sides of (1.6) by

$$x^{w-1}(1-x)^d\ H_{p,q}^{m,n}\left[zx^h(1-x)^k\left|\begin{matrix}(a_p, e_p)\\(b_q, f_q)\end{matrix}\right.\right]$$

and integrating it with respect to x between limits 0 to 1, we get

$$(3.1)\quad \int_0^1 x^{w-1}(1-x)^d F_4[u, v; 1+c, 1+d; -xt, (1-x)t]$$

$$\times H_{p,q}^{m,n}\left[zx^h(1-x)^k\left|\begin{matrix}(a_p, e_p)\\(b_q, f_q)\end{matrix}\right.\right]dx$$

$$=\int_0^1\left\{\sum_{r=0}^{\infty}\frac{(u,r)(v,r)t^r}{(1+c,r)(1+d,r)}P_r^{(c,\ d)}(1-2x)\right\}$$

$$\times\ x^{w-1}(1-x)^d\ H_{p,q}^{m,n}\left[zx^h(1-x)^k\left|\begin{matrix}(a_p, e_p)\\(b_q, f_q)\end{matrix}\right.\right]dx.$$

Now on interchanging the order of integration and summation on the right-hand side of (3.1), evaluating the x-integral with the help of (1.7) and finally, intepreting with the help of (1.1), we arrive at the required result (2.1).

The integrals from (2.2) to (2.4) can be obtained in a similar manner with the help of appropriate integrals similar to (1.7).

It may be remarked that (2.2) may be obtained from (2.1) and (2.4) from (2.3) and vice-versa by using known transformation formulae of the H-function.

4. Special Cases. (i) Taking $h \to 0$, in (2.1) and using (1.2), we get

$$(4.1) \quad \int_0^1 x^{w-1} (1-x)^d F_4 [u, v ; 1+c, 1+d ; -xt, (1-x)t] \times H_{p,q}^{m,n}\left[z(1-x)^k \middle| \begin{matrix} (a_p, e_p) \\ (b_q, f_q) \end{matrix} \right] dx$$

$$= \frac{\Gamma(w)}{\Gamma(1+c-w)} \sum_{r=0}^{\infty} \frac{(u, r)(v, r)\, \Gamma(1+c+r-w) t^r}{(1+c, r)(1+d, r)\, r!} \times H_{p+1,q+1}^{m,n+1}\left[z \middle| \begin{matrix} (-d-r, k), (a_p, e_p) \\ (b_q, f_q), (-w-d-r, k) \end{matrix} \right],$$

where $k>0$, $R(w)>0$, $R(d)>-1$, $|t|<1$, $R(k+d\, b_i/f_i)>-1$, $i=1, \ldots, m$, $T>0$, $|\arg z| < \frac{1}{2}T\pi$.

(ii) Substituting $k \to 0$, in (2.1) and using (1.2), we get a result recently obtained by Mital [6].

(iii) Putting $h \to 0$, in (2.2) and using (1.3) and (1.4), we have

$$(4.2) \quad \int_0^1 x^{w-1} (1-x)^d F_4 [u, v ; 1+c, 1+d ; -xt, (1-x)t] \times H_{p,q}^{m,n}\left[z (1-x)^{-k} \middle| \begin{matrix} (a_p, e_p) \\ (b_q, f_q) \end{matrix} \right] dx$$

$$= \frac{\Gamma(w)}{\Gamma(1+c-w)} \sum_{r=0}^{\infty} \frac{\Gamma(1+c+r-w)(u, r)(v, r)\, t^r}{(1+c, r)(1+d, r)\ r!} \times H_{p+1,q+1}^{m+1,n}\left[z \middle| \begin{matrix} (a_p, e_p), (1+w+d+r, k) \\ (1+d+r, k), (b_q, f_q) \end{matrix} \right],$$

where $k>0$, $R(w)>0$, $R(d)>-1$, $|t|<1$, $R\{d-k(a_i-1)/e_i\}>-1$, $i=1, \ldots, n$, $T>0$, $|\arg z| < \frac{1}{2}T\pi$.

(iv) With $k \to 0$, in (2.2) and using (1.4), we obtain

$$(4.3) \quad \int_0^1 x^{w-1} (1-x)^d F_4 [u, v ; 1+c, 1+d ; -xt, (1-x)t] \times H_{p,q}^{m,n}\left[zx^{-h} \middle| \begin{matrix} (a_p, e_p) \\ (b_q, f_q) \end{matrix} \right] dx$$

$$= \sum_{r=0}^{\infty} \frac{(u, r)(v, r)\, \Gamma(1+d+r) t^r}{(1+c, r)(1+d, r)\ r!}$$

$$\times H_{p+1,q+2}^{m+1,n+1}\left[z \middle| \begin{matrix} (w-c-r, h), (a_p, e_p), (1+w+d+r, h) \\ (w, h), (b_q, f_q), (w-c-h) \end{matrix} \right],$$

where $h>0$, $R(w)>0$, $R(d)>-1$, $|t|<1$, $R\{w-h(a_i-1)/e_i\}>0$, $i=1, \ldots, n$, $T>0$, $|\arg z| < \frac{1}{2}T\pi$.

(v) Letting $h=k$, in (2.4) and using (1.5) we get

$$\text{(4.4)} \qquad \int_0^1 x^{w-1}(1-x)^d F_4[u, v; 1+c, 1+d; -xt, (1-x)t] \times H_{p,q}^{m,n}\left[zx^h(1-x)^{-h}\left|\begin{matrix}(a_p, e_p)\\(b_q, f_q)\end{matrix}\right.\right]dx$$

$$=\sum_{r=0}^{\infty}\frac{(u,r)(v,r)\,t^r}{(1+c,r)(1+d,r)\,\Gamma(1+w+d+r)\,r!}$$

$$\times H_{p+2,q+2}^{m+2,n+1}\left[z\left|\begin{matrix}(1-w,h),(a_p,e_p),(1+c-w,h)\\(1+d+r,h),(1+c+r-w,h),(b_q,f_q)\end{matrix}\right.\right],$$

where $h>0$, $R(w)>0$, $R(d)>-1$, $|t|<1$, $R(w+h\,b_i/f_i)>0$, $R\{d-h(a_{i'}-1)/e_{i'}\}>-1$, $i=1, \ldots, m$; $i'=1, \ldots, n$, $T>0$, $|\arg z|<\frac{1}{2}T\pi$.

(vi) Since the G-function and several other functions are special cases of the H-function, the integral for the various other functions can be deduced as special cases of our main results.

ACKNOWLEDGEMENTS. I am thankful to Prof. C.B. Rathie for his guidance in the preparation of this paper and to the University Grants Commission, New Delhi, for financial support.

REFERENCES

1. F. Brafman, Generating function of Jacobi and related polynomials, Proc. Amer. Math. Soc. 2 (1951), 942–949.

2. B. L. J. Braaksma, Asymptotic expansion and analytic continuation for a class of Barnes integral, Compositio Math. 15 (1963), 239–341.

3. T. J. I' A. Bromwich, An introduction to the theory of infinite series, Macmillan and Co., (1965).

4. A. Erdelyi et al., Higher transcendental functions, Vol. II (1954), Mc Graw-Hill, New York.

5. C. Fox, The G and H functions as symmetrical Fourier Kernels, Trans. Amer. Math. Soc., 98 (1961), 395–429.

6. P. K. Mital, Ph. D. Thesis approved by the University of Rajasthan, Jaipur, India (1974).

Jñānabha Sect. A, Vol 6, July 1976.

On Uniform Summability–Λ of Orthonormal Expansions of Functions of Class L^{∞}

By **Syed A. Husain**

Department of Mathematics, University of Wyoming, Laramie, Wyoming 82071, *U.S.A.*

(Received : October 14, 1975)

1. Introduction. Let $\| \lambda_{\nu\mu} \|$ $(\nu, \mu=0, 1, 2, \ldots)$ be an infinite matrix of real numbers. Let $f(x)$ be an integrable function and $\{\phi_\nu(x)\}$, $\nu=0, 1, 2, \ldots$, be an orthonormal system in $[0, 1]$.

Let

$$(1) \qquad f(x) \sim \sum_{\nu=0}^{\infty} c_\nu \phi_\nu(x)$$

where

$$c_\nu = \int_0^1 f(t)\phi_\nu(t)\, dt \ (\nu=0, 1, 2, \ldots),$$

be an orthonormal expansion of $f(x)$ in the system $\{\phi_\nu(x)\}$.

If

$$\Lambda_{\mu, x}(f) = \sum_{\nu=0}^{\infty} \lambda_{\nu\mu} c_\nu \phi_\nu(x)$$

is uniformly covergent in x, for each $\mu=0, 1, 2, \ldots,$

and
$$\lim_{\mu \to \infty} \Lambda_{\mu, x}(f) = \Lambda_x(f)$$

exists uniformly in x, then (1) is said to be uniformly summable–Λ. The Λ–summation method is not necessarily regular.

Now let $\{\lambda_\nu\}$ be an infinite sequence of real numbers. If the series $\sum_{\nu=0}^{\infty} \lambda_\nu c_\nu \phi_\nu(x)$ converges uniformly in x, then $\{\lambda_\nu\}$ is called a sequence of uniform convergence factors of orthonormal expansions of $f(x)$.

If (1) holds, then

$$S_n(f\,;x)=\sum_{\nu=0}^{n}\ \lambda_\nu c_\nu \phi_\nu\ (x)=\int_0^1 f(t)\Big(\sum_{\nu=0}^{n}\ \lambda_\nu\phi_\nu(x)\phi_\nu\ (t)\Big)\ dt$$

$$=\int_0^1\ f(t)K_n\ (x,\,t)\ dt$$

where

$$K_n(x,\,t)=\sum_{\nu=0}^{n}\ \lambda_\nu\phi_\nu(x)\phi_\nu(t).$$

In this paper, the orthonormal system $\{\phi_\nu(x)\}$ is considered closed in C, the class of continuous functions on $[0,\,1]$, and $|\ \phi_\nu(x)\ | < A_\nu$, where the A_ν are independent of x.

In [3] the following theorem on uniform convergence factors is proved.

Theorem 1. The necessary and sufficient conditions that $\{\lambda_\nu\}$ be a sequence of uniform convergence factors of orthonormal expansions of $f(x) \in L^\infty$, are

(2) that there exists an M such that

$$\int_0^1 |\ K_n(x,\,t)\ |\ dt < M$$

for all x and n.

(3) For every $\epsilon > 0$, there exist N and $\eta > 0$ such that for any measurable subset $H \cup [0,\,1]$ of $m(H) < \eta$, we have

$$\left|\int_H (K_m\ (x,\,t)-K_n(x,\,t))\ dt\ \right| < \epsilon$$

for all $m,\,n > N$ and for all x.

We note that if (1) holds, then

$$\Lambda_{n,\,\mu,\,x}\,(f)=\sum_{\nu=0}^{n}\ \lambda_{\nu\mu}c_\nu\phi_\nu(x)=\int_0^1 f(t)\left(\sum_{\nu=0}^{n}\ \lambda_{\nu\mu}\phi_\nu(x)\phi_\nu(t)\ dt\right)$$

$$=\int_0^1 f(t)K_{n,\,\mu}\ (x,\,t)\ dt$$

where

$$K_{n,\,\mu}\,(x,\,t)=\sum_{\nu=0}^{n}\lambda_{\nu\mu}\ \phi_\nu(x)\phi_\nu(t).$$

We also write

$$\Lambda_{\mu,\,x}(f)=\lim_{n\to\infty}\Lambda_{n,\,\mu,\,x}(f)=\sum_{\nu=0}^{\infty}\lambda_{\nu\mu}\,c_{\nu}\,\phi_{\nu}(x)$$

$$=\lim_{n\to\infty}\int_0^1 f(t)\,K_{n,\,\mu}(x,\,t)\,dt.$$

Now if

$$\lim_{n\to\infty}\int_0^1 f(t)K_{n,\,\mu}(x,\,t)\,dt$$

exists for every $f(t)\in L^{\infty}$, then $\{K_{n,\,\mu}(x,\,t)\}$ converges weakly in L (class of summable functions). But since L is weakly complete ([2], p. 240), there exists a $K_{\mu}(x,\,t)\in L$ such that

$$\Lambda_{\mu,\,x}(f)=\lim_{n\to\infty}\int_0^1 f(t)K_{n,\,\mu}(x,\,t)\,dt=\int_0^1 f(t)K_{\mu}(x,\,t)\,dt.$$

2. Main Result.

Theorem 2. The necessary and sufficient conditions for uniform summability-Λ of orthonormal expansions of $f(x)\in L^{\infty}$, are

(4) $\lim_{\mu\to\infty}\lambda_{\nu\mu}$ exists $(\nu=0,\,1,\,2,\,\ldots)$

(5) $\int_0^1\left|K_{n,\,\mu}(x,\,t)\right|dt< M_{\mu}$ $(\mu=0,\,1,\,2,\,\ldots)$

for all x and n.

(6) For every $\epsilon>0$, there exists N and $\eta>0$ such that for any measurable subset $H\subset[0,\,1]$ of $m(H)<\eta$, we have

$$\left|\int_H (K_{m,\,\mu}(x,\,t)-K_{n,\,\mu}(x,\,t))\,dt\right|<\epsilon\ (\mu=0,\,1,\,2,\,\ldots)$$

for all $m,\,n>N$ and for all x.

(7) $$\int_0^1\left|K_m(x,\,t)\right|dt<M$$

for all x and μ.

(8) For every $\epsilon>0$, there exists μ and $\eta>0$ such that for any measurable subset $H\subset[0,\,1]$ of $m(H)<\eta$, we have

$$\left|\int_H (K_{\mu'}(x,\,t)-K_{\mu''}(x,\,t))\,dt\right|<\epsilon$$

for all $\mu',\,\mu''>\mu$ and for all x.

PROOF. In order that $\{\lambda_{\nu\mu}\}$ for each fixed $\mu=0, 1, 2, \ldots$, be a sequence of uniform convergence factors of orthonormal expansions of $f(x) \in L^{\infty}$, by Theorem 1, the conditions (5) and (6) are necessary and sufficient. We first establish the necessity of conditions (4), (7) and (8).

Since

$$\Lambda_{\mu, x}(f)$$

converges uniformly in x, for every $f(t) \in L^{\infty}$, and by hypothesis

$$|\phi_k(t)| < A_k, \text{ independent of } t,$$

therefore,

$$\lim_{\mu \to \infty} \Lambda_{\mu, x}(\phi_k(t)) = \lim_{\mu \to \infty} \int_0^1 \phi_k(t) \left(\sum_{\nu=0}^{\infty} \lambda_{\nu\mu} \phi_{\mu}(x)\, \phi_{\nu}(t) \right) dt$$

$$= \lim_{\mu \to \infty} \lambda_{k\mu}\ \phi_k(x)$$

exists uniformly in x, for every k.

This implies that

$$\lim_{\mu \to \infty} \lambda_{k\mu}$$

exists for every k.

The necessity of condition (4) is established.

Now suppose we contradict (7). Then there exists a sequence of indices $\{\mu_i\}$ and a sequence of points $\{x_i\}$ such that

$$\overline{\lim}_{i \to \infty} \int_0^1 \left| K_{\mu_i}(x_i, t) \right| dt = +\infty . \tag{9}$$

But, by Banach-Steinhaus Theorem ([2], p. 80), for every $f(t) \in L^{\infty}$,

$$\overline{\lim}_{i \to \infty} \int_0^1 f(t) K_{\mu_i}(x_i, t) dt = +\infty \tag{10}$$

which contradicts the uniform convergence of $\Lambda_{\mu, x}(f)$.

Hence the necessity of condition (7).

Now we contradict (8). Then for some $\epsilon > 0$, there exist two increasing sequences of positive integers $\{\mu_i'\}$ and $\{\mu_i''\}$, a sequence of points $\{x_i\}$, and a sequence of sets $\{H_i\}$ with $m(H_i) \to 0$ as $i \to \infty$, such that

$$\left| \int_{H_i} (K_{\mu_i'}(x_i, t) - K_{\mu_i''}(x_i, t))\, dt \right| \geqslant \epsilon . \tag{11}$$

Now put $K_{\mu_i'}(x_i, t)-K_{\mu_i''}(x_i, t)=\alpha_i(t)$.

Since

$$\lim_{\mu\to\infty}\Lambda_{\mu, x}(f)$$

exists uniformly in x, it follows that

$$\lim_{i\to\infty}\int_0^1 f(t)\,\alpha_i(t)dt=0$$

for all $f(t)\in L^\infty$.

Therefore, by the Theorem of Lebesgue ([2], pp. 7-8), *viz.*, for every $\epsilon>0$, there exists an $\eta>0$ such that for any subset $H\subset[0, 1]$ of $m(H)<\eta$, we have

$$\left|\int_H \alpha_i(t)\,dt\right|<\epsilon \quad (i=1, 2, 3,\ldots)$$

which contradicts (11). Hence the necessity of condition (8).

Now it remains to show the sufficiency of conditions (4), (7) and (8). Let $\epsilon>0$, then by Luzin's Theorem, for every $f(t)\in L^\infty$, there exist $h(t)\in C$ and an $\eta>0$ such that

(12) $$|f(t)-h(t)|<\epsilon/2,$$

except in a set $H\subset[0, 1]$, $m(H)<\eta$.

Since $\{\phi_\nu(t)\}$ is closed in C, there exists $g(t)=\sum_{\rho=0}^{k}\gamma_\rho\,\phi_\rho(t)$ such that

(13) $$|h(t)-g(t)|<\epsilon/2$$

in $[0, 1]$.

Combining (12) and (13), we get

(14) $$|f(t)-g(t)|<\epsilon$$

for all $t\in[0, 1]-H$, where $m(H)<\eta$.

Above we note that

(15) $$\Lambda_{\mu, x}(g)=\lim_{n\to\infty}\Lambda_{n, \mu, x}(g)$$

$$=\lim_{n\to\infty}\int_0^1\Big(\sum_{\rho=0}^{k}\gamma_\rho\,\phi_\rho(t)\Big)\Big(\sum_{\nu=0}^{n}\lambda_{\nu\mu}\phi_\nu(x)\phi_\nu(t)\Big)\,dt$$

$$=\sum_{\rho=0}^{k}\gamma_\rho\,\lambda_{\rho\mu}\,\phi_\rho(x).$$

Now consider

$$(16)\quad \left|\Lambda_{\mu', x}(f)-\Lambda_{\mu'', x}(f)\right|$$

$$=\left|\int_0^1 f(t)\,(K_{\mu'}(x, t)-K_{\mu''}(x, t))dt\right|$$

$$\leqslant\left|\int_{[0,\,1]-H}(f(t)-g(t))\,(K_{\mu'}(x, t)-K_{\mu''}(x, t))\,dt\right|$$

$$+\left|\int_{H}(f(t)-g(t))(K_{\mu'}(x, t)-K_{\mu''}(x, t))\,dt\right|$$

$$+\left|\int_0^1 g(t)(K_{\mu'}(x, t)-K_{\mu''}(x, t))\,dt\right|=I_1+I_2+I_3.$$

From (14) and (7)

$$(17)\quad I_1\leqslant\int_{[0,\,1]-H}\left|f(t)-g(t)\right|\,\left|K_{\mu''}(x, t)-K_{\mu'}(x, t)\right|dt$$

$$\leqslant \in\left(\int_0^1\left|K_{\mu'}(x, t)\right|dt+\int_0^1\left|K_{\mu''}(x, t)\right|dt\right)\leqslant 2M\in.$$

(18) Let $H=H_1+H_2$, such that

$$(f(t)-g(t))\,(K_{\mu'}(x, t)-K_{\mu''}(x, t))\geqslant 0 \text{ on } H_1$$

and

$$(f(t)-g(t))\,(K_{\mu'}(x, t)-K_{\mu''}(x, t))<0 \text{ on } H_2.$$

Obviously, $m(H_1)<\eta$ and $m(H_2)<\eta$.

Now by (18) and (8)

$$(19)\quad I_2\leqslant\left|\int_{H_1}(f(t)-g(t))\,(K_{\mu'}(x, t)-K_{\mu''}(x, t))\,dt\right|$$

$$+\left|\int_{H_2}(f(t)-g(t))\,(K_{\mu'}(x, t)-K_{\mu''}(x, t)\,dt\right|$$

$$\leqslant(\text{ess sup}\,|f(t)|+\sup|g(t)|)$$

$$\times\left\{\left|\int_{H_1}(K_{\mu'}(x, t)-K_{\mu''}(x, t))\,dt\right|\right.$$

$$\left.+\left|\int_{H_2}(K_{\mu'}(x, t)-K_{\mu''}(x, t))\,dt\right|\right\}$$

$$\leqslant(\text{ess sup}\,|f(t)|+\sup|g(t)|)\,2\in$$

for all $\mu', \mu''>\mu_1$, and for all x.

From (15) and (4) it is clear that

$$(20)\qquad I_3=0$$

for all $\mu', \mu''>\mu_2$, and for all x.

Thus (16) in conjunction with (17), (19) and (20) yields

$$\left| \Lambda_{\mu', x}(f) - \Lambda_{\mu'', x}(f) \right| \leqslant 2M\in + (\text{ess sup}) f \mid (t) \mid + \sup \mid g(t) \mid) 2\in$$

for all μ', $\mu'' > \max(\mu_1, \mu_2)$, and for all x.

Hence $\Lambda_{\mu, x}(f)$ converges uniformly in x.

This evidently completes the proof of Theorem 2.

REFERENCES

1. S. Aljancic, Uber Summierbarkeit von orthogonalentwicklungen Stetigen Funktion. Publ. Inst. Math. Acad. Serbe Sci. **10** (1956), 121–130. [*MR* **18**, 479].

2. Stefan Banach, Theorie des operations lineaires, Chelsea Publishing Company, New York 1955 [*MR* **17**, 175].

3. Syed A. Husain, On uniform convergence factors of orthonomal expansions of functions of class L^∞, J. Reine Angew. Math. **238** (1969), 217–220. [*MR* 30, 2284].

4. S. A. Husain and D. Waterman, Uniform convergence factors of orthogonal expansions, Publ. Inst. Math. (Beograd) (N. S.) **3 (17)** (1963), 89–92. [*MR* 30, 2284].

Jñānabha Sect. A, Vol. 6, 1976.

A Fixed Point Theorem for a Semigroup of Mappings with Contractive Iterates

By **S. A. Husain and V. M. Sehgal**
Department of Mathematics, University of Wyoming, Laramie, Wyoming 82071, *U.S.A.*

(Received : November 25, 1975)

1. Introduction. Recently, a number of papers have been published ([1]-[9]) extending the well-known contraction principle. The purpose of this paper is to obtain a theorem that contains several of these results and also improves an earlier result of Sehgal and Thomas [7].

Throughout this paper, (X, d) is a complete metric space and M a non-empty subset of X. Let F be a commutative semigroup of self-mappings of M. We shall further assume that F satisfies the following condition :

(1) for each $x \in M$ there exists an $f_x \in F$ such that for all $y \in M$,

$$d(f_x y, f_x x) \leqslant \alpha(d(y, f_x y) + d(x, f_x x)) + \beta(d(y, f_x x) + d(x, f_x y)) + \phi(d(x, y)),$$

where $\alpha, \beta \in R^+$ (non-negative reals) and $\phi : R^+ \to R^+$ is a mapping which is nondecreasing, continuous on the right and satisfies $\phi(t) < (1-3(\alpha+\beta))t$, if $t > 0$.

Note that the hypothesis on ϕ in (1) implies $3(\alpha+\beta) < 1$.

2. Main result.

Theorem 1. Let M be a closed subset of X and suppose F satisfies (1). If there exists an $x_0 \in M$ such that

(2) $$\sup \{d(x_0, fx_0) : f \in F\} < \infty$$

then, there exists a unique $u \in M$ such that $fu = u$ for each $f \in F$. Moreover, there is a sequence $\{g_n\} \subseteq F$ with $g_n x \to u$ for each $x \in M$.

We need the following lemmas to prove Theorem 1.

Lemma 1. Let $\phi : R^+ \to R^+$ be a mapping which is continuous on the right and satisfies $\phi(t) \leqslant ct$ for some c with $0 \leqslant c \leqslant 1$, $t \in R^+$. Then for each $r \in R^+$, $\lim_n \phi^n(r) = 0$.

Proof. Since $\phi(r)<r$ for $r>0$, the right continuity of ϕ implies that $\phi(0)=0$. Now for any $r>0$, $\{\phi^n(r)\}$ is a nonincreasing sequence in R^+ and hence there is an $s\geqslant 0$ such that $\phi^n(r)\to s$. If this $s>0$, then

$$s=\lim_n \phi^{n+1}(r)=\phi(\lim_n \phi^n(r))=\phi(s)<s,$$

a contradiction. Thus $\lim_n \phi^n(r)=0$.

Lemma 2. Let F satisfy (1). Suppose there exists a $u\in M$ with $f_u u=u$, then (*a*) u is the unique point such that $fu=u$ for each $f\in F$. Further, if the sequence $\{x_n\}$ in M defined by $x_{n+1}=f_{x_n} x_n$, $x_0\in M$, is convergent to u, then (*b*) there is a sequence $\{g_n\}\subseteq F$ with $g_n x\to u$ for each $x\in M$.

Proof. If $v\neq u$ satisfies $f_u v=v$, then by (1)

$$d(v,u)=d(f_u v, f_u u)\leqslant 2\beta d(v,u)+\phi(d(v,u)),$$

that is,

$$d(v,u)<(2\beta+(1-3(\alpha+\beta))\,d(v,u)\leqslant d(v,u),$$

a contradiction. Thus u is the unique fixed point of the mapping f_u. Furthermore, by the commutativity of F, we have, for any $f\in F$

$$fu=f(f_u u)=f_u(fu),$$

and hence $fu=u$. To prove (*b*) let for each $n\in I$ (non-negative integers) $f_x=f_{x_n}$ and $g_n=f_n.f_{n-1}\dots f_0$. Then $g_n\in F$. We show $g_n x\to u$ for each $x\in M$. We have

$$d(g_n x, u)\leqslant d(g_n x, x_{n+1})+d(x_{n+1}, u).$$

Since by hypothesis $x_n\to u$, to prove (*b*) it suffices to show that $d(g_n x, x_{n+1})\to 0$. However, by (1),

$$\begin{aligned} d(g_n x, x_{n+1}) &= d(f_n(g_{n-1}x), f_n x_n)\\ &\leqslant \alpha(d(g_{n-1}x, g_n x)+d(x_n, x_{n+1}))+\beta(d(g_{n-1}x, x_{n+1})\\ &\qquad +d(x_n, g_n x))+\phi(d(g_{n-1}x, x_n))\\ &\leqslant \alpha(d(g_{n-1}x, x_n)+2d(x_n, x_{n+1})+d(x_{n+1}, g_n x))\\ &\quad +\beta(d(g_{n-1}x, x_n)+2d(x_n, x_{n+1})+d(x_{n+1}, g_n x)\\ &\qquad +\phi(d(g_{n-1}x, x_n)). \end{aligned}$$

It follows from the last inequality that

$$(3)\quad (1-\alpha-\beta)\,d(g_n x, x_{n+1})\leqslant 2(\alpha+\beta)\,d(x_n, x_{n+1}) +(\alpha+\beta)d(g_{n-1}x, x_n)+\phi(d(g_{n-1}x, x_n)).$$

We consider cases (i) $\alpha+\beta=0$ and (ii) $\alpha+\beta\neq 0$. If $\alpha+\beta=0$, then putting $r=d(g_0 x, x_1)$ we have by (3)

$$d(g_n x, x_{n+1})\leqslant\phi(d(g_{n-1}x, x_n))\leqslant\dots\leqslant\phi^n(r)\to 0 \text{ as } n\to\infty.$$

If $\alpha+\beta\neq 0$, then, since $\phi(d(g_{n-1}x, x_n))\leqslant(1-3(\alpha+\beta))d(g_{n-1}x, x_n)$, it follows by (3) that

(4) $$d(g_n x, x_{n+1})\leqslant\gamma\, d(x_n x_{n+1})+\delta\, d(g_{n-1}\, x, x_n)$$

where $$\gamma=(1-(\alpha+\beta))^{-1}\,(2\alpha+2\beta)<1$$

and $$\delta=(1-(\alpha+\beta))^{-1}(1-2(\alpha+\beta))<1.$$

Now let $\epsilon>0$ be given. Since $d(x_n, x_{n+1})\to 0$, there exists an integer N such that $d(x_n, x_{n+1})<\epsilon$ for all $n\geqslant N$. Using the relation (4) repeatedly, it is easy to verify that for any positive integer k,

$$d(g_{N+k}\, x, x_{N+k+1})\leqslant(1-\delta)^{-1}\,\gamma\epsilon+\delta^k\, d(g_N x, x_{N+1}).$$

Since ϵ is arbitrary, it follows from the above inequality that $d(g_n x, x_{n+1})\to 0$ as $n\to\infty$.

Proof of Theorem 1. Let $\{x_n\}$ be the sequence constructed in Lemma 2 where $x_0\in M$ satisfies (2). Let for each $n\in I$, $f_n=f_{x_n}$ where $x_{n+1}=f_n(x_n)$ and let

(5) $$O(x_n)=\{fx_n\ :\ f\in F\},$$

Since for any positive integer J, $x_{k+J}=fx_J$ where

$f=f_{k+J-1}\cdot f_{k+J-2}\dots f_k\in F$, therefore $x_{k+J}\in O\,(x_k)$ for each positive integer J. Thus if $m, n\in I$ with $m\geqslant n$, then it follows that

(6) $$O(x_m)\subseteq O(x_n).$$

Let for each $n\in I$,

$$d_n=\sup\,\{d\,(x_n, y):\ y\in O\,(x_n)\}$$

By (2) $d_0<\infty$ and it follows by (6) that $d_n<\infty$ for each $n\in I$. We show that $d_n\to 0$ as $n\to\infty$. To prove this, let y be any element of $O(x_{n+1})$. Then $y=f\,(x_{n+1})$ for some $f\in F$. Hence,

$$d(x_{n+1}, y)=d\,(x_{n+1}, fx_{n+1})=d(f_n x_n, f_n(fx_n)),$$

and therefore by (1),

(7) $$\begin{aligned} d(x_{n+1}, y)\leqslant\alpha\,(d(x_n, f_n x_n)&+d\,(fx_n, f_n\,(fx_n))\\ &+\beta(d(x_n, fx_n)+d(fx_n, f_n x_n))\\ &+\phi(d(x_n, fx_n)). \end{aligned}$$

Now, since $d(fx_n, f_n\,(fx_n))\leqslant d\,(x_n, fx_n)+d(x_n, (f_n\cdot f)x_n)\leqslant 2d_n$, and $d(fx_n, f_n x_n)\leqslant d(x_n, fx_n)+d\,(x_n, f_n x_n)\leqslant 2d_n$, it follows by (7) and the hypothesis on ϕ that

(8) $\quad d_{n+1} \leqslant 3(\alpha+\beta)\, d_n + \phi(d_n) \leqslant (3(\alpha+\beta)+(1-3(\alpha+\beta))\, d_n = d_n.$

Thus $\{d_n\}$ is a non-increasing sequence in R^+ and hence $d_n \to r \geqslant 0$. Clearly $r=0$ for otherwise (8) together with the right continuity of ϕ would imply

$$r \leqslant 3(\alpha+\beta)\, r + \phi(r) < r,$$

a contradiction. Thus

(9) $$d_n \to 0 \quad \text{as} \quad n \to \infty.$$

It now follows immediately from (9) that $\{x_n\}$ is a Cauchy sequence in M, in fact if $m, n \in I$ with $m > n$, then by (6) $x_m \in O\,(x_n)$ and hence

$$d(x_n, x_m) \leqslant d_n \to 0 \quad \text{as} \quad n \to \infty.$$

Since M is a closed subset of X, there exists a $u \in M$ such that $x_n \to u$. We assert that

(10) $$\lim_n d(f_u x_n, f_u\, u) = d(u, f_u\, u).$$

To prove this assertion, we note that

$$d(f_u\, x_n, f_u\, u) \leqslant d(f_u\, x_n, x_n) + d(x_n, u) + d\,(u, f_u\, u),$$

and $$d(u, f_u\, u) \leqslant d(u, x_n) + d\,(x_n, f_u\, x_n) + d(f_u\, x_n, f_u u).$$

Hence, we have

$$|\, d(f_u\, x_n, f_u\, u) - d(u, f_u\, u)\,| \leqslant d(x_n, f_u\, x_n) + d(x_n, u) \leqslant d_n + d(x_n, u)$$

Therefore, as $n \to \infty$, the last inequality implies (10). Now by (1)

(11) $$\begin{aligned} d\,(f_u x_n, f_u u) &\leqslant \alpha\,(d\,(x_n, f_u x_n) + d\,(u, f_u u)) \\ &\qquad + \beta\,(d\,(x_n, f_u u) + d\,(u, f_u x_n)) + \phi\,(d\,(x_n, u)) \\ &\leqslant \alpha\,(d\,(x_n, f_u x_n) + d\,(u, f_u\, u)) \\ &\qquad + \beta\,(2d\,(x_n, u) + d\,(u, f_u\, u) + d\,(x_n, f_u x_n)) + \phi\,(d\,(x_n, u)). \end{aligned}$$

Since $d\,(x_n, f_u x_n) \leqslant d_n$, it follows by (9), (10) and (11) that as $n \to \infty$,

(12) $$d\,(u, f_u u) \leqslant (\alpha+\beta)\, d\,(u, f_u u).$$

Since $\alpha+\beta<1$, (12) implies that $f_u u = u$. Lemma 2, now completes the proof of Theorem 1.

Note that Theorem 1 improves Theorem 1 in [7]. The following special case of Theorem 1 provides a generalization of anearlier result of Guseman [3] and a recent result of Khazanchi [5] and Iseki [4]. (See also Reich [6].)

Corollary 1. Let $f: X \to X$ be a mapping satisfying the condition : for each $x \in X$, there exists a positive integer $n = n\,(x)$ such that for all $y \in X$

(13) $$d(f^n y, f^n x) \leqslant \alpha\, d(x, f^n x) + d(y, f^n y)) + \beta\, (d(y, f^n x) + d(x, f^n y)) + \gamma\, d(x, y)$$

where α, β, $\gamma \in R^+$ with $3\alpha + 3\beta + \gamma < 1$, then f has a unique fixed point $u \in X$ and $f^n x \to u$ for each $x \in X$.

PROOF. Let $F = \{f^n : n > 0\}$. Then F is a commutative semigroup satisfying (1) with $f_x = f^{n(x)}$ and $\phi(t) = \gamma t$ there. It is easy to show (see [4]) that (13) implies (2) for each $x_0 \in X$. Thus by Theorem 1 there is a unique $u \in X$ with $fu = u$. Setting

$$\delta = (1 - \alpha - \beta)^{-1} (\alpha + \beta + \gamma) < 1,$$

it follows from (13) that for sufficiently large $n \in I$ and $x \in X$, $d(f^n x, u) \leqslant \delta\, d(f^{n-\nu(x)} x, u)$ and this implies that $f^n x \to u$.

It may be noted that in [4] Corollary 1 is proved with the additional hypothesis that f is continuous.

If M is a bounded subset of X, then, since (2) holds for each $x_0 \in M$, we have

Corollary 2. Let M be a closed bounded subset of X and suppose F satisfies (1). Then there exists a unique $u \in M$ with $fu = u$ for $f \in F$ and there exists a sequence $\{g_n\} \subseteq F$ such that $g_n x \to u$ for each $x \in M$.

Note that Corollary 2 improves an earlier result of Browder [2].

REFERENCES

1. D.W. Boyd and J.S.W. Wong, On nonlinear contractions, Proc. Amer. Math. Soc. 20 (1969), 458–464.
2. Felix E. Browder, On the convergence of successive approximations for nonlinear functional equations, Indag. Math. 30 (1968), 27–35.
3. L.F. Guseman, Jr., Fixed point theorems for mappings with a contractive iterate at a point, Proc. Amer. Math. Soc. 26 (1970), 615–618.
4. K. Iseki, A generalization of Sehgal-Khazanchi's fixed point theorem, Math. Seminar Notes XV, Kobe University, vol. 2, No. 2 (1974).
5. L. Khazanchi, Results on fixed points in complete metric spaces, Math. Japon (to appear).
6. S. Reich, Some remarks concerning contraction mappings, Canad. Math. Bull. 1 (1971), 121–124.
7. V.M. Sehgal and J.W. Thomas, A fixed point theorem for semigroups of mappings with a contractive iterate, Proc, Amer. Math. Soc. 29 (1971), 121–124.
8. S.P. Singh, Some theorems on fixed points, Yokohama Math. J. 18 (1970), 23-25.
9. S.P. Singh, Some results on fixed point theorems, ibid. 17 (1969), 61-64.

Jñānabha Sect. A, Vol. 6, 1976.

Fixed Point Theorems in the Completion of Quasi-Gauge Space

By K. P. Gupta and N. P. S. Wawa

Department of Mathematics, Government Science College,

Rewa–486 001, *M.P.*, *India*

(Received : March 29, 1976 ; in revised form : February 21, 1977)

Baidyanath Ray has proved a fixed point theorem for a sequence of continuous maps on a metric space (see [4], Theorem 2.1). We prove the existence of fixed point in complete quasi-gauge space in [1]. We also extend the result for these incomplete quasi-gauge spaces which can be completed.

We first give a few definitions :

DEFINITION 1. (Reilly [5]). A quasi-gauge structure for topological space (X, T) is a family P of quasi-pseudometrics on X such that T has as a subbase the family P of quasi-pseudometrics on X such that T has a subbase the family $\{B(x, p, \epsilon) : x \text{ in } X, p \text{ in } P, \epsilon > 0\}$ where $B(x, p, \epsilon)$ is the set $\{Y \text{ in } X / p(x, y) < \epsilon\}$. If a topological space (X, T) has a quasi-gauge structure P, it is called a quasi-gauge space and is denoted by (X, P).

Remark. Every topological space is a quasi-gauge space (see [5] Theorem 2.6). If (X, d) is a metric space we may take P to consist of d alone.

DEFINITION 2. A continuous mapping $T : X \to X$ of a quasi-gauge space X into itself is called densifying, if for every $x, y \in X$ such that $p(x, y) > 0$, we have

$$p(Tx, Ty) < \lambda\, p(x, y).$$

We now prove the following theorem for the complete quasi-gauge spaces.

Theorem 1. Let $\{T_n\}$ be a sequence of densifying and weakly p-contractive mappings, each mapping being on a left (right) sequentially complete quasi-gauge space (X, P) into itself such that—

(i) For any two operators T_i, T_j, $p(T_i x, T_j y) < \lambda\, p(x, y)$ where $0 < \lambda < 1$ and $x, y \in X$ with $x \neq y$ and

(ii) There is a point x_0 in X such that any two consecutive members of $\{x_n = T_n x_{n-1}\}$ are distinct.

Then T has a unique common fixed point.

Proof. The above theorem has already been proved in paper [1]. We define the completion of quasi-gauge space (X, P) as follows.

Suppose (X, P) is a quasi-gauge space and (X^*, P^*) is a completion of it. Then X^* consists of all equivalence classes of Cauchy sequences in (X, P) under an equivalence relation defined as follows. If $\{x_n\}$ and $\{y_n\}$ are P-Cauchy in (X, P) then $\{x_n\}$ is equivalent to $\{y_n\}$ if and only if $\lim_{n\to\infty} p(x_n, y_n)=0$. Let $T : (X, P)\to(X, P)$ be a contraction map. Then define $T^* : (X^*, P^*)\to(X^*, P^*)$ as follows. If $x^*\in X^*$ with $\{x_n : x_n\in X\}=x^*$ then $T^* x^*=$the equivalence class containing $\{Tx_n\}$.

Next we prove two lemmas.

Lemma 1. Suppose $\{T_n\}$ is a sequence of densifying mappings, each mapping being from a quasi-gauge space (X, P) into itself, such that

(i) If $x, y \in X$ with $x\neq y$ then $p(T_i x, T_j y) < \lambda\, p(x, y)$ where $0 < \lambda < 1$ and $i, j = 1, 2, 3,\ldots$, and

(ii) There is a point x_0 in X such that any two consecutive members of x are distinct and $x_n=T_n X_{n-1}$, for $n=1, 2,\ldots,$. Then (i) and (ii) are true for $\{T_n^*\}$ in X^*.

Proof. (i) Let x^*, y^* be any two points of X^* such that $x^*\neq y^*$ and let $\{x_n\} \in x^*$ and $\{y_n\} \in y^*$.

Without loss of generality, let $x_n\neq y_n$ for all n. Now for any two members T_i and T_j,

$$p(T_i x, T_j y)<\lambda\, p(x, y).$$

We have

$$\lim_{n\to\infty} p(T_i x_n, T_j y_n)<\lambda \lim_{n\to\infty} p(x_n, y_n)$$

i.e.

$$p(T_i^* x^*, T_j^* y^*)<\lambda\, p^*(x^*, y^*).$$

(ii) Take $x_0^*=x_0'=$the equivalence class containing $\{x_0, x_0, \ldots, x_0, \ldots,\}$. Let $x_1^*=T_1^* x_0'$, the equivalence class containing $\{T_1x_0, T_1x_0, \ldots, T_1x_0, \ldots,\}$ *i.e.*, the class containing $\{x_1, x_1, \ldots, x_1, \ldots\}$ $=x_1'$. Similarly $x_n^*=x_n'$ where $n=1, 2, \ldots$. Since $x_r\neq x_{r+1}$ for any r, so $x_r'\neq x'_{r+1}$ and hence $x_r^*\neq x_{r+1}^*$ for any r.

The proof is thus completed.

Note. Under the hypothesis of Lemma 1, we can apply Theorem 1 to show that $\{T_n^*\}$ each mapping (X^*, P^*) into itself has a unique common fixed point u^* in X^*.

Lemma 2. If $\{T_n\}$ is a sequence of densifying mappings, each mapping being a quasi-gauge space (X, P) into itself and there is a dense subset X of X such that for any two operators T_i, T_j, $p(T_i x, T_j y) < \lambda\, p(x, Y)$, where $0 < \lambda < 1$ and $x, y \in X_0$ with $x \neq y$ then $p(T_i\, x, Tj\, y) < \lambda\, p(x, y)$ for all $x, y \in X$ with $x \neq y$.

PROOF. Suppose $x \neq y$ are any two points of X. Since X_0 is dense in X, there are two sequences $\{x_n\}$ and $\{y_n\}$ of points of X_0 such that $\lim_{n \to \infty} x_n = x$, $\lim_{n \to \infty} y_n = y$ and $x_n \neq y_n$ for all n. Since $x_n, y_n \in X$ and $x_n \neq y_n$, we have $p(T_i x_n, T_j\, y_n) < \lambda\, p(x, y)$. Using continuity of T_i, T_j, we get

$$p(T_i x, T_j y) < \lambda\, p(x, y).$$

Theorem 2. Let $\{T_n\}$ be sequence of continuous densifying mappings, each mapping being on a quasi-gauge space (X, P) into itself and suppose there is a dense subset X_0 of X, such that

(i) For any two maps T_i, T_j, $p(T_i x, T_j y) < \lambda\; p(x, y)$ where $0 < \lambda < 1$ and $x, y \in X_0$ with $x \neq y$ and

(ii) There is a point x_0 in X such that any two consecutive members of $\{x_n = T_n x_{n-1}\}$ are distinct.

Then a necessary and sufficient condition for the existence of a unique fixed point u common to $\{T_n\}$ is that there is a sequence $\{u_n\}$ in u^*, with $T_n{}^*\, u^* = v^*$, for $(n = 1, 2, 3, \ldots\}$ converging to u.

PROOF. By Lemma 2 (i) holds for every pair x, y of points of X with $x \neq y$. Then by the note following Lemma 1 there is a unique fixed point u^* in X^* common to $\{T_n{}^*\}$. For sufficient part, let $\{u_n\} \in u^*$ converging to a point u in X. For a fixed m,

$$\begin{aligned} p(u, T_m u) &= p^*\,(u', (T_m u)') \\ &< p^*\,(u', u^*) + p^*(u^*, T_m{}^*\, u^*) \\ &\quad + p^*(T_m{}^* u^*, (T_m u_n)') + p^*\,((T_m u_n)', (T_m u)'). \end{aligned}$$

By the continuity of T_m, the terms on the right-hand side tend to zero as $n \to \infty$, the $\lim_{n \to \infty} (v, T_m\, v) = 0$ *i.e.*, $T_m\, u = u$. So we prove that u is a fixed point common to $T_n (n = 1, 2, 3, \ldots)$. Uniqueness u of follows from that of u^*.

For the necessary part, if u is a fixed point common to $\{T_n\}$ then $u' =$ the equivalence class containing $\{u, u, \ldots, u, \ldots\}$ is a fixed point common to T_n, $\{n = 1, 2, \ldots,\}$. Since u is unique, u^* contains $\{u, u, \ldots, u \ldots,\}$. Define $u_n = u$ for every n. Then $\{u_n\} \in u^*$ and converges to u.

This evidently completes the proof of Theorem 2.

ACKNOWLEDGEMENES. We are thankful to Dr. R.C. Varma for the advice in the preparation of this paper, and to Prof. H.M. Srivastava of the University of Victoria for his valuable suggestions for the improvements in the paper.

REFERENCES

1. K.P. Gupta, Fixed point theorem in left (right) sequentially complete quasi-gauge space (to appear).
2. R. Kannan, Some results on fixed points, Bull. Calcutta Math. Soc. 60 (1968), 71-76.
3. R. Kannan, Some results on fixed points. II, Amer. Math. Monthly 76 (1969), 405–408.
4. Baidyanath Ray, On a paper of Kannan, Bull. Calcutta Math. Soc. 63 (1971), 7–10.
5. L. Reilly, Quasi-gauge spaces, J. London Math. Soc. (2) 6 (1973), 481-487.
6. P.V. Subrahmanyam, Remarks on some fixed point theorems related to Banach's contraction principle, J. Mathematical and Physical Sci. 8 (1974) 445-457.

Jñānabha Sect. A, Vol. 6, 1976.

PERIODIC ORBITS OF COLLISION IN THE THREE-DIMENSIONAL CIRCULAR RESTRICTED PROBLEM OF FOUR BODIES

By **K. B. Bhatnagar**

Department of Mathematics, Delhi College

University of Delhi, Delhi–110006, *India*

AND

Z. A. Taqvi

Department of Mathematics, Jamia College,

Jamia Millia, New Delhi–110025, *India*

(Received : May 17, 1975 ; in final form : October 6, 1976)

Abstract. In this paper periodic orbits of collision in the three-dimensional circular restricted problem of four bodies have been studied. It has been assumed that the infinitesimal mass is moving under the gravitational field of three finite masses which are in circular orbits for which a particular solution is an equilateral triangular configuration. The study has been made under the restriction that the masses of the three primaries are μ, $\alpha\mu$ and $1-\mu(1+\alpha)$ and the third coordinate ξ_3 of the infinitesimal mass is of the order μ and α has to be a significant quantity.

1. Equations of Motion. Let P_1, P_2, P_3 be three bodies of masses m_1, m_2, m_3 moving in their mutual gravitational field. We know that a particular solution of the problem is an equilateral triangular configuration. Let l be the length of the side of the equilateral triangle, w the constant angular velocities of the bodies and K^2 the Gaussian constant of graviation. Then w is given by

$$w^2 l^3 = K^2(m_1+m_2+m_3) = K^2 m \text{ (say)}.$$

Consider the motion of an infinitesimal mass P in gravitational field of P_1, P_2, P_3. The primaries move in a plane and the mass of P is so small that the triangular configuration is not changed. Let C be the geometric centre of the triangular configuration $P_1P_2P_3$ and G the centre of mass of the masses m_1, m_2, m_3 situated at P_1, P_2, P_3 respectively. Take the line GX parallel to CP_1 as X-axis and the line

GY perpendicular to GX and in the sense of rotation as the Y-axis and the line through G and perpendicular to the plane $P_1P_2P_3$ as the Z-axis. Let the coordinates of P be (X, Y, Z) and those of P_j as $(X_j, Y_j, 0)$; $(j=1, 2, 3)$ in the rotating $GXYZ$ system. Let l be the unit of length, $1/w$ the unit of time and m the unit of mass.

Let (ξ, η, ζ) be the coordinates of P in the fixed $GXYZ$ system. Then

$$\left.\begin{aligned} \xi &= X\cos t - Y\sin t, \\ \eta &= X\sin t + Y\cos t, \\ \zeta &= Z. \end{aligned}\right\} \tag{1}$$

Thus the kinetic energy of the infinitesimal mass is given by

$$T=\tfrac{1}{2}\,[\dot\xi^2+\dot\eta^2+\dot\zeta^2]=\tfrac{1}{2}[\dot X^2+\dot Y^2+\dot Z^2-2(\dot XY-X\dot Y)+(X^2+Y^2)]$$

Let $$(p_1, p_2, p_3)=\left(\frac{\partial T}{\partial \dot X}, \frac{\partial T}{\partial \dot Y}, \frac{\partial T}{\partial \dot Z}\right)=\dot X-Y,\ \dot Y+X,\ \dot Z).$$

Then $$T=\tfrac{1}{2}\,[p_1^2+p_2^2+p_3^2].$$

If ρ_1, ρ_2, ρ_3 be the distances of P from P_1, P_2, P_3 respectively, then the potential function of P is given by

$$V=-\left[\frac{\mu_1}{\rho_1}+\frac{\mu_2}{\rho_2}+\frac{\mu_3}{\rho_3}\right],$$

where $\mu_j=m_j/(m_1+m_2+m_3)$; $(j=1, 2, 3)$.

Now the Hamiltonian H is given by

$$H=-(T-V)+p_1\dot X+p_2\dot Y+p_3\dot Z,$$

or equivalently

$$H=\tfrac{1}{2}\,[p_1^2+p_2^2+p_3^2+2(p_1Y-p_2X)]-\frac{\mu_1}{\rho_1}-\frac{\mu_2}{\rho_2}-\frac{\mu_3}{\rho_3}. \tag{2}$$

We, now, shift the origin to C, the geometric centre of the equilateral triangle $P_1P_2P_3$ through parallel axes. Let (x_1, x_2, x_3) be the coordinates of P and $(\bar x_1, \bar x_2, 0)$ the coordinates of the centre of mass referred to $Cx_1x_2x_3$ system. We have $(x_1, x_2, x_3)=(X+\bar x_1, Y+\bar x_2, Z)$ and therefore

$$\left.\begin{aligned} p_1&=\dot X-Y=(\dot x_1-x_2-(\dot{\bar x}_1-\bar x_2), \\ p_2&=\dot Y+X=(\dot x_2+x_1)-(\dot{\bar x}_2+\bar x_1)\ ;\ p_3=\dot Z=\dot x_3. \end{aligned}\right\} \tag{3}$$

In canonical form the equations of motion are

$$\dot x_i=\frac{\partial H}{\partial p_i}\ ;\ \dot p_i=-\frac{\partial H}{\partial x_i},\ (i=1, 2, 3), \tag{4}$$

where, from (2)

$$H=\tfrac{1}{2}\ [p_1^2+p_2^2+p_3^2]+p_1(x_2-\bar{x}_2)-p_2(x_1-\bar{x}_1)-\frac{\mu_1}{\rho_1}-\frac{\mu_2}{\rho_2}-\frac{\mu_3}{\rho_3}$$

($=c$, c being the energy constant) ([1], p. 583). (5)

Also $$c=c\ (\mu_1,\ \mu_2,\ \mu_3),$$

or $$c=c_0+\mu_1c_1+\mu_2c_1+\mu_3c_3+o(\mu^2), \tag{6}$$

2. **Regularisation of the Solution.** We regularise the problem by Levi-Civita's transformation which may be generated by ([1], p. 583)

$$S=(A+\xi_1^2-\xi_2^2)p_1+2\xi_1\xi_2p_2+\xi_3p_3,$$

such that $$x_i=\frac{\partial S}{\partial p_i},\ \pi_i=\frac{\partial S}{\partial \xi_i}\ (i=1,\ 2,\ 3).$$

where π_i $(i=1, 2, 3)$ are the momenta associated with the new coordinates ξ_i, and $A=1/\sqrt{3}$. We have

$$\pi_1=\frac{\partial S}{\partial \xi_1}=2\xi_1p_1+2\xi_2p_2,$$

$$\pi_2=\frac{\partial S}{\partial \xi_2}=-2\xi_2p_1+2\xi_1p_2\ ;$$

$$\pi_3=\frac{\partial S}{\partial \xi_3}=p_3.$$

Solving these equations for p_1, p_2, p_3 we obtain

$$\left.\begin{aligned} &p_1=\frac{\pi_1\xi_1-\pi_2\xi_2}{2(\xi_1^2+\xi_2^2)};\ p_2=\frac{\pi_1\xi_2+\pi_2\xi_1}{2(\xi_1^2+\xi_2^2)}\ ;\ p_3=\pi_3. \\ \text{Further}\quad &x_1=\frac{\partial S}{\partial p_1}=A+\xi_1^2-\xi_2^2\ ;\ x_2=\frac{\partial S}{\partial p_2}=2\xi_1\xi_2\ ; \\ &x_3=\frac{\partial S}{\partial p_3}=\xi_3. \end{aligned}\right\} \tag{7}$$

Thus the Hamiltonian H, given by (5), becomes

$$H=\frac{\pi^2}{8\xi^2}+\frac{1}{2}\ \pi_3^2+\frac{1}{2}\ (\pi_1\xi_2-\pi_2\xi_1)-\frac{A}{2\xi^2}(\pi_1\xi_2+\pi_2\xi_1)$$

$$-\frac{\pi_1\xi_1-\pi_2\xi_2}{2\xi^2}\ \bar{x}_2+\frac{\pi_1\xi_2+\pi_2\xi_1}{2\xi^2}\ \bar{x}_1-\frac{\mu_1}{\rho_1}-\frac{\mu_2}{\rho_2}-\frac{\mu_3}{\rho_3}=c \tag{8}$$

where $\pi^2=\pi_1^2+\pi_2^2$; $\xi^2=\xi_1^2+\xi_2^2$.

Since the coordinates of P_1, P_2, P_3 in the $Cxyz$ system are $(A, 0, 0)$; $\left(A \cos\frac{2\pi}{3}, A \sin\frac{2\pi}{3}, 0\right)$ and $\left(A \cos\frac{4\pi}{3}, A \sin\frac{4\pi}{3}, 0\right)$, therefore,

$$\left.\begin{aligned} \rho_1^2&=(\xi_1^2+\xi_2^2)^2+\xi_3^2=\xi^4+\xi_3^2 ; \\ \rho_2^2&=1+\xi^4+3A(\xi_1^2-\xi_2^2)-2\xi_1\xi_2+\xi_3^2, \\ \rho_3^2&=1+\xi^4+3A(\xi_1^2-\xi_2^2)+2\xi_1\xi_2+\xi_3^2. \end{aligned}\right\} \tag{9}$$

The equations of motion (4) become

$$\dot{\xi}_i=\frac{\partial H}{\partial \pi_i} ; \dot{\pi}_i=-\frac{\partial H}{\partial \xi_i}, \quad (i=1, 2, 3), \tag{10}$$

where H is given by (8).

We now introduce a new independent variable τ instead of t given by

$$dt=\rho_1 d\tau, \ \tau=0 \text{ when } t=0. \tag{11}$$

The equations of motion (10) become

$$\frac{d\xi_i}{d\tau}=\frac{\partial K}{\partial \pi_i} ; \frac{d\pi_i}{d\tau}=-\frac{\partial K}{\partial \xi_i} (i=1, 2, 3), \tag{12}$$

where K is new Hamiltonian given by

$$\begin{aligned} K&=\rho_1(H-c) \\ &=\frac{\pi^2}{8}\cdot\frac{\rho_1}{\xi^2}+\frac{1}{2}\rho_1\pi_3^2+\frac{1}{2}\rho_1(\pi_1\xi_2-\pi_2\xi_1)-\frac{A}{2}\cdot\frac{\rho_1}{\xi^2}(\pi_1\xi_2+\pi_2\xi_1) \\ &-\frac{1}{2}(\pi_1\xi_1-\pi_2\xi_2)\,\bar{x}_2\cdot\frac{\rho_1}{\xi^2}+\frac{1}{2}(\pi_1\xi_2+\pi_2\xi_1)\bar{x}_1\cdot\frac{\rho_1}{\xi^2} \\ &-\mu_1-\mu_2\frac{\rho_1}{\rho_2}-\mu_3\frac{\rho_1}{\rho_3}-\rho_1(c_0+\mu_1c_1+\mu_2c_2+\mu_3c_3). \end{aligned} \tag{13}$$

We have put the value of c from (6). Here

$$\bar{x}_1=\frac{2\mu_1-\mu_2-\mu_3}{2}A ; \ \bar{x}_2=\frac{\mu_2-\mu_3}{2} ; \ \bar{x}_3=0.$$

Suppose μ_1, μ_2, μ_3 are the masses of the Sun, the Earth and the Moon, respectively.

Let $\mu_1=1-\alpha\mu-\mu$; $\mu_2=\alpha\mu$; $\mu_3=\mu$, where α is a constant of the order 10^2. Let us take ξ_3 to be of $o(\mu)$; then

$$\rho_1=\xi^2+o(\mu) ; \ \bar{x}_1=\frac{2-3\alpha\mu-3\mu}{2}A ; \ \bar{x}_2=\frac{\alpha\mu-\mu}{2} ; \ \bar{x}_3=0.$$

The Hamiltonian (13) becomes

$$K=\frac{\pi^2}{8}+\frac{1}{2}\xi^2\pi_3^2+\frac{1}{2}\xi^2(\pi_1\xi_2-\pi_2\xi_1)-\frac{A}{2}(\pi_1\xi_2+\pi_2\xi_1)$$

$$-\frac{1}{2}(\pi_1\xi_1-\pi_2\xi_2)\left(\frac{\alpha\mu-\mu}{2}\right)$$

$$+\frac{1}{2}(\pi_1\xi_2+\pi_2\xi_1)\left(\frac{2-3\alpha\mu-3\mu}{2}A\right)-(1-\alpha\mu-\mu)$$

$$-\alpha\mu\frac{\rho_1}{\rho_2}-\mu\frac{\rho_1}{\rho_3}-\rho_1\{c_0+(1-\alpha\mu-\mu)c_1+\alpha\mu c_2+\mu c_3\},$$

or

$$K=\frac{\pi^2}{8}+\frac{1}{2}\xi^2\pi_3^2+\frac{1}{2}\xi^2(\pi_1\xi_2-\pi_2\xi_1-2c_0')-1$$

$$+\mu[-\tfrac{1}{4}(\alpha-1)(\pi_1\xi_1-\pi_2\xi_2)$$

$$-\frac{3A}{4}(\alpha+1)(\pi_1\xi_2+\pi_2\xi_1)+(\alpha+1)-\alpha\frac{\rho_1}{\rho_2}-\frac{\rho_1}{\rho_3}$$

$$+\rho_1(\alpha c_1'+c_2')]+o(\mu^2),$$

where $c_0'=c_0+c_1$; $c_1'=c_1-c_2$; $c_2'=c_1-c_3$. Putting K in the form $K_0+\mu K_1$, we have

$$K_0=\frac{\pi^2}{8}+\frac{1}{2}\xi^2\pi_3^2+\frac{1}{2}\xi^2(\pi_1\xi_2-\pi_2\xi_1-2c_0')-1=-\epsilon. \qquad (14)$$

and

$$K_1=-\frac{1}{4}(\alpha-1)(\pi_1\xi_1-\pi_2\xi_2)-\frac{3A}{4}(\alpha+1)(\pi_1\xi_2+\pi_2\xi_1)$$

$$+(\alpha+1)-\alpha\frac{\rho_1}{\rho_2}-\frac{\rho_1}{\rho_3}+\rho_1(\alpha c_1'+c_2').$$

We note that ([5], p. 386)

(i) The form K_0 ensures that the orbits which are analytically continued from the two-body orbits will belong to the $K=0$ manifold, that is, are solutions of the regularised equations of the restricted problem.

(ii) K_0 is supposed to be negative $=-\epsilon$

(iii) $|\epsilon|<1$.

3. **Generating Solution.** For generating solution we choose K_0 for our Hamiltonion function. Since it is not involved explicitly, therefore, the Hamilton-Jacobi equation may be written as

$$\frac{1}{8}\left[\left(\frac{\partial w}{\partial\xi_1}\right)^2+\left(\frac{\partial w}{\partial\xi_2}\right)^2\right]+\frac{1}{2}\xi^2\left(\frac{\partial w}{\partial\xi_3}\right)^2$$

$$+\frac{1}{2}\xi^2\left[\xi_2\frac{\partial w}{\partial\xi_1}-\xi_1\frac{\partial w}{\partial\xi_2}-2c_0'\right]=\alpha, \qquad (15)$$

where $\alpha=1-\epsilon>0$, being taken for an arbitrary constant. Putting $\xi_1=\xi\cos\phi$, $\xi_2=\xi\sin\phi$, the equation (15) becomes

$$\frac{1}{8}\left[\left(\frac{\partial w}{\partial\xi}\right)^2+\frac{1}{\xi^2}\left(\frac{\partial w}{\partial\phi}\right)^2\right]+\frac{1}{2}\xi^2\left(\frac{\partial w}{\partial\xi_3}\right)^2$$
$$+\frac{1}{2}\xi^2\left(-\frac{\partial w}{\partial\phi}-2c_0'\right)=\alpha', \tag{16}$$

where α' is another arbitrary constant.

This differential equation is the same as in ([1], p. 583), except that instead of c_0 in that equation, we are getting c_0'. Therefore, the solution of (16) can be written as

$$w=U(\xi)+2G\phi+\bar{H}\xi_3\ ;\ z=\xi^2, \tag{17}$$

$$U(z,G,\alpha')=[\bar{H}^2-2(G+c_0')]^{\frac{1}{2}}\int_{z_1}^{z}\frac{\sqrt{f(z)}}{z}\,dz. \tag{18}$$

Here z_1 is the smaller of the roots of the equation

$$f(z)=-\left[z^2-\frac{2\alpha' z}{\bar{H}^2-2(G+c_0')}+\frac{G^2}{\bar{H}^2-2(G+c_0')}\right]=0. \tag{19}$$

Let the other root of equation (19) be z_2. Let us introduce parameters a, e, l by relations

$$z_1=a(1-e)\ ;\ z_2=a(1+e)\ ;\ z=a(1-e\cos l), \tag{20}$$

where $0\leqslant e\leqslant 1$.

We take a new parameter L instead of α' defined by the relation

$$\alpha'=L[\bar{H}^2-2(G+c_0')]^{1/2}>0. \tag{21}$$

We have

$$a=\frac{L}{\sqrt{\bar{H}^2-2(G+c_0')}}\ ;\ e=\sqrt{1-\frac{G^2}{L^2}}\leqslant 1\ ;$$
$$l=[\bar{H}^2-2(G+c_0')]^{1/2}(\tau-\tau_0). \tag{22}$$

Taking $(l, L, g, G, h, \bar{H})$ as the canonical set corresponding to Hamiltonian K_0 and ξ_3 of $o(\mu)$, the final solution can be taken as

$$\left.\begin{aligned}
\pm\xi_1&=\sqrt{a(1-e\cos l)}\cos\phi\ ;\\
\pm\xi_2&=\sqrt{a(1-e\cos l)}\sin\phi\ ;\\
\xi_3&=h-\frac{\bar{H}\sqrt{L^2-G^2}}{\bar{H}^2-2(G+c_0')}\sin l,\\
\pm\pi_1&=\frac{2e\,L\sin l\cos\phi-2G\sin\phi}{\sqrt{a(1-e\cos l)}}\ ;\\
\pm\pi_2&=\frac{2e\,L\sin l\sin\phi+2G\cos\phi}{\sqrt{a(1-e\cos l)}},\\
\pi_3&=\bar{H},
\end{aligned}\right\} \tag{23}$$

where ϕ is given by

$$\phi=\frac{1}{2}(f+g)-\frac{1}{2}\frac{\sqrt{L^2-G^2}}{\bar{H}^2-2(G+c_0')}\sin l,\ e\neq 1, \tag{24}$$

and when $e=1$, ($G=0$, $f=0$), we have

$$\left.\begin{aligned}
&\xi_1=\pm\sqrt{2a}\ \sin\frac{l}{2}\cos\phi\ ;\\
&\xi_2=\pm\sqrt{2a}\ \sin\frac{l}{2}\sin\phi\ ;\\
&\xi_3=h-\frac{\sqrt{2a}\ \bar{H}L}{\bar{H}_2-2c_0'}\sin l,\\
&\pi_1=\pm\frac{4L}{\sqrt{2a}}\cos\frac{l}{2}\cos\phi\ ;\\
&\pi_2=\pm\frac{4L}{\sqrt{2a}}\cos\frac{l}{2}\sin\phi\ ;\ \pi_0=\bar{H},
\end{aligned}\right\} \tag{25}$$

where ϕ is given by

$$\phi=\frac{1}{2}g-\frac{L}{2(\bar{H}^2-2c_0')}\sin l,\ e=1. \tag{26}$$

The original synodic cartesian coordinates in the uniformly rotating xyz system are obtained from equation (15) ($\mu=0$) and equations (25) as

$$\left.\begin{aligned}
&x_1=A+\xi_1^2-\xi_2^2\ ;\ x_2=2\xi_1\ ;\ x_3=\xi_3,\\
&p_1=\frac{\pi_1\xi_1-\pi_2\xi_2}{2(\xi_1^2+\xi_2^2)}\ ;\ p_2=\frac{\pi_1\xi_2+\pi_2\xi_2}{2(\xi_1^2+\xi_2^2)}\ ;\ p_3=\pi_3.
\end{aligned}\right\} \tag{27}$$

The synodic cartesian coordinates referred to $GXYZ$ system are given by

$$\left.\begin{aligned}
&X=x_1-\bar{x}_1\ ;\ Y=x_2-\bar{x}_2\ ;\ Z=x_3,\\
&\dot{X}=\dot{x}_1\ ;\ \dot{Y}=\dot{x}_2\ ;\ \dot{Z}=\dot{x}_3.
\end{aligned}\right\} \tag{28}$$

It may be noted that $\dot{\bar{x}}_1=\dot{\bar{x}}_2=0$.

The sidereal cartesian coordinates are given by

$$\left.\begin{aligned}
&\xi=X\cos t-Y\sin t\ ;\\
&\eta=X\sin t+Y\cos t\ ;\ \zeta=Z,\\
&\dot{\xi}=\cos t(\dot{X}-Y)-\sin t(\dot{Y}+X)\ ;\\
&\dot{\eta}=\sin t(\dot{X}-Y)+\cos t(\dot{Y}+X)\ ;\ \dot{\zeta}=\dot{Z}
\end{aligned}\right\} \tag{29}$$

Here t is given by the relation

$$dt=r_1 d\tau \text{ or } \Big| \, t \, \Big|_{t_0}^{t} = \int_0^{\tau} \xi^2 \, d\tau + o(\mu).$$

Therefore

$$t-t_0=\frac{a}{[\bar{H}^2-2(G+c_0')]^{1/2}}\,[l-e\sin l],$$

where t_0 is a constant. It may, further, be observed that l is the eccentric anomaly of the two-body problem.

In terms of the canonical elements the complete Hamiltonian may be written as

$$\begin{aligned}
K=L\,[\bar{H}^2-2(G+c_0')]^{1/2}-1+\mu[&-\tfrac{1}{4}(\alpha-1)\,\{2eL\sin l\cos 2\phi-2G\sin 2\phi\}\\
&-\frac{3A}{4}(\alpha+1)\cdot\{2eL\sin l\sin 2\phi+2G\cos 2\phi\}+(\alpha+1)\\
&\quad-\alpha\, a(1-e\cos l)\{1+a^2(1-e\cos l)^2\\
&\quad+3Aa(1-e\cos l)\cos 2\phi-a(1-e\cos l)\sin 2\phi\}^{-1/2},\\
&\quad-a(1-e\cos l)\{1+a^2(1-e\cos l)^2\\
&\quad+3Aa(1-e\cos l)\cos 2\phi+a(1-e\cos l)\sin 2\phi\}^{-1/2}\\
&\quad+a(1-e\cos l)\,(\alpha\, c_1'+c_2')]+o(\mu^2).
\end{aligned} \tag{30}$$

Let us denote the coefficient of μ by R and neglecting μ^2, etc., we have

$$K=L[\bar{H}^2-2(G+c_0')]^{1/2}-1+\mu R. \tag{31}$$

The zero-order terms of R are

$$\begin{aligned}
&\frac{1}{2}(\alpha-1)G\sin 2\phi-\frac{3A}{2}(\alpha+1)G\cos 2\phi+(\alpha+1)\\
&-\alpha\, a\{1+a^2+3Aa\cos 2\phi-a\sin 2\phi\}^{-1/2}\\
&-a\{1+a^2+3Aa\cos 2\phi+a\sin 2\phi\}^{-1/2}+a(\alpha\, c_1'+c_2').
\end{aligned} \tag{32}$$

The equations of motion for the complete Hamiltonian are

$$\left.\begin{aligned}
&\frac{dL}{d\tau}=-\frac{\partial K}{\partial l}=-\mu\frac{\partial R}{\partial l}\,;\quad \frac{dG}{d\tau}=-\frac{\partial K}{\partial g}=-\mu\frac{\partial R}{\partial g}\,;\\
&\frac{\partial \bar{H}}{\partial \tau}=-\frac{\partial K}{\partial h}=-\mu\frac{\partial R}{\partial h}\,;\quad \frac{dl}{d\tau}=\frac{\partial K}{\partial L}=[\bar{H}^2-2(G+c_0')]^{1/2}+\mu\frac{\partial R}{\partial L}\,;\\
&\frac{dg}{d\tau}=\frac{\partial K}{\partial G}=\frac{-L}{[\bar{H}^2-2(G+c_0')]^{1/2}}+\mu\frac{\partial R}{\partial G}
\end{aligned}\right\} \tag{33}$$

$$\frac{dh}{d\tau}=\frac{\partial K}{\partial \bar{H}}=\frac{L\bar{H}}{[\bar{H}^2-2(G+c_0')]^{1/2}}+\mu\frac{\partial R}{\partial \bar{H}}.$$

Equations (33) form the basis of a general perturbation theory for the problem in question.

4. Existence of Periodic Orbits when $\mu \neq 0$. The periodic orbits exist when $\mu \neq 0$ if (cf. [1], [3], [4])

$$\frac{\partial (K_1)}{\partial w_i}=\frac{\partial (K_1)}{\partial a_i}=0,\ (i=1,\ 2,\ 3), \tag{34}$$

and

$$\begin{vmatrix} \frac{\partial^2 (K_1)}{\partial w_2^2} & \frac{\partial^2 (K_1)}{\partial w_3\, \partial w_2} & 0 & 0 & 0 \\ \frac{\partial^2 (K_1)}{\partial w_2\, \partial w_3} & \frac{\partial^2 (K_1)}{\partial w_3^2} & 0 & 0 & 0 \\ \frac{\partial^2 (K_1)}{\partial w_2\, \partial a_1} & \frac{\partial^2 (K_1)}{\partial w_3\, \partial a_1} & \frac{\partial^2 K_0}{\partial a_1^2} & \frac{\partial^2 K_0}{\partial a_2\, \partial a_1} & \frac{\partial^2 K_0}{\partial a_3\, \partial a_1} \\ \frac{\partial^2 (K_1)}{\partial w_2\, \partial a^2} & \frac{\partial^2 (K_1)}{\partial w_3\, \partial a_2} & \frac{\partial^2 K_0}{\partial a_1\, \partial a_2} & \frac{\partial^2 K_0}{\partial a_2^2} & \frac{\partial^2 K_0}{\partial a_3\, \partial a_2} \\ \frac{\partial^2 (K_1)}{\partial w_3\, \partial a_3} & \frac{\partial^2 (K_1)}{\partial w_3\, \partial a_3} & \frac{\partial^2 K_0}{\partial a_1\, \partial a_3} & \frac{\partial^2 K_0}{\partial a_2\, \partial a_3} & \frac{\partial^2 K_0}{\partial a_3^2} \end{vmatrix} \neq 0, \tag{35}$$

where $x_1=L,\ x_2=G,\ x_3=\bar{H}\ ;\ y_1=l,\ y_2=g,\ y_3=h\ ;\ x_i=a_i,\ y_i=n_i^{(0)}+w$ $(i=1,\ 2,\ 3)$.

To prove that the determinant (35) is non-zero, we proceed as follows :

We may note that

$$\frac{\partial^2 (K_i)}{\partial a_j \partial w_i}=0 \text{ for } \frac{\partial (K_1)}{\partial w_i}=0,\ \left[\begin{matrix} i=1,\ 2,\ 3 \\ j=1,\ 2,\ 3 \end{matrix}\ i\neq i\right].$$

Also $\qquad K_0=a_1[a_3^2-2\ (a_2+c_0')]^{1/2}-1.$

Therefore, the value of the determinant obtained from the elements in the 3rd, 4th and 5th rows and columns of (35) is given by

$$-\frac{a_1}{[a_3^2-2(a_2+c_0)]^{3/2}}\neq 0, \tag{36}$$

since $a_1 \neq 0$, assuming that $a_2+c_0<0$.

Again, we consider the determinant, obtained from the elements in the 1st and 2nd rows and columns of (35). From (32), we have

$$\begin{aligned} K_1=\frac{1}{2}\,(\alpha-1)G\ \sin\,2\phi-\frac{3A}{2}\,(\alpha+1)G\ \cos\,2\phi+(\alpha+1) \\ -\alpha\, a\,\{1+a^2+3Aa\ \cos\,2\phi-a\ \sin\,2\phi\}^{-1/2} \\ -a\{1+a^2+3Aa\cos\,2\phi+a\ \sin\,2\phi\}^{-1/2}+a\{\alpha c_1'+c_2'\}, \end{aligned}$$

where $2\phi = y_1 + y_2 = n_1^{(0)} + n_2^{(0)} + w_1 + w_2$ (approx.)

Also

$$\xi_3 = y_3 - \frac{x_3(x_1^2 - x_2^2)}{x_3^2 - 2(x_2 + c_0')} \sin y_1 \ ; \ x_i = a_i \ ; \ y_i = n_1^{(0)} + w_i.$$

Therefore

$$\frac{\partial(K_1)}{\partial w_2} = \frac{1}{2}(\alpha - 1)\, G \cos 2\phi + \frac{3A}{2}(\alpha + 1) G \sin 2\phi$$

$$+ \frac{\alpha a}{2\rho_2^3}\{-3Aa \sin 2\phi - a \cos 2\phi\} + \frac{a}{2\rho_3^3}\{-3Aa \sin 2\phi + a \cos 2\phi\}.$$

Thus $\dfrac{\partial(K_1)}{\partial w_2} = 0$, gives $2\phi = 0$, $G = \dfrac{a^2}{\rho^3}$, where $\rho^2 = 1 + a^2 + 3Aa$.

Here $\rho_2^2 = 1 + a^2 + 3Aa \cos 2\phi - a \sin 2\phi$;

$\rho_3^2 = 1 + a^2 + 3Aa \cos 2\phi + a \sin 2\phi$.

Now $\dfrac{\partial^2(K_1)}{\partial w_2^2} = -\dfrac{3a^3}{2\rho^5}(\alpha + 1) \neq 0$ (since $a \neq 0$, $\alpha = 1 - \epsilon > 0$)

We may also note that K_1, given in (30), can also be put in the form $K_1 = \lambda + \mu \xi_3^2$, where λ, μ are independent of ξ_3.

Therefore $\dfrac{\partial(K_1)}{\partial w_3} = 2\mu\xi_3 . \dfrac{d\xi_3}{dw_3} = 0$, gives

$$\xi_3 = 0, \text{ and } \frac{\partial^2(K_1)}{\partial w_3^2} = 2\mu \neq 0 \text{ at } \xi_3 = 0.$$

Also

$$\frac{\partial^2(K_1)}{\partial w_2 \partial w_3} = 0.$$

Thus the determinant obtained from the elements in the 1st and 2nd rows and columns of (35) is also not equal to zero. Therefore, the determinant (35) is also not equal to zero. Hence the conditions for the existence of the periodic orbits for $\mu \neq 0$ are satisfied.

5. **Periodic Orbits of Collision when $\mu \neq 0$.** Levi-Civita ([6], p.1) has proved, in one of his papers a relation which is satisfied along a collision orbit in the restricted orbit. His main result is that the invariant relation for collision orbits can be analytically continued from the one that corresponds to the problem of two bodies. In our case it is given by $G = 0$. Therefore when $\mu \neq 0$, this condition will change into

$$G + \mu F(l, L, g, G, h, \bar{H}, \mu) = 0. \tag{37}$$

For showing the validity of that continuation orbits, we will consider orbits corresponding to the case when $e=1$ (*i.e.* $G=0$) and the orbit starts as an ejection from the origin and returns to it after a time $\frac{T}{4}$ ([2], p. 829).

Consider, now, a periodic orbit ($\mu \neq 0$) starting at the origin as an ejection orbit. Levi-Civita's condition for the collision is

$$\dot{\theta}+1=\rho' f(\rho', \theta), \tag{38}$$

where $\tan \theta=\dfrac{x_2}{x_1-A}=\dfrac{2\xi_1 \xi_2}{\xi_1^2-\xi_2^2}=\tan 2\phi, \quad \rho'=\sqrt{\rho_1}$; thus, $\theta=2\phi$.

The condition becomes $2\dot{\phi}+1=\rho' f(\rho', 2\phi)$

$$2\frac{d\phi}{dt}+\rho_1=\rho_1^{3/2} f(\rho_1^{1/2}, 2\phi). \tag{39}$$

But $\tan \phi=\dfrac{\xi_2}{\xi_1}\cdot$ Therefore,

$$\frac{d\phi}{d\tau}=\frac{1}{4(\xi_1^2+\xi_2^2)}(\xi_1 \pi_2-\xi_2 \pi_1)-\tfrac{1}{2}\xi^2=\frac{G}{2\xi^2}-\frac{\xi^2}{2}.$$

The condition (39) yields

$$G=\xi^4-\rho_1 \xi^2+\xi^2 \rho_1^{3/2} f(\rho_1^{1/2}, 2\phi). \tag{40}$$

The condition (40) corresponds to (37). At $\tau=0$, equation (40) is satisfied since there both $G(0)$ and $\xi(0)$ are zero.

The orbit chosen will certainly satisfy equation (40) since it starts at the origin. If the ejection angle for $\mu \neq 0$ case is almost the same as for the $\mu=0$ case ; then for sufficiently small value of μ, the two orbits will remain as close as wish in a finite length of time ([8]). This means that after a time $\frac{1}{4}T$ has elapsed, the $\mu \neq 0$ orbit will re-enter an arbitrary small neighbourhood of $\rho_1=0$. Since equation (40) is satisfied along the entire orbit, the infinitesimal body will approach the origin with characteristic of a collision orbit ([6], p. 1).

REFERENCES

1. K. B. Bhatnagar, Periodic orbit of collision in the plane circular problem of four bodies, Indian J. Pure Appl. Math. 2 (1971), 583–596.

2. K. B. Bhatnagar, Periodic orbit of collision in the plane elliptic restricted problem of three bodies, Proc. Nat. Inst. Sci. India Part A 35 (1969), 829–844.

3. **R. K. Choudhry**, Existence of periodic orbits of the third kind in the elliptic restricted problem of the three bodies and the stability of the generating solution, Proc. Nat. Acad. India. Sect. A **36** (1966), 249–264.

4. **G. N. Duboshin,** Celestial Mechanics (Analytical and Qualitative Methods) (Russian), 178–184, 1964.

5. **Georgio E. O. Giacaglia,** Periodic orbits of collision in the restricted problem of three bodies, Astronom. J. **72** (1967), 386–391.

6. **T. Levi-Civita,** Traiettorie Singolari edurti nel problema ristretto dei tre corpi, Ann. Mat. Pura Appl. (3) **9** (1904), 1–21.

7. **T. Levi-Civita,** Sur la re'solution qualitative du proble'me restreint des trois corps, Acta Math. 30 (1906), 305–327.

8. **H. Poincare'**, Lecons de Mecanique Ce'leste, Vol. 1, Gauthier-Villars, Paris, 1905.

Jñānabha Sect. A, Vol. 6, 1976

A WATSON SUM FOR NON-TERMINATING DOUBLE HYPERGEOMETRIC SERIES

By R. P. Singal

Department of Mathematics, Guru Nanak College, Ferozepore–152001, *India*

(Received : February 7, 1976 ; in revised form : September 28, 1976)

1. Recently, Carlitz ([4], [5]) gave two Saalschutzian theorem for double hypergeometric series. Subsequently, the author [7] presented a unification of Carlitz's results and also gave ([7], [8], [9]) what may be regarded as Dixon and Watson sums for terminating double hypergeometric series. The object of this paper is to obtain a Watson sum for the non-terminating double hypergeometric series :

$$F^{0\,:\,3\,;\,3}_{1\,:\,1\,;\,1}\left[\begin{matrix}-:\,d,\,a,\,b\,;\,d,\,a',\,b'\,;\\ 2d:\,(a+b+1)/2\,;\,(a'+b'+1)/2\,;\end{matrix}\right],$$

$$1+2d=a+a',\ \min\{Re\,(a-b'),\ Re\,(a'-b)\}>0.$$

Following the notation due to Burchnall and Chaundy [3], as in [7], the Kampe' de Fe'riet hypergeometric series can be defined as (cf. also [11], p. 423, Eq. (26) et seq.)

$$F^{p\,:\,q\,;\,Q}_{r\,:\,s\,;\,S}\left[\begin{matrix}(a_p):(b_q)\,;\,(B_Q)\,;\\ (c_r):(d_s)\,;\,(D_S);\end{matrix}\ X,\,Y\right]$$
$$=\sum_{m,n=0}^{\infty}\frac{((a_p))_{m+n}\,((b_q))_m((B_Q))_n}{((c_r))_{m+n}\,((d_s))_m((D_S))_n}\,\frac{X^mY^n}{m!\,n!},$$

where (a_p) stands for the sequence $a_1, \ldots, a_p$, $(\lambda)_m=\Gamma(\lambda+m)/\Gamma(\lambda)$, and in general, $q\neq Q$, $s\neq S$.

2. The result to be proved is

$$F^{0\,:\,3\,;\,3}_{1\,:\,1\,;\,1}\left[\begin{matrix}-:\,d,\,a,\,b\,;\,d,\,a',\,b'\,;\\ 2d:\,(1+a+b)/2\,;\,(1+a'+b')/2\,;\end{matrix}\right]$$

$$=\Gamma\left(\begin{matrix}\tfrac12,\,\tfrac12,\,\tfrac12,\,\tfrac12(1+2d),\,\tfrac12(1+a+b),\,\tfrac12(1+a'+b'),\\ \tfrac12a,\,\tfrac12a',\,\tfrac12(1+a),\,\tfrac12(1+a'),\,\tfrac12(1+b),\,\tfrac12(1+b'),\end{matrix}\right.$$
$$\left.\begin{matrix}\tfrac12(a-b'),\,\tfrac12(a'-b),\,\tfrac12(1-b-b')\\ \tfrac12(1-b),\,\tfrac12(1-b'),\,\tfrac12(1+2d-b-b')\end{matrix}\right)$$

$$-\Gamma\begin{pmatrix}\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}(1+2d), \frac{1}{2}(1+a+b), \frac{1}{2}(1+a'+b'), \\ \frac{1}{2}a, \frac{1}{2}a', \frac{1}{2}b, \frac{1}{2}b', \frac{1}{2}(1+a), \frac{1}{2}(1+a'),\end{pmatrix.$$

$$\left.\begin{matrix}\frac{1}{2}(a-b'), \frac{1}{2}(a'-b), \frac{1}{2}(1-b-b') \\ 1-\frac{1}{2}b, 1-\frac{1}{2}b', 1+d-\frac{1}{2}b-\frac{1}{2}b'\end{matrix}\right), \tag{2.1}$$

provided $1+2d=a+a'$, $Re\,(a-b')>0$, and $Re\,(a'-b)>0$, where, for convenience,

$$\Gamma\begin{pmatrix}a, b, c, \ldots \\ p, q, r, \ldots\end{pmatrix}=\frac{\Gamma(a)\,\Gamma(b)\,\Gamma(c)\ldots}{\Gamma(p)\,\Gamma(q)\,\Gamma(r)\ldots}.$$

Proof of (2.1). Consider the double series

$$F\equiv F^{0\,:\,3\,;\,3}_{1\,:\,1\,;\,1}\left[\begin{matrix}-: a, b, d\,; a', b', d'\,; \\ c: \quad e\,; \quad e'\,;\end{matrix}\right]$$

$$=\sum_{m,\,n=0}^{\infty}\frac{(a)_m\,(b)_m\,(d)_m\,(a')_n\,(b')_n\,(d')_n}{(c)_{m+n}\,(e)_m\,(e')_n\,m\,!\,n\,!}.$$

Using the well-known sum (cf. [2], p. 28, Eq. (3))

$$\sum_{k=0}^{m,n\,(m,\,n)}\frac{(c-1)_k\,(c)_{2k}\,(-m)_k\,(-n)_k\,(-1)^k}{(c-1)_{2k}\,(c+m)_k\,(c+n)_k\,k\,!}=\frac{(c)_m\,(c)_n}{(c)_{m+n}},$$

and after some simplification, we have

$$F=\sum_{k=0}^{\infty}\frac{(a)_k\,(b)_k\,(d)_k\,(a')_k\,(b')_k\,(d')_k\,(c-1)_k\,(-1)^k}{(e)_k\,(e')_k\,(c)_{2k}\,(c-1)_{2k}\,k\,!}$$

$$\times{}_3F_2\begin{pmatrix}a+k, b+k, d+k\,; \\ e+k, c+2k\,;\end{pmatrix}{}_3F_2\begin{pmatrix}a'+k, b'+k, d'+k\,; \\ e'+k, c+2k\,;\end{pmatrix}. \tag{2.2}$$

Now taking $c=2d=2d'$, $e=(1+a+b)/2$, $e'=(1+a'+b')/2$, using the familiar Watson sum ([2], p.16) and on separating the even and odd terms, it is expressible as the difference of two ${}_7F_6\,(1)$ series. On choosing $1+2d=a+a'$, each ${}_7F_6\,(1)$ reduces to ${}_5F_4\,(1)$, which can be summed by using the known formula ([10], p. 56, Eq. (2.3.4.5)), giving us the desired result (2.1).

REMARK. Formula (2.1) reduces to the Watson sum when $b'=0$.

3. The ${}_3F_2\,(1)$ series in (2.2) can also be summed by Saalschutz's theorem, which after some simplification gives

$$F^{0\,:\,3\,;\,3}_{1\,:\,1\,;\,1}\left[\begin{matrix}-: a, b, -n \quad ; a', b', -m \quad ; \\ c: 1+a+b-c-n\,; 1+a'+b'-c-m\,;\end{matrix}\right]$$

$$=\frac{(c-a)_n\,(c-b)_n\,(c-a')_m\,(c-b')_m}{(c)_n\,(c)_m\,(c-a-b)_n\,(c-a'-b')_m}$$

$$\times{}_8F_7\begin{pmatrix}c-1, \frac{1}{2}(c+1), a, a', b, b', -n, -m\,; \\ \frac{1}{2}(c-1), c-a, c-a', c-b, c-b', c+n, c+m\,;\end{pmatrix}-1\Bigg).$$

There exist no sums for ${}_8F_7\,(-1)$. The known sum for ${}_6F_5$ gives no elegant result. It can be summed if it is reduced to ${}_4F_3$ in the following three ways :

(*i*) $c=a+a'=b+b'$;

(*ii*) $c=a-n=a'-m$;

(*iii*) $c=a-m=a'-n$.

In the first case we get the result due to Jain [6]. The second case gives us

$$F^{0\,:\,3\,;\,3}_{1\,:\,1\,;\,1}\left[\begin{matrix} -: & a,\, b,\, -n\,; & a',\, b',\, -m\,; \\ c: & 1+b\,; & 1+b'\,; \end{matrix}\right]$$

$$=\frac{\Gamma(a-b)\,\Gamma(a'-b')\,(1+b+b'-a')_m\,n\,!\,m\,!}{\Gamma(a)\,\Gamma(a'-b-b')\,(1+b)_n\,(1+b')_m(1-a')_m}. \tag{3.1}$$

In the third case, the sum will be zero unless $n=m$, in which case $a=a'$, and hence we will get a particular case of the result (3.1) above.

ACKNOWLEDGEMENT. I express my sincere thanks to Professor H. M. Srivastava for pointing out the references [3] and [11] below.

REFERENCES

1. P. Appell et J. Kampe´ de Fe´riet, Fonction hyperge´ome´triques et hypers phe´riques ; Polynomes d'Hermite, Gauthier-Villars, Paris 1926

2. W. N. Bailey, Generalized hypergeometric series, Cambridge, 1935.

3. J. L. Burchnall and T. W. Chaundy, Expansions of Appell's double hypergeometric functions (II), Quart. J. Math. (Oxford) 12 (1941), 112–128.

4. L. Carlitz, A Saalschutzian theorem for double series, J. London Math. Soc. 38 (1963), 415–418.

5. L. Carlitz, Another Saalschutzian theorem for double series, Rend. Sem. Mat. Univ. Padova 34 (1964), 200–203.

6. R. N. Jain, Sum of a double hypergeometric series, Matematiche (Catania) 21 (1966), 300–301.

7. R. P. Singal, Transformation formulae for Kampe´ de Fe´riet hypergeometric function, Indian J. Pure Appl. Math. 2 (1971), no 4, 610–614.

8. R P. Singal, A transformation formula for double hypergeometric series, Rocky Mountain J. Math 3 (1973), 377–381.

9. R. P. Singal, A Watson sum for the double hypergeometric series, Ganita to appear ; see also Math. Student 39 (1971), 247.

10. L. J. Slater, Generalized hypergeometric series, Cambridge. 1966.

11. H. M. Srivastava and R. Panda, An integral representation for the product of two Jacobi polynomials, J. London Math. Soc. (2) 12 (1976), 419–425.

Jñānabha Sect. A, Vol 6, 1976

SOME IDENTITIES AND HYPERGEOMETRIC FUNCTIONAL EXPANSIONS

By **B. M. Singhal**

Department of Mathematics, Government Science College, Gwalior–474001, *M.P., India*

(Received : September 15, 1975 ; in revised form : March 26, 1976)

1. Introduction. Some years ago, Srivastava [5] gave various classes of expansions for the generalized Appell functions of two variables (cf., *e.g.*, [1], p. 150). Subsequently, Ragab [4] rediscovered an obvious special form of one of Srivastava's expansions ([5], p. 48, Eq. (7)). These expansion formulas seem to have been motivated by similar results for the ordinary Appell functions which were given earlier by Burchnall and Chaundy ([2], [3]). The object of the present note is to derive certain general identities, involving double series with arbitrary terms, from which some of Srivastava's expansions would follow as special cases. Indeed, if we let $\{f(n)\}$, $\{g(n)\}$ and $\{h(n)\}$ be sequences of arbitrary complex numbers, then the identities to be established may be unified formally as :

$$(1.1) \qquad \sum_{r=0}^{\infty} A_r \sum_{m,n=0}^{\infty} B_{r,m,n} f(m+n+2r)\, g(m+r)\, h(n+r)$$

$$= \sum_{m,n=0}^{\infty} f(m+n)\, g(m)\, h(n) \sum_{r=0}^{\infty} A_r B_{r,m-r,n-r}$$

for arbitrary coefficients A_r and $B_{r,m,n}$.

The following known forms of certain familiar hypergeometric summation theorems will be required in our analysis (cf. [2]) :

$$(1.2) \qquad \sum_{r=0}^{\infty} \frac{(-m)_r(-n)_r}{r!\,(\alpha)_r} = \frac{(\alpha)_{m+n}}{(\alpha)_m\,(\alpha)_n} ;$$

$$(1.3) \qquad \sum_{r=0}^{\infty} \frac{(-m)_r(-n)_r}{r!\,(-\alpha-m-n+1)_r} = \frac{(\alpha)_m\,(\alpha)_n}{(\alpha)_{m+n}} ;$$

Of these (1.2) is valid if m, n are positive integers and (1.3) is valid when one of m, n is an integer. Also the following summation theorems are valid only when one of m, n is a positive integer :

(1.4) $$\sum_{r=0}^{\infty} (-1)^r \frac{(\alpha)_{2r}\,(-m)_r\,(-n)_r}{r!\,(\alpha+r-1)_r\,(m+\alpha)_r\,(n+\alpha)_r} = \frac{(\alpha)_m\,(\alpha)_n}{(\alpha)_{m+n}};$$

(1.5) $$\sum_{r=0}^{\infty} \frac{(\gamma-\alpha)_r\,(\gamma)_{2r}\,(-m)_r\,(-n)_r}{r!\,(\gamma+r-1)_r\,(m+\gamma)_r\,(n+\gamma)_r\,(\alpha)_r} = \frac{(\alpha)_{m+n}(\gamma)_m(\gamma)_n}{(\gamma)_{m+n}(\alpha)_m(\alpha)_n};$$

(1.6) $$\sum_{r=0}^{\infty} \frac{(\alpha-\gamma)_r\,(-m)_r\,(-n)_r}{r\,!\,(\alpha)_r\,(-\gamma-m-n+1)_r} = \frac{(\alpha)_{m+n}\,(\gamma)_m(\gamma)_n}{(\gamma)_{m+n}\,(\alpha)_m(\alpha)_n}.$$

2. Reducible forms of (1.1). By suitably specializing the coefficients A_r and B_r, m, n, the innermost series on the right-hand side of (1.1) can be summed by one or the other of the known results (1.2) to (1.6), and we thus arrive at the following identities :

(2.1) $$\sum_{r=0}^{\infty} \frac{(-1)^r\,(\alpha)_r\,(\gamma-\alpha)_r}{r\,!\,(\gamma)_r}$$
$$\times \sum_{m,\,n=0}^{\infty} \frac{(\alpha+r)_{m+n}}{m\,!\,n\,!} f(m+n+2r)\, g\,(m+r)\, h\,(n+r)$$
$$= \sum_{m,\,n=0}^{\infty} \frac{(\gamma)_{m+n}\,(\alpha)_m\,(\alpha)_n}{m\,!\,n\,!\,(\gamma)_m\,(\gamma)_n} f\,(m+n)\, g\,(m)\, h\,(n);$$

(2.2) $$\sum_{r,\,m,\,n=0}^{\infty} \frac{f(m+n+2r)\, g(m+r)\, h(n+r)}{r\,!\,m\,!\,n\,!\,(\gamma)_r}$$
$$= \sum_{m,\,n=0}^{\infty} \frac{(\gamma)_{m+n}}{(\gamma)_m\,(\gamma)_n\,m\,!\,n\,!} f(m+n)\, g(m)\, h(n);$$

(2.3) $$\sum_{r=0}^{\infty} \frac{(-1)^r\,(\alpha)_r}{r\,!}$$
$$\times \sum_{m,\,n=0}^{\infty} \frac{(\alpha+r)_{m+n}}{m\,!\,n\,!} f\,(m+n+2r)\, g(m+r)\, h(n+r)$$
$$= \sum_{m,\,n=0}^{\infty} \frac{(\alpha)_m\,(\alpha)_n}{m\,!\,n\,!} f(m+n)\, g(m)\, h(n);$$

(2.4) $$\sum_{r=0}^{\infty} \frac{(-1)^r}{r\,!\,(\gamma+r-1)_r\,(\gamma)_{2r}} \sum_{m,\,n=0}^{\infty} \frac{f(m+n+2r)\, g(m+r)\, h(n+r)}{(\gamma+2r)_m\,(\gamma+2r)_n\,m\,!\,n\,!}$$
$$= \sum_{m,\,n=0}^{\infty} \frac{f(m+n)\, g(m)\, h(n)}{m\,!\,n\,!\,(\gamma)_{m+n}};$$

$$(2.5)\qquad \sum_{r=0}^{\infty} \frac{(\gamma-\alpha)_r}{r!\,(\gamma+r-1)_r\,(\alpha)_r\,(\gamma)_{2r}}$$

$$\times \sum_{m,\,n=0}^{\infty} \frac{f(m+n+2r)\,g(m+r)\,h(n+r)}{(\gamma+2r)_m\,(\gamma+2r)_n\,m!\,n!}$$

$$= \sum_{m,\,n=0}^{\infty} \frac{(\alpha)_{m+n}}{(\gamma)_{m+n}\,(\alpha)_m\,(\alpha)_n} \cdot \frac{f(m+n)\,g(m)\,h(n)}{m!\,n!},$$

provided that the various series involved are absolutely convergent.

3. Applications. By appropriately choosing the arbitrary sequences $\{f(n)\}$, $\{g(n)\}$ and $\{h(n)\}$, the identities (2.1) to (2.5) of the preceding section can be applied to derive some of the expansion formulas given earlier by Srivastava [5]. For example, if we let

$$(3.1)\qquad f(n)=\frac{\prod_{j=1}^{A}(a_j)_n}{(\gamma)_n \prod_{j=1}^{C}(c_j)_n}, \qquad g(n)=\frac{\prod_{j=1}^{B}(b_j)_n}{\prod_{j=1}^{D}(d_j)_n}x^n,$$

and

$$(3.2)\qquad h(n)=\frac{\prod_{j=1}^{B'}(b')_{jn}}{\prod_{j=1}^{D'}(d'_j)_n}\,y^n, \qquad n=0,1,2,\ldots,$$

our identity (2.1) will yield the following result due to Srivastava ([5], p. 48, Eq. (7)) :

$$(3.3)\qquad F\left[\begin{matrix}(a):(b),\alpha\,;(b'),\alpha\,;\\(c):(d),\gamma\,;(d'),\gamma\,;\end{matrix}\ x,y\right]$$

$$=\sum_{r=0}^{\infty}\frac{(-1)^r(\alpha)_r(\gamma-\alpha)_r\,f(2r)\,g(r)\,h(r)}{r!\,(\gamma)_r}$$

$$\times F\left[\begin{matrix}(a)+2r,\alpha+r\ :(b)+r\,;(b')+r\,;\\(c)+2r,\gamma+2r:(d)+r\,;(d')+r\,;\end{matrix}\ x,y\right],$$

where f, g, h are given by (3.1) and (3.2), (a) abbreviates the sequence of A parameters $a_1, \ldots, a_A$, with similar interpretations for (b), (b'), etc., and the notation used for the double hypergeometric functions occurring on either side of (3.3) is due to Burchnall and Chaundy [3, p. 112].

Remark. An obvious special form of Srivastava's expansion (3.3) above happens to be the main result (12) on page 346 of a latter paper by Ragab [4].

ACKNOWLEDGEMENTS. The author wishes to thank Dr. B. M. Agrawal for his kind supervision and to Professor H. M. Srivastava for some helpful suggestions.

REFERENCES

1. P. Appell and J. Kampe′ de Fe′riet, Fonctions hyperge′ome′triques et hypersphe′riques, Paris, 1926.
2. J. L. Burchnall and T. W. Chaundy, Expansions of Appell's double hypergeometric functions (I) Quart. J. Math. (Oxford), **11** (1940) 249–270.
3. J. L. Burchnall and T. W. Chaundy, Expansions of Appell's double hypergeometric functions (II), Quart. J. Math. (Oxford) **12** (1941), 112–128.
4. F. M. Ragab, Expansions of Kampe′ de Fe′riet's double hypergeometric functions of higher order, J. Natur. Sci. Math. **12** (1972), 343–361.
5. H. M. Srivastava, Certain pairs of inverse series relations, J. Reine Angew. Math. **245** (1970), 47–54.

Jñānabha Sect. A, Vol. 6, 1976

Generalized Gauss-Codazzi Equations for the Curvature Tensors $R^{\;i}_{jhk}$ $(x, \dot{x})$ in a Hypersurface of a Finsler Space

By **H. D. Pande and A. Kumar**†

Department of Mathematics, University of Gorakhpur,

Gorakhpur–273001, *U. P., India*

(Received : February 20, 1975 ; in final form : April 15, 1976)

Summary. Rund [1] derived Gauss-Codazzi equations for the curvature tensor $K_{\alpha\beta\gamma\epsilon}$ in a Finsler space. Srivastava and subsequent authors (cf. [3], [4] and [5]) have derived them for Cartan's first and second curvature tensors in a subspace of a Finsler space. The object of the present paper is to derive these equations for the curves of congruences associated with a hypersurface of an n-dimensional Finsler space.

1. Introduction. Let us consider an n-dimensional Finsler space F_n with the fundamental metric tensor $F(x, \dot{x})$ homogeneous of degree 1 in $\dot{x}^i$. The metric tensor is given by $g_{ij}\ (x, \dot{x}) \overset{\text{def}}{=} \frac{1}{2}\dot{\partial}_i\, \dot{\partial}_j\ F^2(x, \dot{x})$ and is symmetric in i and j. The symbols ∂_i, $\dot{\partial}_i$ and $\dot{\partial}_{ij}{}^2$ denote the partial derivatives $\partial/\partial x^i$, $\partial/\partial \dot{x}^i$ and $\partial^2/\partial \dot{x}^i\, \partial \dot{x}_j$, respectively.

The covariant derivatives of $T^i_j\ (x, \dot{x})$ with respect to x^k in the sense of Cartan are given by

$$(1.1)a \qquad T^i_{j|k} = \partial_k T^i_j - \left(\dot{\partial}_m\, T^i_j\right) G^m_k + T^h_j\, \Gamma^{*i}_{hk} T - {}^i_h \Gamma^{*h}_{jk}$$

† Present address : Department of Applied Sciences, M. M. M. Engineering College, Gorakhpur-273001, U. P., India.

and

(1.1)b $$T^i_j\|_k = F\dot{\partial}_k T^i_j + T^h_j A^i_{hk} - T^i_h A^h_{jk},$$

where the connection coefficient $\overset{*}{\Gamma}{}^i_{hk}(x, \dot{x})$ and tensor field $A^i_{hk}(x, \dot{x})$ have their usual meanings. For details see Rund [2].

A hypersurface F_{n-1} of F_n may be represented parametrically in the form $x^i = x^i(u^\alpha)$, $\alpha = 1, \ldots\ldots, n-1$, where the $n-1$ parameters u^α form the coordinate system of F_{n-1} and it will be assumed that the matrix of the projection factors $B^i = \partial_\alpha x^i$ has the rank $n-1$. Throughout this discussion we shall denote $B^i_{\alpha\beta} = \partial^2_{\alpha\beta} x^i$ and $B^{ij\ldots k}_{\alpha\beta\ldots\gamma} = B^i_\alpha B^j_\beta \; B^k_\gamma$.

It is easy to verify that the vector $\dot{u}^\alpha$, the induced metric tensor $g_{\alpha\beta}(u, \dot{u})$ and the induced symmetric connection defined in the hypersurface F_{n-1} are related with the corresponding quantities in the space F_n by the following relations.

(1.2) $$\dot{x}^i = \dot{u}^\alpha B^i_\alpha,$$

(1.3) $$g_{\alpha\beta} = g_{ij} B^{ij}_{\alpha\beta},$$

(1.4)a $$\overset{*}{\Gamma}{}^\alpha_{\beta\gamma}(u, \dot{u}) = B^\alpha_i \left(B^i_{\beta\gamma} + \overset{*}{\Gamma}{}^i_{hk} B^{hk}_{\beta\gamma} \right).$$

Furthermore,

$$B^\alpha_i = g^{\alpha\epsilon} B^s_\epsilon \, g_{is}, \quad B^\alpha_i B^i_\beta = \delta^\alpha_\beta.$$

If $N^i(x, \dot{x})$ denotes the unit normal at each point P of F_{n-1} with respect to the tangential direction $\dot{x}^i$ at P, we then have

(1.4)b $$g^{ij} N_i N_j = N_i N^i = 1, \; N_i B^i_\alpha = g_{ij} N^j B^i_\alpha = 0,$$

$$N^i B^\alpha_i = 0, \text{ and } g_{ij} N^i N^j = 1.$$

The induced mixed covariant derivative of T^i_α as denoted by $T^i_{\alpha\|\gamma}$ is given by

$$(1.5) \qquad T^i_{\alpha\,\|\,\gamma}=\partial_\gamma T^i_\alpha-\left(\dot{\partial}_\beta T^i_\alpha\right)\overset{*}{\Gamma}{}^{\beta}_{\rho\gamma}\,\dot{u}^\rho+T^s_\alpha\overset{*}{\Gamma}{}^{i}_{sh}\,B^h_\gamma-T^i_\epsilon\overset{*}{\Gamma}{}^{\epsilon}_{\alpha\gamma}.$$

With the help of mixed covariant derivative $T^i_{\alpha\,\|\,\gamma}$ we can construct a mixed tensor [5]

$$(1.6) \qquad I^i_{\alpha\beta}=B^i_{\alpha\,\|\,\beta}=B^i_{\alpha\beta}-B^i_\epsilon\overset{*}{\Gamma}{}^{\epsilon}_{\alpha\beta}+\overset{*}{\Gamma}{}^{i}_{hk}\,B^{hk}_{\alpha\beta}\,,$$

which are regarded as a vector of imbeding space F_{n-1} and are also normal to F_{n-1}. Thus, we can write

$$(1.7) \qquad I^i_{\alpha\beta}=N^i\,\overline{\Omega}_{\alpha\beta}\,,$$

where $\overline{\Omega}_{\alpha\beta}$ is the second fundamental form symmetric in α and β.

Verma and Sinha [5] obtained the induced derivative $N^i_{\,\|\,\beta}$ in the form

$$(1.8) \qquad N^i_{\,\|\,\beta}=-\,\overline{\Omega}_{\alpha\beta}\,g^{\alpha\delta}\,B^i_\delta+E^i_m\,I^m_{\rho\beta}\,\dot{u}^\rho,$$

where

$$(1.9) \qquad E^i_m=N^iM_m-2M^i_m,\; M^i_m=C^i_{mp}\,N^p\,,$$

$$M_m=C_{pkm}\,N^pN^k=M_{km}\,N^k.$$

2. Generalized Gauss-Codazzi Equations. We consider congruence of curves such that each of them passes through each point of the hypersurface. Let λ^i be the contravariant components of the unit vector in the direction of the curve of congruences. This vector may be expressed linearly in terms of B^i_α and normal vectors N^i of F_{n-1}. Thus, we write

$$(2.1) \qquad \lambda^i=t^\alpha B^i_\alpha+dN^i\,,$$

where d and t are the parameters. By taking the induced covariant derivative of type (1.5) of the equation (2.1), we have

$$(2.2) \qquad \lambda^i_{\,\|\,\beta}=B^i_\alpha\,t^\alpha_{\,\|\,\beta}+t^\alpha\,B^i_{\alpha\,\|\,\beta}+d_{\,\|\,\beta}\,N^i+d\,N^i_{\,\|\,\beta}.$$

Now differentiating (2.2) covariantly with respect to u^γ in the sense of (1.5), we obtain

(2.3) $\lambda^{i}_{\parallel [\beta\gamma]} = B^{i}_{\alpha} t^{\alpha}_{\parallel [\beta\gamma]} + t^{\alpha} B^{i}_{\alpha \parallel [\beta\gamma]} + d N^{i}_{\parallel [\beta\gamma]} + N^{i} d_{\parallel [\beta\gamma]}$.

Also from equations (1.6), we can write

(2.4) $$B^{i}_{\alpha \parallel [\beta\gamma]} = I^{i}_{\alpha[\beta \parallel \gamma]}.$$

On substituting the value of $I^{i}_{\alpha\beta}$ and $N^{i}_{\parallel \gamma}$ from (1.7) and (1.8), we obtain

(2.5) $$B^{i}_{\epsilon \parallel [\beta\gamma]} = N^{i} \bar{\Omega}_{\epsilon[\beta \parallel \gamma]} + \bar{\Omega}_{\epsilon[\beta} \bar{\Omega}_{\gamma]\alpha} g^{\alpha\delta} B^{i}_{\delta} + E^{i}_{m} I^{m}_{\rho[\gamma} \bar{\Omega}_{\beta]\epsilon} \dot{u}^{\rho}.$$

Similarly, with the help of equations (1.7) and (1.8), we obtain

(2.6) $$N^{i}_{\parallel [\beta\gamma]} = B^{i}_{\delta} \bar{\Omega}_{\alpha[\gamma \parallel \beta]} g^{\alpha\delta} + E^{i}_{m} (I^{m}_{\rho[\beta} \dot{u}^{\rho})_{\parallel \gamma]} + I^{m}_{\rho[\beta} E_{<m>\gamma]} \dot{u}^{\rho}.$$

On differentiating the equation $\lambda^{i}_{\parallel \beta} = \lambda^{i}_{| h} B^{h}_{\beta}$ with respect to u^{γ} and using the commutation formula

$$2\lambda^{i}_{| [hk]} = R^{i}_{jhk} \lambda^{j} - K^{j}_{mhk} e^{m} \lambda^{i}_{| j},$$

we get

(2.7) $$2\lambda^{i}_{\parallel [\beta\gamma]} = \left(R^{i}_{jhk} \lambda^{j} - K^{j}_{mhk} l^{m} \lambda^{i}_{|j} \right) B^{hk}_{\beta\gamma}.$$

Similarly, we have

(2.8) $$2t^{\delta}_{\parallel [\beta\gamma]} = R^{\delta}_{\alpha\beta\gamma} t^{\alpha} - K^{\alpha}_{\epsilon\beta\gamma} l^{\epsilon} t^{\delta}_{,\alpha}$$

where a comma denotes the induced covariant derivative of (1.1)*b* and

(2.9) $$R^{\delta}_{\alpha\beta\gamma}(u, \dot{u}) = 2\Big\{ \partial_{[\gamma} \overset{*}{\Gamma}{}^{\delta}_{\beta]\alpha} - \Big(\dot{\partial}_{[\rho} \overset{*}{\Gamma}{}^{\delta}_{\alpha[\beta} \Big) G^{\alpha}_{\gamma]} + C^{\delta}_{\alpha\epsilon} (\partial_{[\gamma} \dot{\partial}_{\beta]} G^{\epsilon} - G^{\epsilon}_{\rho[\beta} G^{\rho}_{\gamma]} + \overset{*}{\Gamma}{}^{\delta}_{\epsilon[\gamma} \overset{*}{\Gamma}{}^{\delta}_{\beta]\alpha} \Big\},$$

(2.10) $$K^{\delta}_{\alpha\beta\gamma}(u, \dot{u}) = 2 \Big\{ \partial_{[\gamma} \overset{*}{\Gamma}{}^{\delta}_{\beta]\alpha} - \Big(\dot{\partial}_{\rho} \overset{*}{\Gamma}{}^{\delta}_{\alpha[\beta} \Big) G^{\rho}_{\gamma]} + \overset{*}{\Gamma}{}^{\delta}_{\epsilon[\gamma} \overset{*}{\Gamma}{}^{\epsilon}_{\beta]\alpha} \Big\}$$

We know that

(2.11) $$d_{\parallel \beta} = \partial_{\beta} d - (\dot{\partial}_{\alpha} d) G^{\alpha}_{\beta}.$$

Differentiating (2.11) covariantly with respect to u^γ in the sense of (1.5) again and subtracting the equation thus obtained by interchanging the indices β and γ, we obtain

$$d_{\parallel[\beta\gamma]} = \dot{\partial}_\alpha \, d\left(\partial_{[\beta} d G^\alpha_{\gamma]} + G^\alpha_{\rho[\beta} G^\rho_{\gamma]}\right). \tag{2.12}$$

By substituting equations (2.5), (2.6), (2.7), (2.8) and (2.12) in (2.3) and after arranging the terms, we get

$$\begin{aligned}(2.13)\quad R^i_{jhk}(x,\dot{x})\lambda^j B^{hk}_{\beta\gamma} &= R^\delta_{\alpha\beta\gamma}(u,\dot{u})t^\alpha B^i_\delta + K^j_{mhk}\, l^m \lambda^i\big|_j B^{hk}_{\beta\gamma}\\ &\quad - B^i_\delta K^\alpha_{\epsilon\beta\gamma}\, l^\epsilon t^\delta_{,\alpha} + 2\Big[B^i_\delta\big\{\alpha\,\overline{\Omega}_{\alpha[\gamma\parallel\beta]} + t^\epsilon\,\overline{\Omega}_{\epsilon[\beta}\overline{\Omega}_{\gamma]\alpha}\big\}g^{\alpha\delta}\\ &\quad + N^i\big\{\dot{\partial}_\epsilon\, d\big(\partial_{[\beta} dG^\epsilon_{\gamma]} + G^\epsilon_{\rho[\beta} G^\rho_{\gamma]}\big) + t^\epsilon\,\overline{\Omega}_{\epsilon[\beta\parallel\gamma]}\big\}\\ &\quad + E^i_m\big\{t^\epsilon\,\overline{\Omega}_{\epsilon[\beta} I^m_{<\rho>\gamma]}\dot{u}^\rho + d\big(I^m_{\rho[\beta}\dot{u}^\rho\big)_{\parallel\gamma]}\big\}\\ &\quad + d\, I^m_{\rho[\beta} E^i_{<m>\parallel\gamma]}\dot{u}^\rho\Big].\end{aligned}$$

Multiplying equation (2.13) by B^δ_i and noting (1.9), we have

$$\begin{aligned}(2.14)\quad & R^i_{jhk}(x,\dot{x})\lambda^j B^\delta_i B^{hk}_{\beta\gamma}\\ &= R^\delta_{\alpha\beta\gamma}(u,\dot{u})\,t^\alpha + K^j_{mhk}\, l^m\lambda^i\big|_j B^\delta_i B^{hk}_{\beta\gamma} - K^\alpha_{\epsilon\beta\gamma}\, l^\epsilon t^\delta_{,\alpha}\\ &\quad + 2\Big[g^{\alpha\delta}\big\{d\,\overline{\Omega}_{\alpha[\gamma\parallel\beta]} + t^\epsilon\overline{\Omega}_{\epsilon[\beta}\overline{\Omega}_{\gamma]\alpha}\big\}\\ &\quad - B^\delta_i\Big\{2M^i_m\big(t^\epsilon\overline{\Omega}_{\epsilon[\beta} I^m_{<\rho>\gamma]}\dot{u}^\rho + d\big(I^m_{\rho[\beta}\dot{u}^\rho\big)_{\parallel\gamma]}\big)\\ &\quad - d\, I^m_{\rho[\beta} E^i_{<m>\parallel\gamma]}\dot{u}^\rho\Big\}\Big]\end{aligned}$$

Again multiplying (2.13) by N_i and using (1.4)b, we get

$$\begin{aligned}(2.15)\quad R^i_{jhk}(x,\dot{x})N_i\lambda^j B^{hk}_{\beta\gamma} &= K^j_{mhk} N_i l^m\lambda^i\big|_j B^{hk}_{\beta\gamma}\\ &\quad + 2\Big[\dot{\partial}_\epsilon d\big\{\partial_{[\beta} dG^\epsilon_{\gamma]} + G^\epsilon_{\rho[\beta} G^\rho_{\gamma]}\big\} + t^\epsilon\overline{\Omega}_{\epsilon[\beta\parallel\gamma]}\\ &\quad + N_i\Big\{E^i_m\, t^\epsilon\overline{\Omega}_{\epsilon[\beta} I^m_{<\rho>\gamma]}\dot{u}^\rho + d\big(I^m_{\rho[\beta}\dot{u}^\rho\big)_{\parallel\gamma]}\\ &\quad + d\, I^m_{\rho[\beta} E^i_{<m>\parallel\gamma]}\dot{u}^\rho\Big\}\Big].\end{aligned}$$

Equations (2.14) and (2.15) which are based on a vector λ^i of most general nature can be regarded as generalization of Gauss-Codazzi equations in a hypersurface imbeded in a Finsler space F_n.

3. Particular Cases. Since the vector λ^i is a linear combination of B^i_α and N^i, therefore we can consider a congruence of curves in three different ways. Firstly, it is such that the vector λ^i in the direction of the curves of the congruences is normal to F_{n-1}, *i.e.*, $\lambda^i = dN^i$; secondly, it lies in the space spanned by B^i_α, *i.e.*, $\lambda^i = t^\alpha B^i_\alpha$; and thirdly, it is tangential to the curves $\dot{x}^i = \dot{u}^\alpha B^i_\alpha$.

REFERENCES

1. H. Rund, Hypersurfaes of a Finsler space, Canad. J. Math. **8** (1956), 487–503.
2. H. Rund, The differential geometry of Finsler spaces, Springer-Verlag, Berlin, 1959.
3. B. B. Sinha and S. P. Singh, A generalization of Gauss-Codazzi equations for the first curvature tensor of Cartan, Progress Math. (Allahabad) 6 (1972), 1–6.
4. S. C. Srivastava and R. S. Sinha, Generalized Gauss-Codazzi equations for a subspace of a Finsler space, Univ. Nac. Tucuman Rev. Ser. A **19** (1969), 123-129.
5. M. Verma and R. S. Sinha, Generalized Gauss-Codazzi equations for second of Cartan's curvature tensor, Tensor (N. S.) **22** (1971), 321–325.

Jñānabha Sect. A, Vol. 6, 1976

THERMAL STRESSES DUE TO PRESCRIBED FLUX OF HEAT WITHIN A CIRCULAR REGION IN AN INFINITE ELASTIC HALF SPACE

By **Mange Lal**
Christ Church College, Kanpur
AND
A. K. Chaturvedi
Pt. J. L. N. College, Banda, India

(Received : June 24, 1975)

Abstract. In this paper an exact solution has been obtained for the thermoelastic problem of an isotropic material occupying a half space with stress free edges subjected to two different temperature distributions, *i.e.* (i) a constant flux of heat within a circular region of exposure, the exterior of circular region being free from any flux of heat and (ii) paraboloid distribution of temperature within the circular region, the exterior being insulated. Numerical results have been given.

1. Introduction. The thermal stress problem in an elastic half-space at constant temperature $T=T_0$ inside a circle of radius a, the exterior of the circle being thermally insulated, was considered by Nowacki [2]. Recently Bhattacharya [1] has considered the problem of determining the thermal stresses due to prescribed flux of heat on the surface of thick plate. The object of this paper is to find the exact solution of the thermoelastic problem of an isotropic material with stress free edges subjected to two different temperature distributions. In the first case, we assume a constant flux of heat within a circular region of exposure, the exterior of the circular region being free from any flux of heat. Secondly, we assume a paraboloid distribution of flux within the circular region, the exterior being insulated. The numerical calculations for the variation of $(\widehat{\rho\rho}+\widehat{\theta\theta})$ on he free surface have also been obtained in both cases.

2. Solutions of the Equations of Thermoelasticity. We shall consider the temperature and displacement field in a perfectly elastic solid which is conducting heat. With regards to both its mechanical and thermal properties the solid will be assumed to be isotropic and homogeneous. It will be assumed that there is symmetry about z-axis and any point of the solid may be expressed in terms of cylindrical co-ordinates (ρ, θ, z). For symmetrical deformations of the solid the displacement and the only nonvanishing components of stress tensor will be $(u, 0, w)$ and $\widehat{\rho\rho}$, $\widehat{\theta\theta}$, $\widehat{zz}$, $\widehat{\rho z}$ respectively.

The temperature field is given by the Laplace's equation

$$\nabla^2 T \equiv \frac{\partial^2 T}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial T}{\partial \rho} + \frac{\partial^2 T}{\partial z^2} = 0, \tag{2.1}$$

in the steady state and in the absence of thermal sources. Stress components are obtained by using the potential of thermoelastic displacement ϕ given by the equations

$$u_T = \frac{\partial \phi}{\partial \rho} \quad \text{and} \quad w_T = \frac{\partial \phi}{\partial z}. \tag{2.2}$$

From the equations of equilibrium and the stress strain relations in problem of thermal stresses, we have

$$\frac{\partial^2 \phi}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial \phi}{\partial \rho} + \frac{\partial^2 \phi}{\partial z^2} = \beta T\ ;\ \beta = \frac{1+\eta}{1-\eta}\,\alpha, \tag{2.3}$$

where T is the deviation of the absolute temperature from the temperature of the solid in a state of zero stress and strain, α is the coefficient of linear expansion of the solid and η is its poisson ratio.

A particular integral of the equation (2.3) is given by

$$\phi = \frac{\beta}{2}\int_0^\infty \xi^{-3} A(\xi)(1+\xi z)e^{-\xi z} J_0(\xi\rho)\, d\xi, \tag{2.4}$$

where $A(\xi)$ is a function of ξ only.

From the relations (2.3) and (2.4), we obtain

$$T = \int_0^\infty \xi^{-1} A(\xi)\, e^{-\xi z} J_0(\xi\rho)\, d\xi \tag{2.5}$$

which satisfies equation (2.1).

The components of displacement and stress can now be written as

$$u_T = -\frac{\beta}{2}\int_0^\infty \xi^{-2} A(\xi)\,(1+\xi z)e^{-\xi z} J_1(\xi\rho)\, d\xi, \tag{2.6}$$

$$w_T = -\frac{\beta}{2}\, z \int_0^\infty \xi^{-1} A(\xi)\, e^{-\xi z} J_0(\xi\rho)\, d\xi, \tag{2.7}$$

$$\widehat{\rho z}_T = 2\mu\frac{\partial^2 \phi}{\partial \rho \partial z} = \mu\beta z \int_0^\infty A(\xi)\, e^{-\xi z} J_1(\xi\rho)\, d\xi. \tag{2.8}$$

(2.9)
$$\widehat{zz}_T=-2\mu\left(\frac{\partial^2\phi}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial\phi}{\partial\rho}\right)$$
$$=\mu\beta\int_0^\infty \xi^{-1}A(\xi)(1+\xi z)e^{-\xi z}J_0'(\xi\rho)\,d\xi,$$

(2.10)
$$\widehat{\rho\rho}_T=-2\mu\left(\frac{\partial^2\phi}{\partial z^2}+\frac{1}{\rho}\frac{\partial\phi}{\partial\rho}\right)$$
$$=\mu\beta\left\{\int_0^\infty \xi^{-2}A(\xi)(1+\xi z)\,e^{-\xi z}\frac{J_1(\xi\rho)}{\rho}\,d\xi\right.$$
$$\left.+\int_0^\infty \xi^{-1}A(\xi)(1-\xi z)\,e^{-\xi z}J_0(\xi\rho)\,d\xi\right\}$$

(2.11)
$$\widehat{\theta\theta}_T=-2\mu\left(\frac{\partial^2\phi}{\partial z^2}+\frac{\partial^2\phi}{\partial\rho^2}\right)$$
$$=\mu\beta\left\{2\int_0^\infty \xi^{-1}A(\xi)\,e^{-\xi z}J_0(\xi\rho)\,d\xi\right.$$
$$\left.-\frac{1}{\rho}\int_0^\infty \xi^{-2}A(\xi)(1+\xi z)e^{-\xi z}J_1(\xi\rho)\,d\xi\right\}.$$

The subscript T denotes that the stresses are due to the thermal, expansion, μ being the modulus of rigidity.

We observe that the shearing stress $\widehat{\rho z}_T$ vanishes for $z=0$, and the stress $\widehat{zz}_T$ does not vanish.

To satisfy the boundary conditions on the plane $z=0$, we superimpose an elementary stress system. The components of stresses and displacement are expressed by means of Love's function ψ by relations

(2.12)
$$u_c=\frac{1}{(1-2\eta)}\frac{\partial^2\psi}{\partial\rho\partial z}\;;\; w_c=\frac{1}{(1-2\eta)}\left[2(1-\eta)\nabla^2\psi-\frac{\partial^2\psi}{\partial z^2}\right],$$

(2.13)
$$\widehat{\rho\rho}_c=\frac{2\mu}{(1-2\eta)}\frac{\partial}{\partial z}\left[\eta\nabla^2\psi-\frac{\partial^2\psi}{\partial\rho^2}\right],$$
$$\widehat{\theta\theta}_c=\frac{2\mu}{(1-2\eta)}\frac{\partial}{\partial z}\left[\eta\nabla^2\psi-\frac{1}{\rho}\frac{\partial\psi}{\partial\rho}\right],$$
$$\widehat{\rho z}_c=\frac{2\mu}{(1-2\eta)}\frac{\partial}{\partial\rho}\left[(1-\eta)\nabla^2\psi-\frac{\partial^2\psi}{\partial z^2}\right],$$
$$\widehat{zz}_c=\frac{2\mu}{(1-2\eta)}\frac{\partial}{\partial z}\left[(2-2\eta)\nabla^2\psi-\frac{\partial^2\psi}{\partial z^2}\right],$$

the function ψ satisfying the biharmonic equation

(2.14)
$$\nabla^2\nabla^2\psi=0.$$

The solution of the equation (2.14) can be assumed in the form

$$\psi=\int_0^\infty \xi^{-3}\,\beta(\xi)\,(2\eta+\xi z)\,e^{-\xi z}J_0(\xi\rho)\,d\xi, \tag{2.15}$$

where $B(\xi)$ is the function of ξ only.

The components of complementary stresses are given by

$$\widehat{\rho z}_c=\frac{2\mu z}{(1-2\eta)}\int_0^\infty \xi\, B(\xi)e^{-\xi z}J_1(\xi\rho)\,d\xi, \tag{2.16}$$

$$\widehat{zz}_c=\frac{2\mu}{(1-2\eta)}\int_0^\infty B(\xi)\,e^{-\xi z}(1+\xi z)\,J_0(\xi\rho)\,d\xi. \tag{2.17}$$

Now the boundary conditions to be satisfied are

$$[\widehat{\rho z}_c]_{z=0}=0\ ;\ [\widehat{zz}_T+\widehat{zz}_c]_{z=0}=0. \tag{2.18}$$

The first relation of (2.18) will be satisfied for $z=0$ and from equations (2.9) and (2.17), we have

$$B(\xi)=-\frac{(1-2\eta)}{2}\,\beta\,\xi^{-1}A(\xi) \tag{2.19}$$

Consequently the remaining stresses and the displacements are given by

$$\left.\begin{aligned} u_c&=-\frac{\beta}{2}\int_0^\infty \xi^{-2}A(\xi)(2\eta+\xi z-1)e^{-\xi z}J_1(\xi\rho)\,d\xi,\\ w_c&=\frac{\beta}{2}\int_0^\infty \xi^{-2}A(\xi)(2-2\eta+\xi z)\,e^{-\xi z}J_0(\xi\rho)\,d\xi, \end{aligned}\right. \tag{2.20}$$

$$\widehat{\rho\rho}_c=-\mu\beta\left[\int_0^\infty \xi^{-1}A(\xi)\,(1-\xi z)\,e^{-\xi z}J_0(\xi\rho)\,d\xi + \int_0^\infty \xi^{-2}A(\xi)\,(2\eta+\xi z-1)\,e^{-\xi z}J_1(\xi\rho)\,d\xi\right], \tag{2.21}$$

$$\widehat{\theta\theta}_c=-\mu\beta\left[2\eta\int_0^\infty \xi^{-1}A(\xi)\,e^{-\xi z}J_0(\xi\rho)\,d\xi - \frac{1}{\rho}\int_0^\infty \xi^{-2}A(\xi)\,(2\eta+\xi z-1)\,e^{-\xi z}J_1(\xi\rho)\,d\xi\right]. \tag{2.22}$$

Adding (2.6), (2.7) and (2.20), we find

$$\left.\begin{aligned} u&=-\beta\int_0^\infty \xi^{-2}A(\xi)(\eta+\xi z)\,e^{-\xi z}J_1(\xi\rho)\,d\xi,\\ w&=\beta\,(1-\eta)\int_0^\infty \xi^{-2}A(\xi)\,e^{-\xi z}J_0(\xi\rho)\,d\xi. \end{aligned}\right. \tag{2.23}$$

Also adding (2.10), (2.21) and (2.11), (2.22), we have finally

$$\widehat{\rho\rho}=2\mu\beta\,(1-\eta)\int_0^\infty \xi^{-2}A(\xi)\,e^{-\xi z}\frac{J_1(\xi\rho)}{\rho}\,d\xi,$$

(2.24) $$\widehat{\theta\theta}=2\mu\beta\,(1-\eta)\left[\int_0^\infty \xi^{-1}\,A(\xi)\,e^{-\xi z}J_0(\xi\rho)\,d\xi -\frac{1}{\rho}\int_0^\infty \xi^{-2}\,A(\xi)\,e^{-\xi z}J_1\,(\xi\rho)\,d\xi\right].$$

Hence we have

(2.25)
$$\widehat{\rho\rho}+\widehat{\theta\theta}=2\mu\beta(1-\eta)\int_0^\infty \xi^{-1}\,A(\xi)e^{-\xi z}J_0(\xi\rho)\,d\xi$$
$$\widehat{\rho\rho}-\widehat{\theta\theta}=2\mu\beta\,(1-\eta)\int_0^\infty \xi^{-1}\,A(\xi)e^{-\xi z}\left\{J_0\,(\xi\rho)-\frac{2J_1(\xi\rho)}{\xi\rho}\right\}d\xi.$$

3. Temperature Distribution. In this section we shall suppose that there is prescribed flux of heat within a circular region $0\leqslant\rho<1$, on the free surface $z=0$, the rest of the surface being free from any flux of heat. So the boundary conditions are, on the plane $z=0$

(3.1) $$\frac{\partial T}{\partial z}=-f(\rho)\;;\;0\leqslant\rho<1,$$
$$=0\;;\;\rho>1.$$

Now from the equation (2.5), we have

$$T=\int_0^\infty \xi^{-1}\,A(\xi)\,e^{-\xi z}J_0(\xi\rho)\,d\xi.$$

Therefore

$$\frac{\partial T}{\partial z}=-\int_0^\infty A(\xi)\,e^{-\xi z}J_0\,(\xi\rho)\,d\xi.$$

So that on the boundary surface $z=0$, we have

$$\int_0^\infty \xi\,\frac{A(\xi)}{\xi}\,J_0\,(\xi\rho)d\xi=\begin{cases}f\,(\rho), & 0<\rho<1,\\ 0, & \rho>1.\end{cases}$$

Hence by Hankel's inversion theorem

(3.2) $$\frac{A(\xi)}{\xi}=\int_0^1 \rho\,f(\rho)\,J_0\,(\xi\rho)\,d\rho.$$

Under the same set of transformations, we have on $z=0$

(3.3)
$$u=-\beta\eta\int_0^\infty \xi^{-2}\,A(\xi)\,J_1\,(\xi\rho)\,d\xi,$$
$$w=\beta(1-\eta)\int_0^\infty \xi^{-2}\,A(\xi)\,J_0(\xi\rho)\,d\xi,$$

$$\widehat{\rho\rho}+\widehat{\theta\theta}=2\mu\beta\,(1-\eta)\int_0^\infty \xi^{-1}\,A(\xi)\,J_0(\xi\rho)\,d\xi,$$

(3.4)

$$\widehat{\rho\rho}-\widehat{\theta\theta}=-2\mu\beta(1-\eta)\int_0^\infty \xi^{-1}\,A(\xi)\left\{J_0(\xi\rho)-\frac{2J_1(\xi\rho)}{\xi\rho}\right\}d\xi.$$

Case (i). Let us assume that the flux function is constant so that $f(\rho)=K$. Then from (3.2), we have

$$\frac{A(\xi)}{\xi}=K\int_0^1 \rho J_0\,(\xi\rho)\,d\rho,$$

$$=K\frac{J_1\,(\xi)}{\xi}.$$

Hence

(3.5) $$A(\xi)=KJ_1\,(\xi).$$

This value of $A(\xi)$ substituted in the relations of displacement and stress components gives complete solutions.

We find the values of $(\widehat{\rho\rho}+\widehat{\theta\theta})_{z=0}$ and $(\widehat{\rho\rho}-\widehat{\theta\theta})_{z=0}$ with this expression for $A(\xi)$ as

$$(\widehat{\rho\rho}+\widehat{\theta\theta})_{z=0}=2\mu\,(1+\eta)\,\alpha K\int_0^\infty \frac{J_1(\xi)J_0(\xi\rho)d\xi}{\xi},$$

and

$$(\widehat{\rho\rho}-\widehat{\theta\theta})_{z=0}=-2\mu(1+\eta)\alpha K\int_0^\infty\left\{\frac{J_0(\xi\rho)J_1(\xi)}{\xi}-\frac{2J_1(\xi)J_1(\xi\rho)}{\xi^2\,\rho}\right\}d\xi.$$

If we assume that $2\mu(1+\eta)\alpha K=\delta$, then

$$(\widehat{\rho\rho}+\widehat{\theta\theta})_{z=0}=\delta\begin{cases}\dfrac{1}{2\rho}\,F\left(\dfrac{1}{2},\,\dfrac{1}{2},2\,;\dfrac{1}{\rho^2}\right) & \text{for } \rho>1,\\ 2/\pi & \text{for } \rho=1,\\ F\left(\dfrac{1}{2},\,\dfrac{1}{2},\,1\,;\,\rho^2\right) & \text{for } \rho<1.\end{cases}$$

$$(\widehat{\rho\rho}-\widehat{\theta\theta})_{z=0}=-\delta\begin{cases}\dfrac{1}{2\rho}\,F\left(\dfrac{1}{2},\,\dfrac{1}{2},\,2\,;\dfrac{1}{\rho^2}\right)-F\left(\dfrac{1}{2}\,,-\dfrac{1}{2},\,2\,;\dfrac{1}{\rho^2}\right) & \text{for } \rho>1,\\ -2/3\pi & \text{for } \rho=1,\\ F\left(\dfrac{1}{2},\,\dfrac{1}{2},\,1\,;\,\rho^2\right)-\rho F\left(\dfrac{1}{2},\,-\dfrac{1}{2},\,2\,;\,\rho^2\right) & \text{for } \rho<1.\end{cases}$$

Case (ii). Next we assume that if there is parabolic flux given by $f(\rho)=K(1-\rho^2)$, then we have from (3.2)

$$\frac{A(\xi)}{\xi}=K\int_0^1 \rho(1-\rho^2)\,J_0(\xi\rho)\,d\rho$$

(3.6)

$$=K\left[\frac{4J_1(\xi)}{\xi^3}-\frac{2}{\xi^2}\,J_0(\xi)\right].$$

Therefore from (3.4),

$$(\widehat{\rho\rho}-\widehat{\theta\theta})_{z=0}=\delta\int_0^\infty\left[\frac{4J_1(\xi)J_0(\xi\rho)}{\xi^3}-\frac{2J_0(\xi)J_0(\xi\rho)}{\xi^2}\right]d\xi,$$

and

$$(\widehat{\rho\rho}-\widehat{\theta\theta})_{z=0}=-\delta\left\{\int_0^\infty\left[\frac{4J_1(\xi)J_0(\xi\rho)}{\xi^3}-\frac{2J_0(\xi)\,J_0(\xi\rho)}{\xi^2}\right]d\xi\right.$$
$$\left.-\frac{2}{\rho}\int_0^\infty\left[\frac{4J_1(\xi)J_1(\xi\rho)}{\xi^4}-\frac{2J_0(\xi)J_1(\xi\rho)}{\xi^3}\right]d\xi\right\}.$$

Hence in this case

$$(\widehat{\rho\rho}+\widehat{\theta\theta})_{z=0}=\delta\,I_1,$$
$$(\widehat{\rho\rho}-\widehat{\theta\theta})_{z=0}=-\delta\left[I_1-\frac{2}{\rho}\,I_2\right].$$

where

$$I_1=4\int_0^\infty\frac{J_1(\xi)J_0(\xi\rho)}{\xi^3}\,d\xi\;-2\int_0^\infty\frac{J_0(\xi\rho)J_0(\xi)}{\xi^2}\,d\xi$$
$$=\begin{cases}2\rho\left\{F\left(-\frac{1}{2},\frac{-1}{2},1;\frac{1}{\rho^2}\right)-F\left(\frac{-1}{2},\frac{-1}{2},2;\frac{1}{\rho^2}\right)\right\} & \text{for } \rho>1,\\ 8/9\pi & \text{for } \rho=1,\\ \frac{2}{3}\left\{3F\left(\frac{-1}{2},\frac{-1}{2},1;\rho\right\}-2F\left(\frac{-1}{2},\frac{-3}{2},1;\rho^2\right)\right\} & \text{for } \rho<1.\end{cases}$$

$$I_2=\left[4\int_0^\infty\frac{J_1(\xi)J_1(\xi\rho)}{\xi^4}-2\int_0^\infty\frac{J_0(\xi)J_1(\xi\rho)}{\xi^3}\right]d\xi$$
$$=\begin{cases}\frac{2\rho^2}{3}\left\{F\left(\frac{-1}{2},\frac{-3}{2},1;\frac{1}{\rho^2}\right)-F\left(\frac{-1}{2},\frac{-3}{2},2;\frac{1}{\rho^2}\right)\right\} & \text{for } \rho>1,\\ 32/45\pi & \text{for } \rho=1,\\ \frac{\rho}{3}\left\{3F\left(\frac{-1}{2},\frac{-1}{2},2;\rho^2\right)-2F\left(\frac{-1}{2},\frac{-3}{2},2;\rho^2\right)\right\} & \text{for } \rho<1.\end{cases}$$

4. **Numerical Results.** The variation of $\dfrac{[\widehat{\rho\rho}+\widehat{\theta\theta}]_{z=0}}{2\mu(1+\eta)\alpha K}$ for different values of ρ within the circle $\rho\leqslant 1$ in both the cases when $f(\rho)$ is (i) constant, and (ii) parabolic flux is given by the following table :

ρ		0·0	0·2	0·4	0·6	0·8	1
$\dfrac{[\widehat{\rho\rho}+\widehat{\theta\theta}]_{z=0}}{2\mu(1+\eta)\alpha K}$	(i)	1·0000	0·9899	0·9588	0·9039	0·8208	·6373
	(ii	0·6666	0·6468	0·5896	0·4854	0·3856	·2828

ACKNOWLEDGEMENT. Authors wish to express grateful thanks to Dr. D. Pande, Christ Church College, Kanpur for his kind help and guidance during the preparation of this paper.

REFERENCES

1. Bhattacharya, S. P. (1963), J. Sci. Engg. Research, VII, p. 306.

2. Nowacki, W. (1957), Arch. Mech. Stos. IX, P. 319.

3. Watson, G. N. (1958), A treatise on the theory of Bessel functions, 2nd ed., Cambridge University Press.

Jñānabha Sect. A, Vol. 6, 1976

DOUBLE SERIES FOR THE H-FUNCTION OF TWO VARIABLES

By K. C. Gupta

Department of Mathematics, M. R. Engineering College, Jaipur–302004, *India*

AND

S. P. Goyal

Department of Mathematics, B. V. College of Arts and Science, Banasthali Vidyapith–304022, *India*

(Received : May 28, 1976 ; in revised form : April 20, 1977)

The aim of this paper is to establish three interesting double series relations for the H-function of two variables. By suitably specializing the various parameters involved, these relations would yield the corresponding double series relations for a large number of special functions, and orthogonal polynomials occurring frequently in physics, applied mathematics and statistics. For sake of illustration, we have obtained in this paper, corresponding double series relations for Kampe de Feriet function, Fox's H-function and $M_{k,m}$-function.

1. Introduction. The parameters of the H-function of two variables [7, p. 117], occurring in the present paper, will be displayed in the following contracted notation (analogous to the notation of Srivastava and Panda [8, p. 266, Eq. (1.5) et seq.]) :

$$H\begin{smallmatrix} 0,\, n_1 \,:\, m_2,\, n_2\,;\, m_3,\, n_3 \\ p_1,\, q_1 \,:\, p_2,\, q_2\,;\, p_3,\, q_3 \end{smallmatrix}\left[\begin{matrix} x \\ y \end{matrix}\left|\begin{matrix} ((a_{p_1};\alpha_{p_1}, A_{p_1})) : ((c_{p_2}, \epsilon_{p_2})) ; ((e_{p_3}, E_{p_3})) \\ ((b_{q_1};\beta_{q_1}, B_{q_1})) : ((d_{q_2}, \delta_{q_2})) ; ((f_{q_3}, F_{q_3})) \end{matrix}\right.\right]$$

$$=(2\pi i)^{-2}\int_{L_1}\int_{L_2}\phi(u, v)\,\theta_1(u)\,\theta_2(v)\,x^u\,y^v\,du\,dv \tag{1.1}$$

where $((a_{p_1};\alpha_{p_1}, A_{p_1}))$ denotes the sequence of parameters

$$(a_1\,;\alpha_1, A_1), \ldots, (a_{p_1}\,;\alpha_{p_1}, A_{p_1}),\ \text{and so on.}$$

Also, $\phi(u, v) = \prod_{j=1}^{n_1} \Gamma(1-a_j+\alpha_j u + A_j v)$

$$\times\left[\prod_{j=n_1+1}^{p_1} \Gamma(a_j-\alpha_j\, u-A_j\, v) \prod_{j=1}^{q_1} \Gamma(1-b_j+\beta_j u+B_j v)\right]^{-1},$$

and

$$\theta_1(u) = \prod_{j=1}^{n_2} \Gamma(1-c_j+\epsilon_j u) \prod_{j=1}^{m_2} \Gamma(d_j-\delta_j u)$$

$$\times\left[\prod_{j=n_2+1}^{p_2} \Gamma(c_j-\epsilon_j u) \prod_{j=m_2+1}^{q_2} \Gamma(1-d_j+\delta_j u)\right]^{-1},$$

with $\theta_2(v)$ defined analogously in terms of the parameter sets $((e_{p_3}, E_{p_3}))$, $((f_{q_3}, F_{q_3}))$. An empty product is interpreted as unity, and all the greek and capital letters are positive. The nature of contours L_1 and L_2 in (1.1), the conditions on parameters of this function, its asymptotic expansions, particular cases etc., can be found in a recent paper by Goyal [5].

To save space, three dots ... appearing at a particular place in any H-function of two variables will display that the parameters in that position are the same as that of the H-function of two variables defined by (1.1).

2. Double Series Relations.

$$\sum_{r=0}^{m} \sum_{s=0}^{n} \frac{(-m)_r(-n)_s(b)_{r+s}}{r\,!\, s\,!\,(b)_r(b)_s} H^{0,\ n_1\ :\ m_2,\ n_2+1\ ;\ m_3,\ n_3+1}_{p_1,\ q_1+1\ :\ p_2+1,\ q_2\ ;\ p_3+1,\ q_3}\left[\begin{matrix} x \\ y \end{matrix}\left|\begin{matrix} \ldots & : (1-c-r,\ \alpha), \ldots\ ;\ (1-d-s,\ \beta), \ldots \\ \ldots, (1-c-d-r-s\ ;\ \alpha,\ \beta) & : \ldots\ ;\ \ldots \end{matrix}\right.\right]$$

$$= \frac{(b)_{m+n}}{(b)_m\,(b)_n} H^{0,\ n_1\ :\ m_2,\ n_2+1\ ;\ m_3,\ n_3+1}_{p_1,\ q_1+1\ :\ p_2+1,\ q_2\ ;\ p_3+1,\ q_3}\left[\begin{matrix} x \\ y \end{matrix}\left|\begin{matrix} \ldots & : (1-c-n,\ \alpha), \ldots\ ;\ (1-d-m,\ \beta), \ldots \\ \ldots, (1-c-d-m-n\ ;\ \alpha,\ \beta) & : \ldots\ ;\ \ldots \end{matrix}\right.\right] \quad (2.1)$$

$$\sum_{r=0}^{m} \sum_{s=0}^{n} \frac{(-m)_r(-n)_s}{r!\ s!} H^{0,\ n_1\ :\ m_2,\ n_2+2\ ;\ m_3,\ n_3+1}_{p_1,\ q_1+1\ :\ p_2+2,\ q_2+2\ ;\ p_3+1,\ q_3}\left[\begin{matrix} x \\ y \end{matrix}\right|$$

$$\left|\begin{matrix} \ldots & : (1-c-r,\ \alpha), (1-b-r-s,\ \delta), \ldots ; (1-b-s,\ \beta), \ldots \\ \ldots, (1-c-d-r-s\ ;\ \alpha,\ \beta) & : \ldots, (1-b-r,\ \delta),\ (1-b-s,\ \delta)\ ;\ \ldots \end{matrix}\right]$$

$$=H\begin{matrix}0, & n_1 & : m_2, n_2+2 \,;\, m_3, n_3+1\\ p_1, & q_1+1 & : p_2+2, q_2+2 \,;\, p_3+1, \; q_3\end{matrix}$$
$$\left[\begin{matrix}x\\ y\end{matrix}\middle|\begin{matrix}\ldots & :\\ \ldots, (1-c-d-m-n\,;\,\alpha,\beta) & :\end{matrix}\right.$$
$$\left.\begin{matrix}:(1-c-n,\alpha), (1-b-m-n,\delta),\ldots\,;\,(1-d-m,\beta),\ldots\\ :\ldots,(1-b-m,\delta),(1-b-n,\delta)\quad;\ldots\end{matrix}\right] \quad (2.2)$$

$$\sum_{r=0}^{m}\sum_{s=0}^{n}\frac{(-m)_r(-n)_s(b)_{r+s}}{r!\,s!}\,H\begin{matrix}0, & n_1 & : m_2, n_2+1 & ; m_3, n_3+1\\ p_1, & q_1+1 & : p_2+1, q_2+1 & ; p_3+1, q_3+1\end{matrix}$$
$$\left[\begin{matrix}x\\ y\end{matrix}\middle|\begin{matrix}\ldots & :(1-c-r,\alpha),\ldots\\ \ldots,(1-c-d-r-s\,;\,\alpha,\beta) & :\ldots,(-b+c-s+n,\alpha)\end{matrix}\right.$$
$$\left.\begin{matrix};(1-d-s,\beta),\ldots\\ ;\ldots,(-b+d-r+m,\beta)\end{matrix}\right]$$
$$=(-1)^{m+n}\;H\begin{matrix}0, & n_1+1 & : m_2, n_2+1\,;\, m_3, n_3+1\\ p_1+1, & q_1+2 & : p_2+2, q_2 \;\;;\, p_3+2, \; q_3\end{matrix}$$
$$\left[\begin{matrix}x\\ y\end{matrix}\middle|\begin{matrix}(1-c-d+b-m-n\,;\,\alpha,\beta),\ldots\\ \ldots,(1-c-d-m-n\,;\,\alpha,\beta),(1-c-d+b\,;\,\alpha,\beta)\end{matrix}\right.$$
$$\left.\begin{matrix}:(1-c-n,\alpha),\ldots,(1-c+b,\alpha)\,;\,(1-d-m,\beta),\ldots,(1-d+b,\beta)\\ :\ldots \qquad ;\ldots\end{matrix}\right]$$
$$(2.3)$$

Conditions corresponding appropriately to the conditions (i) to (vi) on p. 119 in the paper by Mittal and Gupta [7] are assumed to be satisfied by all the H-functions of two variables occurring in the above relations.

Proof of (2.1). Writing the Mellin-Barnes integral of the H-function of two variables from (1.1) in the left-hand side of (2.1), and changing the order of integration and summation, which is justified as the series involved are finite, we find that

$$l.h.s.=(2\pi i)^{-2}\int_{L_1}\int_{L_2}\frac{\phi(u,v)\,\theta_1(u)\,\theta_2(v)\Gamma(c+\alpha u)\,\Gamma(d+\beta v)\,x^u\,y^v}{\Gamma(c+d+\alpha u+\beta v)}$$
$$\times\left[\sum_{r=0}^{m}\sum_{s=0}^{n}\frac{(-m)_r\,(-n)_s\,(b)_{r+s}\,(c+\alpha u)_r\,(c+\beta v)_s}{r!\,s!\,(b)_r\,(b)_s\,(c+d+\alpha u+\beta v)_{r+s}}\right]du\,dv$$

Using the following result due to Carlitz [2, p. 232, Eq. (1.1)] :

$$\sum_{r=0}^{m}\sum_{s=0}^{n}\frac{(-m)_r\,(-n)_s\,(b)_{r+s}\,(c)_r\,(d)_s}{r!\,s!\,(b)_r\,(b)_s\,(c+d)_{r+s}}=\frac{(b)_{m+n}\,(d)_m\,(c)_n}{(b)_m\,(b)_n\,(c+d)_{m+n}}, \quad (2.4)$$

in the above expression and interpreting the result thus obtained with the help of (1.1), we get the right hand side (2.1).

The result (2.2) can be easily proved by proceeding on the lines similar to those of the result (2.1). To prove (2.3), we also proceed as in (2.1) except that here we use the following result of Carlitz [3, p. 416, Eq. (9)] :

$$\sum_{r=0}^{m} \sum_{s=0}^{n} \frac{(-m)_r\,(-n)_s\,(b)_{r+s}\,(c)_r\,(d)_s}{r!\,s!\,(c+d)_{r+s}\,(b-c-n+1)_s\,(b-d-m+1)_r}$$

$$=\frac{(c+d-b)_{m+n}\,(d)_m\,(c)_n}{(c+d)_{m+n}\,(d-b)_m\,(c-b)_n},$$

instead of (2.4).

3. Special Cases of (2.1). (*a*) On taking $n_1=p_1$, $n_2=n_3=p_2=p_3$, $q_2=q_3$, $m_2=m_3=1$, $d_1=f_1=0$, replacing q_2 by q_2+1, putting all greek and capital letters equal to one in (2.1), and using a recent result due to Goyal [5, p. 119] therein, we get the following double series for the Kampe de Feriet function [1]:

$$\sum_{r=0}^{m} \sum_{s=0}^{n} \frac{(-m)_r\,(-n)_s\,(b)_{r+s}\,(c)_r\,(d)_s}{r!\,s!\,(b)_r\,(b)_s\,(c+d)_{r+s}}$$

$$\times F_{q_1+1,\,q_2}^{p_1,\,p_2+1}\left[\begin{matrix}(a_{p_1}) & : (c_{p_2}),\, c+r\,; & (e_{p_2}),\, d+s \\ (b_{q_1}),\, c+d+r+s : (d_{q_2}) & ; & (f_{q_2})\end{matrix}\,\middle|\,\begin{matrix}x\\y\end{matrix}\right]$$

$$=\frac{(b)_{m+n}(c)_n\,(d)_m}{(b)_m(b)_n(c+d)_{m+n}}$$

$$F_{q_1+1,\,q_2}^{p_1,\,p_2+1}\left[\begin{matrix}(a_{p_1}) & : (c_{p_2}),\, c+n\,; & (e_{p_2}),\, d+m \\ (b_{q_1}),\, c+d+m+n : (d_{q_2}) & ; & (f_{q_2})\end{matrix}\,\middle|\,\begin{matrix}x\\y\end{matrix}\right] \tag{3.1}$$

(*b*) Further on putting $p_1=p_2=q_1=q_2=0$ in (3.1), we get

$$\sum_{r=0}^{m} \sum_{s=0}^{n} \frac{(-m)_r\,(-n)_s\,(b)_{r+s}\,(c)_r\,(d)_s}{r!\,s!\,(b)_r\,(b)_s\,(c+d)_{r+s}}\,\phi_2\,(c+r,\,d+s,\,c+d+r+s,\,x,\,y)$$

$$=\frac{(b)_{m+n}\,(c)_n\,(d)_m}{(b)_m\,(b)_n\,(c+d)_{m+n}}\,\phi_2\,(c+n,\,d+m,\,c+d+m+n,\,x,\,y) \tag{3.2}$$

where $\phi_2(x, y)$ is the well-known confluent form of Appell's function.

(*c*) Again on taking $n_1=p_1=q_1=n_3=p_3=0$, $m_3=q_3=1$, $f_1=0$, $F_1=1$, $\beta=1$ in (2.1), letting $y\to 0$ and using [5, p. 122] therein, we get the following series for Fox's H-function, which is also believed to be new :

$$\sum_{r=0}^{m}\sum_{s=0}^{n}\frac{(-m)_n\,(-n)_s\,(b)_{r+s}(d)_s}{r!\,s!\,(b)_r\,(b)_s}$$

$$H^{m_2,\,n_2+1}_{p_2+1,\,q_2+1}\left[x\left|\begin{matrix}(1-c-r,\,\alpha),\ ((c_{p_2},\epsilon_{p_2}))\\((d_{q_2},\delta_{q_2})),\,(1-c-d-r-s,\,\alpha)\end{matrix}\right.\right]$$

$$=\frac{(b)_{m+n}(d)_m}{(b)_m\,(b)_n}\,H^{m_2,\,n_2+1}_{p_2+1,\,q_2+1}$$

$$\left[x\left|\begin{matrix}(1-c-n,\,\alpha),\,((c_{p_2},\epsilon_{p_2}))\\((d_{q_2},\delta_{q_2})),\,(1-c-d-m-n,\,\alpha)\end{matrix}\right.\right] \tag{3.3}$$

(d) On putting $m_2=q_2=1$, $n_2=p_2=0$, $d_1=0$, $\delta_1=\alpha=1$ in (3.3) and using the results [6, p. 600, Eq. (4.6) ; 4, p. 264, Eq. (3)] in it, we get the following double series involving $M_{k,\,m}$-function :

$$\sum_{r=0}^{m}\sum_{s=0}^{n}\frac{(-m)_r\,(-n)_s\,(b)_{r+s}\,(c)_r\,(d)_s\,(x)^{-(r+s)/2}}{r!\,s!\,(b)_r\,(b)_s\,(c+d)_{r+s}}$$

$$.\,M_{(d-c-r+s)/2,\ (c+d+r+s-1)/2}(x)$$

$$=\frac{(b)_{m+n}(d)_m(c)_n(x)^{-(m+n)/2}}{(b)_m(b)_n(c+d)_{m+n}}\,M_{(d-c-n+m)/2,\ (c+d+m+n-1)/2}\,(x) \tag{3.4}$$

(e) Lastly, if we take $n=0$ in (2.1), the double series reduces to the following single series for the H-function of two variables :

$$\sum_{r=0}^{m}\frac{(-m)_r}{r!}\,H^{0,\,n_1\ :\ m_2,\,n_2+1\ ;\ m_3,\,n_3+1}_{p_1,\,q_1+1\ :\ p_2+1,\,q_2\ ;\ p_3+1,\,q_3}\left[\begin{matrix}x\\y\end{matrix}\left|\begin{matrix}\ldots & :(1-c-r,\,\alpha),\ldots ; (1-d,\,\beta),\ldots\\ \ldots,(1-c-d-r\,;\,\alpha,\,\beta) & :\ \ldots\qquad ;\ \ldots\end{matrix}\right.\right]$$

$$=H^{0,\,n_1\ :\ m_2,\,n_2+1\ ;\ m_3,\,n_3+1}_{p_1,\,q_1+1\ :\ p_2+1,\,q_2\ ;\ p_3+1,\,q_3}$$

$$\left[\begin{matrix}x\\y\end{matrix}\left|\begin{matrix}\ldots & :(1-c,\,\alpha),\ldots ; (1-d-m,\,\beta),\\ \ldots,(1-c-d-m\,;\,\alpha,\,\beta) & :\ \ldots\qquad ;\ \ldots\end{matrix}\right.\right] \tag{3.5}$$

The special cases for the double series relations (2.2) and (2.3) can be obtained by specializing the parameters of the H-function of two variables as indicated above, but we shall not record them here due to the triviality of the analysis to be applied.

ACKNOWLEDGEMENT. We wish to express our sincere thanks to Professor H. M. Srivastava, University of Victoria, Canada, for his valuable suggestions for the improvement of this paper.

REFERENCES

1. P. Appell et J. Kampe′ de Fe′riet, Fonctions Hyperge′ome′triques et Hypersphe′riques; Polynomes d′ Hermite, Gauthier-Villars, Paris, 1926.

2. L. Carlitz, A summation theorem for double hypergeometric series, Rend. Sem. Mat. Univ. Padova **37** (1967), 230–233.

3. L. Carlitz, A Saalschutzian theorem for double series, J. London Math. Soc. **38** (1963), 415–418.

4. A. Erd′elyi et al., Higher Transcendental Functions, Vol. I, Mc Graw-Hill, New York, 1953.

5. S. P. Goyal, The H-function of two variables, Kyungpook Math. J. **15** (1975), 117–131.

6. K. C. Gupta, U. C. Jain, The H–function II, Proc. Nat. Acad. Sci. India Sect. A **36** (1966), 594–609.

7. P. K. Mittal and K. C. Gupta, An integral involving generalized function of two variables, Proc. Indian Acad. Sci. Sect. A **75** (1972), 117–123.

8. H. M. Srivastava and Rekha Panda, Some bilateral generating functions for a class of generalized hypergeometric polynomials. J. Reine Angew. Math. **283/284** (1976), 265–274.

Jñānabha Sect. A, Vol 6, 1976

ON UNIQUENESS OF WEAK SOLUTIONS OF ABSTRACT DIFFERENTIAL INEQUALITY

By **M. A. Malik**

Department of Mathematics, Concordia University (Sir George Williams Campus), 1455 de Maisonneuve Blvd. West, Montreal, Quebec, Canada H3G 1M8

(Received : March 2, 1976)

1. Let H be a Hilbert space and A be a closed linear operator with domain $D(A)$ dense in H. Let also A^* with domain $D(A^*)$ be the adjoint of A. By $L^2_{\text{loc}}(H)$, we mean the space of functions $u : R \to H$ such that the norm $| u(t) |$ is square integrable on every compact subset of R (real line) ; and by $L^2_{[a,\, b]}(H)$ the Hilbert space of square integrable H-valued functions on the interval $[a, b]$. $C^1_0(H)$ represents the space of H-valued continuously differentiable functions with compact support defined on R and the collection of those $\phi \in C^1_0(H)$ such that $\phi(t) \in D(A^*)$ and the norm $| A^*\phi(t) |$ is square integrable will be denoted by K_{A^*}. Note that supp $A^*\phi(t)$ $\subset$supp $\phi(t)$ and K_{A^*} is a linear subspace. $R(\zeta\,; A)$ and $R(\zeta\,; A^*)$ represent the resolvent of A and A^* respectively ; ζ is complex. Consider these abstract differential operators

$$L = \frac{1}{i}\frac{d}{dt} - A \text{ and } L^* = \frac{1}{i}\frac{d}{dt} - A^* \;;\; i = \sqrt{-1}.$$

For a given $f \in L^2_{\text{loc}}(H)$, $u \in L^2_{\text{loc}}(H)$ is said to be a weak solution of $Lu = f$ if

(1.1) $$\int_R \langle u(t), L^*\phi(t) \rangle dt = \int_R \langle f(t), \phi(t) \rangle dt$$

Research supported by N. R. C. of Canada.

for all $\phi \in K_{A^*}$; $<, >$ represents the scalar product in H. In this case we write $(wL)\, u=f$. It is clear that also for functions $\tilde{f}=f$ almost everywhere and $\tilde{u}=u$ almost everywhere, (1.1) is satisfied. We assume that for one such pair $u(t)$, $(wL)\, u(t)=f(t)$ there holds the abstract differential inequality

$$(1.2) \qquad |\,(wL)\, u(t)\,| \leqslant \phi(u)\, |\, u(t)\,|$$

for all $t \in R$; $\phi(t)$ is a positive scalar valued function.

While discussing the uniqueness for solutions of the Cauchy problem, S. Agmon and L. Nirenberg [1] proved :

Theorem 1. *Let H be a Hilbert space and let $u(t) \in H$ be a solution of*

$$(1.3) \qquad |\, Lu\,(t)\,|_H \leqslant \phi(t)\, |\, u(t)\,|_H$$

on the interval $0 \leqslant t \leqslant T$ with $u(T)=0$. Let $\mathfrak{J}$ be a sequence of lines $\operatorname{Im} \zeta = a_n$, $n=1, 2, \ldots$ *in the ζ-plane with $a_n \to \infty$ and assume that the resolvent $R(\zeta\, ; A)$ is bounded by M outside j intervals of length s on each line of $\mathfrak{J}$ for some appropriate j, s and M. There is a constant c depending only on j, s and M such that* if $\phi(t) \leqslant c$, *then* $u \equiv 0$.

2. The aim of this note is to present a generalization of the above theorem when $u(t)$ satisfy (1.3) 'weakly'—in the sense that (1.2) holds. In fact, we prove :

Theorem 2. *Let the resolvent $R(\zeta\, ; A)$ be bounded by a constant M on a sequence of lines* $\operatorname{Im} \zeta = a_n$, $n=1, 2, \ldots$ *in the ζ-plane with $a_n \to \infty$ and $u \in L^2_{loc}(H)$ satisfy* (1.2). If $\phi(t) \leqslant c$ *where $cM \leqslant 1$ and* supp $u \subset (-\infty, T)$ *for $T < \infty$, then $u \equiv 0$.*

Proof. Choose any two real numbers α and β such that $-\infty < \beta < \alpha < T$ and define $\xi(t)$ a scalar valued continuously differentiable monotonic function as follows.

$$(2.1) \qquad \xi(t) = \begin{cases} 1, & \text{if } t \geqslant \alpha, \\ 0, & \text{if } t \leqslant \beta. \end{cases}$$

Set $v_n(t) = e^{a_n t}\, \xi(t) u(t)$, where u is in $L^2_{loc}(H)$ and satisfies the hypotheses of Theorem 2. It is clear that the supp $v_n \subset [\beta, T]$ and on the interval $[\beta, S]$, $|v_n(t)|$ is a square integrable function. First we note that

(2.2) $\qquad (wL)\nu_n(t)+ia_n\nu_n(t)=e^{a_nt}\xi(t)\,(wL)u(t)-ie^{a_nt}\xi'(t)u(t).$

In fact, for all $\phi\in K_{A^*}$, we have

$$\text{(2.3)}\qquad \begin{aligned}&\int<(wL)\nu_n+ia_n\nu_n,\ \phi>dt\\ &=\int<u,\ e^{a_nt}\xi(L^*\phi-ia_n\phi)>dt\\ &=\int<u,\ L^*e^{a_nt}\xi\phi+ie^{a_nt}\xi'\phi>dt\\ &=\int<e^{a_nt}\ \xi(wL)u-ie^{a_nt}\xi'u,\ \phi>dt.\end{aligned}$$

From (2.2) or (2.3), one has

$$\text{(2.4)}\qquad \int_R<\nu_n,\ \frac{1}{i}\frac{d\phi}{dt}-ia_n\phi-A^*\phi>dt=\int_R<X(t),\ \phi(t)>dt$$

where $X(t)$ represent the right hand side of (2.2). As both $\nu_n(t)$ are of compact support, the Parseval formula in (2.4) leads to

$$\text{(2.5)}\qquad \int_R<\hat{\nu}_n(\lambda),\ (\lambda-ia_n-A^*)\hat{\phi}(\lambda)>d\lambda=\int_R<\hat{X}(\lambda),\ \hat{\phi}(\lambda)>d\lambda,$$

where '^' denotes the Fourier transform. Setting $(L^*-ia_n)\phi=\psi$, we have $(\lambda-ia_n-A^*)\hat{\phi}(\lambda)$. Since from the hypothesis, the resolvent $R(\zeta\ ;\ A)$ exists for $\zeta=\lambda+ia_n$, the resolvent of A^* would exist for $\bar{\zeta}=\lambda-ia_n$ and $R(\zeta\ ;\ A)=R(\bar{\zeta}\ ;\ A^*)$ which further implies that $R(\bar{\zeta}\ ;\ A^*)$ is bounded by M whenever $R(\zeta\ ;\ A)$ is bounded by M; see [2; page 255]. Thus in view of the hypothesis we have $\hat{\phi}(\lambda)$ $=R(\lambda-ia\ ;\ A^*)\hat{\psi}(\lambda)$ for all λ (real) and so in (2.5) one has

$$\text{(2.6)}\qquad \int_R<\hat{\nu}_n(\lambda),\ \hat{\psi}(\lambda)>d\lambda=\int_R<\hat{X}(\lambda),\ R(\lambda-ia_n\ ;\ A^*)\hat{\psi}(\lambda)>d\lambda.$$

In the right-hand of (2.6) we use the Schwartz inequality, the hypothesis on the resolvent and then the Plancheral formula to get

$$\text{(2.7)}\qquad \begin{aligned}&\left|\int<\hat{X}(\lambda),\ R(\lambda-ia_n\ ;\ A^*)\hat{\psi}(\lambda)>d\lambda\right|^2\\ &\leqslant\int|\hat{X}(\lambda)|^2\,d\lambda\int|R(\lambda-ia_n\ ;\ A^*)\hat{\psi}(\lambda)|^2\,d\lambda\\ &\leqslant M^2\int|\hat{X}(\lambda)|^2\,d\lambda\int|\hat{\psi}(\lambda)|^2\,d\lambda\\ &=M^2\int|X(t)|^2\,dt\int|\psi(t)|^2\,dt.\end{aligned}$$

And so, (2.6) and (2.7) give

$$(2.8)\qquad \left|\int_{\beta}^{T} < v_n(t),\ \psi(t) > dt\ \right|^2 \leqslant M^2 \int |\ x(t)\ |^2\ dt \int |\ \psi(t)\ |^2\ dt$$

for $\psi=(L^*-ia_n)\phi$ where ϕ ranges over $K_A{}^*$. Since for any $\psi \in C_0^\infty\ (H)$,

$$\phi(t)=\frac{1}{2\pi}\int_R e^{it\lambda}\ R(\lambda-ia_n\ ;\ A^*)\hat{\psi}(\lambda)\ d\lambda \in K_A{}^*$$

and satisfies $(L^*-ia_n)\phi=\psi$, one concludes that the collection

$$\{\psi\ ;\ (L^*-ia_n)\phi=\psi,\ \phi \in K_A{}^*,\ \text{supp}\ \phi \subset [\beta,\ T]\}$$

is dense in $L^2_{[\beta,\ T]}(H)$. Thus from (2.8) one easily has

$$(2.9)\qquad \int |\ v_n(t)\ |^2\ dt \leqslant M^2 \int |\ X(t)\ |^2\ dt$$

from where

$$(2.10)\qquad \left[\int_{\beta}^{T} |\ e^{a_n t}\xi(t)\ |^2\ dt\right]^{1/2}$$

$$\leqslant M\left\{\left[\int_{\beta}^{T} |\ e^{a_n t}\xi' u\ |^2\ dt\right]^{1/2}+\left[\int_{\beta}^{T} |\ e^{a_n t}\xi(wL)u\ |^2\ dt\ \right]^{1/2}\right\}$$

$$\leqslant M\left\{\left[\int_{\beta}^{T} |\ e^{a_n t}\xi' u\ |^2 dt\right]^{1/2}+c\left[\int_{\beta}^{T} |\ e^{a_n t}\xi\ u\ |^2\ dt\right]^{1/2}\right\}$$

in the last term of the right-hand side of (2.10) we use (1.2) and $\phi(t)\leqslant c$.

As $cM<1$ and $\xi(t)=1$ on $t \geqslant \alpha$ from (2.10) one concludes

$$(2.11)\qquad \int_{\alpha}^{T} |\ e^{a_n t}u(t)\ |^2\ dt \leqslant \text{const.}\ e^{a_n \alpha}$$

Choosing $\alpha' > \alpha$, from (2.11) one therefore has

$$(2.12)\qquad \int_{\alpha'}^{T} |\ u(t)\ |^2\ dt \leqslant \text{const.}\ e^{2a_n(\alpha-\alpha')},$$

Making $a_n \to \infty$, we have $u(t)=0$ on $\alpha < t \leqslant T$. Since α is arbitrary, we conclude that $u(t)\equiv 0$ on $-\infty < t \leqslant T$.

This completes the proof.

I thank Professor S. Zaidman for his kind interest.

REFERENCES

1. S. Agmon and L. Nirenberg, Properties of Solutions of ordinary differential equations in Banach space, *Comm. Pure. Appl. Math.* 16 (1963), 123–129.

2. K. Yosida, *Functional Analysis*, Springer-Verlag, New, York, 1968.

Jñānabha Sect. A, Vol. 6, 1976

ELASTIC ENERGY CALCULATIONS FOR TWO-DIMENSIONAL PROBLEMS

By **Raj Rani Bhargawa**

Department of Applied Mathematics, University of Roorkee, Roorkee–247672, *U.P., India*

(Received : January 20, 1977)

The methods of calculating elastic energy in the plane theory of elasticity are reconsidered in this paper. Available methods are based upon the real-variable methods. But, in the plane strain problems, it is shown that the complex-variable technique is more useful for finding explicit expression for elastic energy and that it gives the results in a simpler manner.

The available literature on elasticity states two methods for the calculation of elastic energy.

In plane strain problems, the first method calculates the strain energy density [1] given by

$$W=\left(\frac{\lambda+\mu}{2}\right)(e_{xx}+e_{yy})^2+\frac{\mu}{2}\left[(e_{yy}-e_{xx})^2+4e_{xy}^2\right]$$

for an elemental volume of unit height of an elastic body under consideration. Here λ, μ are usual Lame's constants and e_{ij} $(i, j=x,y)$ are the components of strain. Integrating the expression over the whole area, the elastic energy of the body per unit height is obtained. Hence, the total energy E, in the body per unit height is given by

$$E=\iint_S W\,dS$$

where S is the area of the cross-section.

In the second method, the work done at the boundary by the traction forces is calculated. Using the classical theorem [1] for the equivalence of the work done on the boundary to twice the potential energy, the elastic energy of the body under consideration is calculated. The kinetic energy is assumed to be zero. In two-dimensional problems, if p_{nx}, p_{ny} are the components of the tractions on the boundary and u, v are the displacement components, then the strain energy in the body per unit height is given by

$$E=\int_s (u\,p_{nx}+v\,p_{ny})\,ds \tag{1}$$

where s is the boundary of the cross-section.

In this paper it is shown that the powerful complex variable technique can be applied to find a formula for the calculation of elastic energy. This is achieved in the following way.

From (1), it may be seen that

$$(u\, p_{nx}+v\, p_{ny})=Re\ (p_{nx}+i\, p_{ny})(u-iv), \tag{2}$$

where Re denotes 'Real-part of'.

Also, using the notation given in Muskhellishvili [2], it is well known that

$$(p_{nx}+i\, p_{ny})\, ds=-id\,[\phi(z)+z\,\overline{\phi'(z)}+\overline{\psi(z)}] \tag{3}$$

and

$$u-iv=-\frac{1}{2\mu}\,[\bar{z}\,\phi'(z)+\psi(z)]+\frac{4\mu+\lambda}{2\lambda\mu}\,\overline{\phi(z)}. \tag{4}$$

Hence, using (2), (3), (4), the elastic energy E is given by

$$E=-\tfrac{1}{2}\, Re\int_S\left[\frac{1}{2\mu}\,\bar{z}\,\phi'(z)+\psi(z)+\frac{4\mu+\lambda}{2\lambda\mu}\,\overline{\phi(z)}\right]$$
$$\times\, id\,[\phi(z)+z\,\overline{\phi'(z)}+\overline{\psi(z)}]. \tag{5}$$

If the region is mapped onto a unit circle (as in the case of many problems of the plane elasticity), the above formulae can be directly transformed. Let the mapping function be $z=\omega(\rho)$, and let

$$\phi(z)=\phi(\omega(\rho))=f(\rho)\,;\ \phi'(z)=\frac{f'(\rho)}{\omega'(\rho)}$$

and

$$\psi(z)=\psi(\omega(\rho))=g(\rho)\,;\ \psi'(z)=\frac{g'(\rho)}{\omega'(\rho)}$$

Substituting these values in (5),

$$E=\frac{1}{2}\,Re\int_\Gamma\left[-\frac{1}{2\mu}\,\overline{\omega(\rho)}\,\frac{f'(\rho)}{\omega'(\rho)}+g(\rho)+\frac{4\mu+\lambda}{2\lambda\mu}\,\overline{f(\rho)}\right]$$
$$\times\, i.d.\left[f(\rho)+\omega(\rho)\,\frac{\overline{f'(\rho)}}{\overline{\omega'(\rho)}}+\overline{g(\rho)}\right],$$

where Γ is now the boundary of the unit circle in the ρ plane. Note that $\rho\bar{\rho}=1$. Explicit results in a simple manner may be derived by writing the known expressions for $f(\rho)$, $\omega(\rho)$, $g(\rho)$ for the case, when the region in the z-plane is finite or infinite, simply-connected, and the finite boundary is subjected to given boundary tractions or displacements.

As an illustration of this method, the elastic energy stored in an infinite plate with an elliptic hole under uniform pressure P is calculated. The problem in classical elasticity is solved by mapping the region outside the ellipse onto a circle $|\rho|=1$. The mapping function is

$$z=\omega(\rho)=R\left(\frac{1}{\rho}+\rho m\right),$$

where $$R=\frac{a+b}{2}, \text{ and } m=\frac{a-b}{a+b},$$

a, b are semi-major and semi-minor axes.

If $f(\rho)$ and $g(\rho)$ are the functions $\phi(z)$ and $\psi(z)$ when z is replaced by $R\left(\frac{1}{\rho}+\rho m\right)$, then for the classical solution of the elliptic hole problem, it is well known [3], that

$$f(\rho)=-PR\rho\ ;\ \ g(\rho)=-PR\rho+\frac{m(\rho^2+m)}{1-m\rho^2}$$

Hence substituting these relations in (5) and evaluating the integrals' we get

$$E=\left(\frac{\pi P^2R^2m^2}{2}\right)\frac{\sigma}{\lambda}+\frac{\pi P^2R^2}{2\mu} \tag{6}$$

which gives the elastic energy stored in an infinite plate with an elliptic hole. σ is poisson's ratio.

From (6), as a particular case, the elastic energy stored in an infinite plate with a circular hole may be obtained by putting $m=0$. This comes out to be $(\pi P^2R^2)/2\mu$.

It may be seen that the energy stored in an infinite plate with an elliptic hole is minimum for the case when the plate has a circular hole of the same area as that of the elliptic hole. It follows from (6), by differentiating (6) with respect to m and since the second derivative is positive, that the energy is minimum for the case of a circular hole.

Results pertaining to elastic energy in the case of a square hole under uniform pressure in an infinite plate can also be found by the above method. The function, mapping an approximate square hole in an infinite plate onto a unit circle, is given in [3]. The mapping function is

$$z=\omega(\rho)=A\left(\frac{1}{\rho}-\frac{1}{6}\rho^3\right),$$

where A=area of square hole.

The values of $\phi(z)$, $\psi(z)$ are not given in the above book nor are available in the literature. The values of these functions can be found and are given as

$$\phi(z)=\frac{PA\rho^3}{6},$$

$$\psi(z)=-\frac{13}{6}PA\frac{\rho}{\rho^4+2}.$$

By calculations similar to those given above, we find the energy in the region per unit height as equal to

$$E=\frac{P^2A^2\pi}{2\mu}\left[\frac{\mu}{3(\lambda+\mu)}+\frac{7}{6}\right].$$

In a subsequent paper it is proposed to apply the above method to another class of problems concerning inhomogeneties.

REFERENCES

1. A. E. H. Love, *A Treatise on the Mathematical Theory of Elasticity*, Dover, 1944.
2. N. I. Muskhellishivili, *Some Basic Problems of the Mathematical Theory of Elasticity*, P. Noordhoff, 1963.
3. I. S. Sokolnikoff, *Mathematical Theory of Elasticity*, Mc Graw-Hill, 1956.

Jñānabha Sect. A, Vol. 6, 1976

PERIODIC ORBITS OF COLLISION IN THE THREE-DIMENSIONAL RESTRICTED PROBLEM OF THREE BODIES

By K. B. Bhatnagar

Department of Mathematics, Delhi College,
Delhi–110007, *India*

AND

Z. A. Taqvi

Department of Mathematics, Jamia College,
Jamia Millia, New Delhi–110001, *India*

(Received : February 15, 1975)

Abstract. The existence of periodic orbits of collision in the three dimensional elliptic restricted problem have been examined. However, the eccentricity of the ellipses described by the primaries and the z-coordinate of the infinitesimal mass are taken of the order μ, where μ is the mass of the smaller primary and $1-\mu$, the mass of the bigger primary.

1. Equations of motion. Let μ and $1-\mu$ be the masses of two primaries (finite masses) which are moving in elliptic orbits around their centre of mass. We take the origin at the centre of mass and the plane of motion of the primaries as xy-plane and the line joining μ and $1-\mu$ as x-axis. Let the coordinate system rotate with the variable angular velocity $\dot{f}$ about z-axis where f is the true anomaly of one primary around the other. This introduction of a non-uniformly rotating and pulsating coordinate system results in a fixed location of the primaries. Let the coordinates of the infinitesimal mass be (x, y, z).

The equations of motion of the infinitesimal mass under the gravitational field of the two primaries are given by

$$\left.\begin{aligned} \frac{d^2x}{df^2}-2\frac{dy}{df}&=\frac{\partial V}{\partial x}, \\ \frac{d^2y}{df^2}+2\frac{dx}{df}&=\frac{\partial V}{\partial y}, \\ \frac{d^2z}{df^2}&=\frac{\partial V}{\partial z}, \end{aligned}\right\} \tag{1}$$

where $$V=\frac{\Omega}{1+e'\cos f},$$

$$\Omega=\tfrac{1}{2}[(1-\mu)r_1^2+\mu r_2^2]+\frac{1-\mu}{r_1}+\frac{\mu}{r_2}-\frac{1}{2}\,(1+e'\cos f)z^2,$$

with
$$r_1^2=(x-\mu)^2+y^2+z^2$$
$$r_2^2=(x-\mu+1)^2+y^2+z^2$$

and e' is the eccentricity of the elliptic orbit described by the primary bodies.

2. Regularisation of the solution. For regularising the solution, let us first reduce the equation (1) to canonical form. For this let us introduce the variables

$$\left.\begin{aligned} &x_1=x-\mu\ ;\ x_2=y\ ;\ x_3=z\\ &p_1=\dot{x}-y=\dot{x}_1-x_2\\ &p_2=\dot{y}+x-\mu=\dot{x}_2+x_1\\ &p_3=\dot{x}_3 \end{aligned}\right\}\tag{2}$$

where $(\dot{\ })$ denotes differentiation with respect to f.

The equations of motion (1) become

$$\left.\begin{aligned} &\frac{dx_i}{df}=\frac{\partial H}{\partial p_i}\\ &\frac{dp_i}{df}=-\frac{\partial H}{\partial x_i}\quad(i=1,2,3) \end{aligned}\right\}\tag{3}$$

where
$$H=\tfrac{1}{2}[p_1^2+p_2^2+p_3^2+x_1^2+x_2^2+x_3^2]+(p_1x_2-p_2x_1)$$
$$-\frac{1}{1+e'\cos f}\left[\{\tfrac{1}{2}(1-\mu)r_1^2+\mu r_2^2\}+\frac{1-\mu}{r_1}+\frac{\mu}{r_2}\right]-\tfrac{1}{2}x_3^2\tag{4}$$

Now for regularisation of the solution, we shall introduce Livi-Civita's (1906) parabolic transformation which may be defined as canonical transformation generated by

$$\left.\begin{aligned} &S=(\xi_1^2-\xi_2^2)p_1+2\xi_1\xi_2p_2+\xi_3p_3\\ &\text{such that}\quad x_i=\frac{\partial S}{\partial p_i},\ \pi_i=\frac{\partial S}{\partial \xi_i}\quad(i=1,2,3) \end{aligned}\right\}\tag{5}$$

where π_i are the momenta associated with the new coordinates ξ_i.

The equations of motion (3) in terms of the new variables become,

$$\left.\begin{aligned} &\frac{d\xi_i}{df}=\frac{\partial H}{\partial \pi_i}\\ &\frac{d\pi_i}{df}=-\frac{\partial H}{\partial \xi_i}\,(i=1,2,3) \end{aligned}\right\}\tag{6}$$

where H is given by

$$H=\frac{\pi^2}{8\xi^2}+\frac{1}{2}\pi_3{}^2+\frac{\xi^4}{2}+\frac{1}{2}(\pi_1\xi_2-\pi_2\xi_1)$$
$$-\frac{1}{1+e'\cos f}\left[\frac{1}{2}\left\{\frac{1}{2}(1-\mu)r_1{}^2+\mu r_2{}^2\right\}+\frac{1-\mu}{r_1}+\frac{\mu}{r_2}\right]$$

and

$$r_1{}^2=\xi^4+\xi_3{}^2,\ \xi^2=\xi_1{}^2+\xi_2{}^2$$
$$r_2{}^2=1+\xi^4+2(\xi_1{}^2-\xi_2{}^2)+\xi_3{}^2$$
$$\pi^2=\pi_1{}^2+\pi_2{}^2.$$

Jacobi's integral (Szebehely, 1967) may be written as

$$H+I=C$$

where
$$I=\int_0^f\frac{e'\Omega\sin f}{(1+e'\cos f)^2}\,df \tag{7}$$

Now, we introduce a new independent variable τ instead of f defined by

$$df=r_1\,d\tau,\ f=0 \text{ at } \tau=0. \tag{8}$$

The equations of motion (6) will be transformed to

$$\left.\begin{aligned}\frac{d\xi_i}{d\tau}&=\frac{\partial K}{\partial\pi_i}\\ \frac{d\pi_i}{d\tau}&=-\frac{\partial K}{\partial\xi_i}\qquad(i=1,2,3)\end{aligned}\right\} \tag{9}$$

where K is the new Hamiltonian, given by

$$K=r_1(H-C)+\int_0^{r_1}I\,dr_1$$
$$=r_1\left[\frac{\pi^2}{8\xi^2}+\frac{\pi_3{}^2}{2}+\frac{\xi^4}{2}+\frac{(\pi_1\xi_2-\pi_2\xi_1)}{2}\right.$$
$$\left.-\frac{1}{1+e'\cos f}\left\{\frac{(1-\mu)r_1{}^2}{2}+\frac{\mu r_2{}^2}{2}+\frac{1-\mu}{r_1}+\frac{\mu}{r_2}\right\}-C\right]$$
$$+\int_0^{r_1}\int_0^f\frac{e'\sin f}{(1+e'\cos f)^2}\left[\frac{1}{2}\left\{(1-\mu)r_1{}^2+\mu r_2{}^2\right\}+\frac{1-\mu}{r_1}+\frac{\mu}{r_2}\right.$$
$$\left.-\frac{1}{2}(1+e'\cos f)\xi_3{}^2\right]dr_1\,df$$

Suppose that e' and ξ_3 are small quantities of the $o(\mu)$, we may take
$$r_1=\xi^2+o(\mu)$$

$e'=\mu e_1$, neglecting second order terms, we get

$$K=K_0+\mu K_1, \text{ where}$$

$$K_0=\frac{\pi^2}{8}+\frac{1}{2}\ \xi^2\pi_3^2+\frac{1}{2}\ \xi^2\ (\pi_1\xi_2-\pi_2\xi_1-2\ C_0)-1=-\epsilon,\ \text{(say)} \tag{10}$$

$$K_1=\frac{r_1^3}{2}-\frac{1}{2}\,r_1\,r_2^2+1\ -\frac{r_1}{r_2}-C_1r_1 \tag{11}$$

The form given to K_0 ensures that the orbits, which are analytically continued from the two-body orbit will belong to $K=0$ manifold, that is, are solutions of the regularised equations of the restricted problem (Giacaglia, 1967).

With Giacaglia, we shall assume that K_0 is negative and so the corresponding two-body problem will admit bounded orbits as a solution in a rotating system of coordinates. It is easy to see that $|\epsilon|<1$.

3. Generating Solution. For Generating Solution, we shall choose K_0 to be our Hamiltonian function Since f is not involved explicitly in the equation (10), so the Hamiltonian-Jacobi equation may be written as

$$\frac{1}{8}\left[\left(\frac{\partial\omega}{\partial\xi_1}\right)^2+\left(\frac{\partial\omega}{\partial\xi_2}\right)^2\right]+\frac{1}{2}\xi^2\left(\frac{\partial\omega}{\partial\xi_3}\right)^2$$
$$+\frac{1}{2}\ \xi^2\left[\xi_2\ \frac{\partial\omega}{\partial\xi_1}-\xi_1\frac{\partial\omega}{\partial\xi_2}-2\ C_0\right]-1=-\epsilon, \tag{12}$$

where $$\pi_i=\frac{\partial\omega}{\partial\xi_i}\ (i=1,\ 2,\ 3)$$

Putting $\xi_1=\xi\cos\phi$, $\xi_2=\xi\sin\phi$, the equation (12) becomes

$$\frac{1}{8}\left[\left(\frac{\partial\omega}{\partial\xi}\right)^2+\frac{1}{\xi^2}\left(\frac{\partial\omega}{\partial\phi}\right)^2\right]+\frac{1}{2}\xi^2\left(\frac{\partial\omega}{\partial\xi_3}\right)^2+\frac{\xi^2}{2}\left[-\frac{\partial\omega}{\partial\phi}-2\ C_0\right]=\alpha, \tag{13}$$

where $$\alpha=1-\epsilon>0.$$

Proceeding as in (Bhatnagar, 1969), the solution of (13) may be written as

$$\omega=u(\xi)+2G\ \phi+\bar{H}\ \xi_3 \tag{14}$$

$$u(z,\ G,\alpha)=[H^2-2(G+C_0)]^{1/2}\int_{z_1}^{z}\sqrt{f(z)}\ \frac{dz}{z} \tag{15}$$

$$z=\xi^2 \tag{16}$$

Here z_1 is the smaller of the roots of the equation

$$f(z)=-\left[z^2-\frac{2\alpha z}{H^2-2(G+C_0)}+\frac{G^2}{H^2-2(G+C_0)}\right] \tag{17}$$

Let the other root be z_2. For a general solution we need only two arbitrary constants and for these constants, we have, α and G. Therefore the solution may be regarded a general solution.

We introduce the parameters a, e, l by the relations

$$z_1=a\,(1-e)\,;\; z_2=a\,(1+e)\,;\; z=a\,(1-e\cos l) \tag{18}$$

and take
$$\alpha=L[\bar{H}^2-2\,(G+C_0)]^{1/2}>0.$$

From equation (18), we get

$$a=\frac{L}{[\bar{H}^2-2(G+C_0)]^{1/2}} \tag{19}$$

$$e=\left[1-\frac{z_1 z_2}{a^2}\right]^{1/2}=\left[1-\frac{G^2}{L^2}\right]^{1/2}\leqslant 1. \tag{20}$$

From (17), (18), (19) and (20), we get

$$f(z)=a^2e^2\sin^2 l \tag{21}$$

The equations of motion associated with K_0 are

$$\left.\begin{aligned}\frac{d\xi_1}{d\tau}&=\frac{\partial K_0}{\partial\pi_1}=\frac{\pi_1}{4}+\frac{1}{2}\,\xi^2\xi_2\\ \frac{d\xi_2}{d\tau}&=\frac{\partial K_0}{\partial\pi_2}=\frac{\pi_2}{4}-\frac{1}{2}\,\xi^2\xi_1\\ \text{and}\quad \frac{d\xi_3}{d\tau}&=\frac{\partial K_0}{\partial\pi_3}=\pi_3\xi^2\end{aligned}\right\} \tag{22}$$

It is easy to see that

$$[\bar{H}^2-2(G+C_0)]^{1/2}\,(\tau-\tau_0)=\int_{z_1}^{z}\frac{dz}{\sqrt{f(z)}},$$

where $z=z_1$ at $\tau=\tau_0$.

or
$$[\bar{H}^2-2(G+C_0)]^{1/2}\,(\tau-\tau_0)=\int_0^l\frac{ae\sin l}{ae\sin l}\,dl=l$$

Hence
$$l=[\bar{H}^2-2\,(G+C_0)]^{1/2}\,(\tau-\tau_0) \tag{23}$$

The parameters a, e, l are, therefore, given by (19), (20) and (23). From equations (14) and (15), we can easily show that

$$\left.\begin{aligned}\frac{\partial\omega}{\partial L}&=\frac{\partial U}{\partial L}=\int_{z_1}^{z}\frac{dz}{\sqrt{f(z)}}=l\\ \frac{\partial\omega}{\partial G}&=2\phi+\frac{\partial U}{\partial G}=2\phi-f-\frac{\sqrt{L^2-G^2}}{\bar{H}^2-2(G+C_0)}\sin l=g\\ f&=\sqrt{1-e^2}\int\frac{dl}{1-e\cos l}\\ \frac{\partial\omega}{\partial H}&=\xi_3+\frac{\bar{H}[L^2-G^2]^{1/2}}{\bar{H}^2-2(G+C_0)}\sin l=h\end{aligned}\right\} \tag{24}$$

Equations (24) establish the canonical set $(l, L\,;\, g, G\,;\, h, \overline{H})$. We may observe that when $e=1$, we have $G=0, f=0$.

Since $K_0=\alpha-1$, it follows that

$$K_0=L[\overline{H}^2-2(G+C_0)]^{1/2}-1>0,$$

and therefore, for the problem generated by this Hamiltonian, we have

$$\left.\begin{aligned}
&\frac{dL}{d\tau}=-\frac{\partial K_0}{\partial L}=0 \qquad L=\text{const.}=L_0\\
&\frac{dG}{d\tau}=-\frac{\partial K_0}{\partial g}=0 \qquad G=\text{const.}=G_0\\
&\frac{d\overline{H}}{d\tau}=-\frac{\partial K_0}{\partial h}=0 \qquad \overline{H}=\text{const}=H_0\\
&\frac{dl}{d\tau}=\frac{\partial K_0}{\partial L}=[\overline{H}^2-2(G+C_0)]^{1/2}=\text{const.}=n_l\\
&\qquad\therefore\quad l=n_l\tau+l_0\\
&\frac{dg}{d\tau}=\frac{\partial K_0}{\partial g}=\frac{-L}{[\overline{H}^2-2(G+C_0)]^{1/2}}=\text{const.}=n_g\\
&\qquad\therefore\quad g=n_g\tau+g_0\\
&\frac{dh}{d\tau}=\frac{\partial K_0}{\partial \overline{H}}=\frac{L\overline{H}}{[\overline{H}^2-2(G+C_0)]^{1/2}}=\text{const.}=n_h\\
&\qquad\therefore\quad h=n_h\tau+h_0,
\end{aligned}\right\} \tag{25}$$

where l_0, g_0, h_0 are the values of l, g, h at $\tau=0$.

The angle ϕ is given by

$$\phi=\tfrac{1}{2}[f+g]+\tfrac{1}{2}\frac{\sqrt{L^2-G^2}}{\overline{H}^2-2(G+C_0)}\sin l,\; e\neq 1 \tag{26}$$

$$\phi=\tfrac{1}{2}g+\frac{L}{2(\overline{H}^2-2C_0)}\sin l \qquad ,\; e=1 \tag{27}$$

We can easily express the variables ξ_i, π_i in terms of the canonical elements $(l, L, g, G, h, \overline{H})$. We have

$$\left.\begin{aligned}
\pm\xi_1&=\sqrt{z}\cos\phi=\sqrt{a(1-e\cos l)}\cos\phi\\
\pm\xi_2&=\sqrt{z}\sin\phi=\sqrt{a(1-e\cos l)}\sin\phi,\\
\xi_3&=h-\frac{\overline{H}(L^2-G^2)^{1/2}}{\overline{H}^2-2(G+C_0)}\sin l,
\end{aligned}\right\} \tag{28}$$

Therefore

$$\left.\begin{aligned}
\pm\pi_1&=\frac{2e\,L\sin l\cos\phi-2G\sin\phi}{\pm\sqrt{a(1-e\cos l)}}\\
\pm\pi_2&=\frac{2e\,L\sin l\sin\phi+2G\cos\phi}{\pm\sqrt{a(1-e\cos l)}}\\
\pi_3&=\overline{H},
\end{aligned}\right\} \tag{29}$$

where ϕ is given by equation (26).

When $e=1$ $(G=0)$, we have

$$\left.\begin{aligned}\pm\xi_1&=\sqrt{2a}\sin l/2\cos\phi\\ \pm\xi_2&=\sqrt{2a}\sin l/2\sin\phi,\\ \xi_3&=h-\frac{\bar{H}L}{\bar{H}^2-2C_0}\sin l,\\ \pm\pi_1&=\frac{4L}{\sqrt{29}}\cos l/2\cos\phi,\\ \pm\pi_2&=\frac{4L}{\sqrt{29}}\cos l/2\sin\phi,\ \pi_3=\bar{H},\end{aligned}\right\}\qquad(30)$$

where ϕ is given by equation (27).

The original synodic cartesian coordinates in a non-uniformly rotating system are obtained from equations (29) or (30) and equation (5). When $\mu=0$, we have

$$\left.\begin{aligned}&x_1=\xi_1^2-\xi_2^2\,;\ x_2=2\xi_1\xi_2\,;\ x_3=\xi_3\\ &p_1=\frac{1}{2z}(\pi_1\xi_1-\pi_2\xi_2)\\ &p_2=\frac{1}{2z}(\xi_1\pi_2+\xi_2\pi_1)\\ &p_3=\pi_3\end{aligned}\right\}\qquad(31)$$

Here $$z=a\,(1-e\cos l).$$

In the uniformly rotating system, the coordinates $(\bar{\xi},\bar{\eta},\bar{\phi})$ are given by

$$\left.\begin{aligned}&\bar{\xi}=rx_1\,;\ \bar{\eta}=rx_2\,;\ \bar{\phi}=rx_3\\ &\dot{\bar{\xi}}=\dot{r}x_1+r\dot{x}_1\\ &\dot{\bar{\eta}}=\dot{r}x_2+r\dot{x}_2\\ &\dot{\bar{\phi}}=\dot{r}x_3+r\dot{x}_3,\end{aligned}\right\}\qquad(32)$$

where $\dot{r}$ is worked out from

$$\frac{a'(1-e'^2)}{r}=1+e'\cos f.$$

The sidreal cartesian coordinate are obtained by considering the transformation

$$\left.\begin{aligned}&X_1=\bar{\xi}\cos f-\bar{\eta}\sin f\\ &X_2=\bar{\xi}\sin f+\bar{\eta}\cos f\\ &X_3=\bar{\phi}\\ &\dot{X}_1=(\dot{\bar{\xi}}-\bar{\eta})\cos f-(\bar{\xi}+\dot{\bar{\eta}})\sin f\\ &\dot{X}_2=(\dot{\bar{\xi}}-\bar{\eta})\sin f+(\bar{\xi}+\dot{\bar{\eta}})\cos f\\ &\dot{X}_3=\dot{\bar{\phi}}\end{aligned}\right\}\qquad(33)$$

All the above differentiations are with respect to f, where f is given by

$$df = r_1 \, d\tau$$

or
$$f - f_0 = \frac{a}{[\overline{H}^2 - 2(G + C_0)]^{1/2}} \, [l - e \sin l], \tag{34}$$

and f_0 is a constant.

In terms of the canonical variables introduced, the complete Hamiltonian may be written as

$$\begin{aligned} K &= K_0 + \mu K_1 \\ &= L\,\overline{H}^2 - 2(G + C_0)]^{1/2} - 1 + \mu \Big[\frac{1}{2} r_1^3 - \frac{r_1 r_2^2}{2} + 1 - \frac{r_1}{r_2} \\ &\qquad + e_1 \cos f \, (1 + \tfrac{1}{2} r_1^3) + r_1 \frac{C_0 - C}{\mu} \\ &\quad + \int_0^{r_1} \int_0^f \frac{e_1 \sin f}{(1 + \mu e_1 \cos f)^2} \times \Big\{ \tfrac{1}{2}(\overline{1 - \mu} r_1^2 + \mu r_2^2) \\ &\qquad + \frac{1 - \mu}{r_1} + \frac{\mu}{r_2} - \frac{1}{2}(1 + \mu e_1 \cos f) \xi_3^2 \Big\} dr_1 \, df \Big] \end{aligned} \tag{35}$$

where ξ_1, ξ_2 and ξ_3 are given by (28). If we denote the coefficient of μ by R after K has been transformed completely in terms of the canonical variabe, we can write the complete Hamiltonian as

$$K = L[\overline{H}^2 - 2\,(G + C_0)]^{1/2} - 1 + \mu \, R + o(\mu^2)$$

The equations of motion for the complete Hamiltonian are

$$\left.\begin{aligned} \frac{dl}{d\tau} &= \frac{\partial K}{\partial L} = [\overline{H}^2 - 2\,(G + C_0)]^{1/2} + \mu \frac{\partial R}{\partial L} \\ \frac{dg}{d\tau} &= \frac{\partial K}{\partial G} = \frac{-L}{[\overline{H}^2 - 2\,(G + C_0)]^{1/2}} + \mu \frac{\partial R}{\partial G} \\ \frac{dh}{d\tau} &= \frac{\partial K}{\partial \overline{H}} = \frac{L\overline{H}}{[\overline{H}^2 - 2(G + C_0)]^{1/2}} + \mu \frac{\partial R}{\partial \overline{H}} \\ \frac{dL}{d\tau} &= -\frac{\partial K}{\partial l} = -\mu \frac{\partial R}{\partial l} \\ \frac{dG}{d\tau} &= -\frac{\partial K}{\partial g} = -\mu \frac{\partial R}{\partial g} \\ \frac{d\overline{H}}{d\tau} &= -\frac{\partial K}{\partial h} = -\mu \frac{\partial R}{\partial h} \end{aligned}\right\} \quad \text{....(36)}$$

These equations form the basis of a general perturbation theory for the problem in question.

4. Existence of periodic orbits when $\mu \neq 0$. We shall follow the procedure as in (Choudhry, 1966) for proving the existence of periodic orbits when $\mu \neq 0$.

When $\mu=0$, the equations (36) become

$$\left.\begin{aligned}
\frac{dl}{d\tau} &= \frac{\partial K_0}{\partial L} = [\bar{H}^2-2(G+C_0)]^{1/2} = \text{const.}\\
\frac{dg}{d\tau} &= \frac{\partial K_0}{\partial G} = \frac{-L}{[\bar{H}^2-2(G+C_0)]^{1/2}} = \text{const.}\\
\frac{dh}{d\tau} &= \frac{\partial K_0}{\partial \bar{H}} = \frac{L\bar{H}}{[\bar{H}^2-2(G+C_0)]^{1/2}} = \text{const.}\\
\frac{dL}{d\tau} &= -\frac{\partial K_0}{\partial l} = 0\\
\frac{dG}{d\tau} &= -\frac{\partial K_0}{\partial g} = 0\\
\frac{d\bar{H}}{d\tau} &= -\frac{\partial K_0}{\partial h} = 0
\end{aligned}\right\} \tag{37}$$

Let $x_1=L,\ x_2=G,\ x_3=\bar{H}\ ;\ y_1=l,\ y_2=g,\ y_3=h.$

The equations (37) may be written as

$$\left.\begin{aligned}
&\frac{dx_i}{d\tau}=0\ ;\ \frac{dy_i}{d\tau}=n_i^{(0)}\ \text{(say)}\\
\text{i.e.}\qquad &x_i=a_i\ ;\ y_i=n_i^{(0)}+\tau+\omega_i \quad (i=1,\ 2,\ 3)
\end{aligned}\right\} \tag{38}$$

These are the generating solution of the problem of two bodies in a three dimensional coordinate system. Here a_i, ω_i $(i=1,\ 2,\ 3)$ are constants.

Further

$$\left.\begin{aligned}
n_1^{(0)} &= \left(-\frac{\partial K_0}{\partial x_1}\right)_{x_1=a_1}\\
n_2^{(0)} &= \left(-\frac{\partial K_0}{\partial x_2}\right)_{x_2=a_2}\\
n_3^{(0)} &= \left(-\frac{\partial K_0}{\partial x_3}\right)_{x_3=a_3}
\end{aligned}\right\} \tag{39}$$

The generating solution will be periodic with the period τ_0 if

$$\left.\begin{aligned}
&x_i\ (\tau_0)-x_i\ (0)=0\\
&y_i\ (\tau_0)-y_i\ (0)=n_i^{(0)}\ \ \tau_0=2K_i\pi
\end{aligned}\right\} \tag{40}$$

Here K_i $(i=1,\ 2,\ 3)$ are integers so that $n_i^{(0)}$ are commensurable.

Let the general solution in the neighbourhood of the generating solution be periodic with the period $\tau_0\ (1+\alpha)$ where α is a negligible quantity of the $o(\mu)$. Let us introduce a new independent variable ϕ by the equation

$$\phi=\frac{\tau}{1+\alpha}.$$

The period of the general solution will then be τ_0. This period coincides with that of the generating solution.

The equations of motion can be written as

$$\left.\begin{aligned} \frac{dx_i}{d\phi} &= -(1+\alpha)\frac{\partial K}{\partial y_i} \\ \frac{dy_i}{d\phi} &= -(1+\alpha)\frac{\partial K}{\partial x_i} \quad (i=1,2,3) \end{aligned}\right\} \tag{41}$$

Let us take the general solution in the neighbourhood of the generating solution as

$$x_i = a_i + \beta_i + \xi_i(\phi)$$
$$y_i = n_i^{(0)}\phi + \omega_i + \nu_i + \eta_i(\phi), \; (i=1,2,3)$$

Then ξ_i, η_i are given by the equations (Bhatnagar, 1969)

$$\frac{\xi_k(\tau_0, \beta_i, \nu_i, \mu)}{-\mu\,\tau_0} = \frac{\partial[K_1]}{\partial\omega_k} + \sum_{i=1}^{3} \frac{\partial^2[K_1]}{\partial\omega_k\,\partial a_i} + \sum_{i=1}^{3} \frac{\partial^2[K_1]}{\partial\omega_k\,\partial\omega_i} = 0 \tag{42}$$

where $$[K_1] = \frac{1}{\tau_0}\int_0^{\tau_0} K_1\{\phi, a_i, n_i^{(0)}\phi + \omega\}\, d\phi$$

$$\eta_1\left(\tau_i, \beta_i, \nu_i, \mu\right) = \alpha\,\tau_0\frac{\partial K_0}{\partial a_1} + \beta_1\,\tau_0\frac{\partial^2 K_0}{\partial a_1^2} + \beta_2\tau_0\frac{\partial^2 K_0}{\partial a_1\,\partial a_2} + \beta_3\,\tau_0\frac{\partial^2 K_0}{\partial a_1\,\partial a_3} + o(\mu) = 0. \tag{43}$$

$$\eta_2\left(\tau_0, \beta_i, \nu_i, \mu\right) = \alpha\,\tau_0\frac{\partial K_0}{\partial a_2} + \beta_1\,\tau_0\frac{\partial^2 K_0}{\partial a_1\,\partial a_2} + \beta_2\,\tau_0\frac{\partial^2 K_0}{\partial a_2^2} + \beta_3\,\tau_0\frac{\partial^2 K_0}{\partial a_2\,\partial a_3} + o(\mu) = 0. \tag{44}$$

$$\eta_3\left(\tau_0, \beta_i, \nu_i, \mu\right) = \alpha\tau_0\frac{\partial K_0}{\partial a_3} + \beta_1\tau_0\frac{\partial^2 K_0}{\partial a_3\,\partial a_1} + \beta_2\,\tau_0\frac{\partial^2 K_0}{\partial a_3\,\partial a_2} + \beta_3\,\tau_0\frac{\partial^2 K_0}{\partial a_3^2} + o(\mu) = 0. \tag{45}$$

Periodic orbits will exist if (Duboslim, 1964)

$$\frac{\partial[K_1]}{\partial\omega_i} = 0 \; (i=1,2,3) \tag{46}$$

$$\frac{\partial[K_1]}{\partial a_i} = 0 \; (i=1,2,3) \tag{47}$$

$$\frac{\partial(\xi_2, \xi_3, \eta_1, \eta_2, \eta_3)}{\partial(\nu_2, \nu_3, \beta_1, \beta_2, \beta_3)} \neq 0, \text{ when } \mu = \beta_i = \nu_i = 0 \tag{48}$$

We may note that

$$\frac{\partial^2[K_1]}{\partial a_j \partial \omega_i}=0, \text{ for } \frac{\partial[K_1]}{\partial \omega_i}=0 \ [i\neq j, \ i, j=1, 2, 3]$$

$$K_0=a_1[a_2{}^2-2(a_2+C_0)]^{1/2}-1$$

and

$$\begin{vmatrix} \frac{\partial^2 K_0}{\partial a_1{}^2} & \frac{\partial^2 K_0}{\partial a_2\,\partial a_1} & \frac{\partial^2 K_0}{\partial a_3\,\partial a_1} \\ \frac{\partial^2 K_0}{\partial a_1 \partial a_2} & \frac{\partial^2 K_0}{\partial a_2{}^2} & \frac{\partial^2 K_0}{\partial a_3\,\partial a_2} \\ \frac{\partial^2 K_0}{\partial a_1 \partial a_3} & \frac{\partial^2 K_0}{\partial a_2\,\partial a_3} & \frac{\partial^2 K_0}{\partial a_3{}^2} \end{vmatrix} = \frac{-a_1}{[a_3{}^2-2(a_2+C_0)]^{3/2}} \neq 0 \quad (49)$$

It can be seen easily from condition (49) that the determinant (48) will not be equal to zero, if

$$\begin{vmatrix} \frac{\partial^2[K_1]}{\partial \omega_2{}^2} & \frac{\partial^2[K_1]}{\partial \omega_3\,\partial \omega_2} \\ \frac{\partial^2[K_1]}{\partial \omega_3\,\partial \omega_2} & \frac{\partial^2[K_1]}{\partial \omega_3{}^2} \end{vmatrix} \neq 0 \qquad (50)$$

For calculating the partial derivatives occurring in (50), we may note from (35), that

$$K_1=\frac{1}{2}r_1{}^3-\frac{r_1\,r_2{}^2}{2}+1-\frac{r_1}{r_2}+e_1 \cos f \left(1+\frac{1}{2}r_1{}^3\right)+r_1.\frac{C_0-C_1}{\mu}$$

$$+\int_0^{r_1}\int_0^{f} \frac{e_1 \sin f}{(1+\mu\, e_1 \cos f)^2}\times\left[\frac{1}{2}\left\{(1-\mu)\, r_1{}^2+\mu r_2{}^2\right\}+\frac{1-\mu}{r_1}\right.$$

$$\left.+\frac{\mu}{r_2}-\frac{1}{2}(1+\mu\; e_1 \cos f)\xi_3{}^2\right] dr_1\, df$$

Taking only zero-order term, we have

$$[K_1]=\frac{1}{2}r_1{}^3-\frac{r_1\,r_2{}^2}{2}+1-\frac{r_1}{r_2},$$

where

$$z=\xi^2=a\ ;\ r_1{}^2=a^2+\xi_3{}^2$$

$$r_2{}^2=1+a^2+2a \cos \phi+\xi_3{}^2,$$

$$\xi_3=\omega_3+\text{terms containing } \omega_1 \text{ and } \omega_2.$$

$$2\phi=\omega_1+\omega_2+n_1{}^{(0)}+n_2{}^{(0)}$$

Now

$$\frac{\partial(K_1)}{\partial \omega_2}=a_1\ r_1 \sin 2\phi\left[1-\frac{1}{r_2{}^3}\right] \qquad (51)$$

$$=0, \text{ gives}$$

$$2\phi=0,\ \pi \text{ or } r_2=1$$

But

$$\frac{\partial^2\,[K_1]}{\partial \omega_2{}^2}=a_1\ r_1 \cos 2\phi\left[1-\frac{1}{r_2{}^3}\right]+a_1\ r_1 \sin\ 2\phi\left(-\frac{3}{r_1{}^4}\right)\left(-\frac{a \sin 2\phi}{r_2}\right)$$

Therefore when $2\phi=0, \pi$ or $r_2=1$,

$$\frac{\partial^2 [K_1]}{\partial \omega_2^2} \neq 0 \tag{52}$$

Again $$\frac{\partial^2 (K_1)}{\partial \omega_2 \, \partial \omega_3} = \frac{\partial}{\partial \omega_3}\left[a_1 \; r_1 \sin 2\phi\left(1-\frac{1}{r_2^3}\right)\right]$$

2ϕ is independent of ω_3, therefore

$$\frac{\partial^2 (K_1)}{\partial \omega_2 \, \partial \omega_3} = a_1 \sin 2\phi \; \frac{\partial}{\partial \omega_3}\left[r_1\left(1-\frac{1}{r_2^3}\right)\right]$$

$$=0, \text{ for } 2\phi=0 \text{ or } \pi. \tag{53}$$

Further $$\frac{\partial K_1}{\partial \omega_3} = \frac{3}{2} r_1^2 \frac{\partial r_1}{\partial \omega_3} - \frac{r_2^2}{2} \frac{\partial r_1}{\partial \omega_3} - \frac{r_1}{2} \cdot 2r_2 \frac{\partial r_2}{\partial \omega_3} - \frac{1}{r_2} \frac{\partial r_1}{\partial \omega_3} + \frac{r_1}{r_2^2} \frac{\partial r_2}{\partial \omega_3}$$

But $$\frac{\partial r_1}{\partial \omega_3} = \frac{\partial r_1}{\partial \xi_3} \cdot \frac{\partial \xi_3}{\partial \omega_3} = \frac{\xi_3}{r_1}$$

and $$\frac{\partial r_2}{\partial \omega_3} = \frac{\partial r_2}{\partial \xi_3} \cdot \frac{\partial \xi_3}{\partial \omega_3} = \frac{\xi_3}{r_2}$$

Thus $$\frac{\partial K_1}{\partial \omega_3} = B\xi_3 \text{ (say)} \tag{54}$$

Therefore $\dfrac{\partial K_1}{\partial \omega_3}=0$, gives, $\xi_3=0$ or $B=0$.

Now $$\frac{\partial^2 [K_1]}{\partial \omega_3^2} = B\frac{\partial \xi_3}{\partial \omega_3} + \xi_3 \frac{\partial B}{\partial \omega_3}$$

$$\neq 0, \text{ when either } \xi_3=0, \text{ or } B=0 \tag{55}$$

From (52), (53), (55), it follows that determinant (50)$\neq 0$.

Hence condition (48) is satisfied.

Now $$\frac{\partial (K_1)}{\partial \omega_1} = \frac{3}{2} r_1^2 \frac{\partial r_1}{\partial \omega_1} - \frac{r_2^2}{2} \frac{\partial r_1}{\partial \omega_1} - \frac{r_1}{2} \cdot 2r_2 \frac{\partial r_1}{\partial \omega_1} - \frac{1}{r_2} \frac{\partial r_1}{\partial \omega_1} + \frac{1}{r_2^2} \cdot \frac{\partial r_2}{\partial \omega_1}$$

But $$\frac{\partial r_1}{\partial \omega_1}=0 \; ; \; \frac{\partial r_2}{\partial \omega_1} = \frac{\partial r_2}{\partial (2\phi)} \cdot \frac{\partial (2\phi)}{\partial \omega_1} = \frac{-2a \sin 2\phi}{2r_2}$$

Thus $$\frac{\partial (K_1)}{\partial \omega_1} = 0 \text{ for } 2\phi=0 \text{ or } \pi \tag{56}$$

This satisfies condition (46).

Let us consider the condition (47). Since K_1 is independent of a_1, a_2, a_3, therefore

$$\frac{\partial [K_1]}{\partial a_i} = 0, \; (i=1, 2, 3)$$

Hence all the conditions, *viz.* (46), (47), (48) are satisfied for the existence of the periodic orbits when $\mu \neq 0$.

5. Periodic Orbit of Collision when $\mu \neq 0$. In this section we shall prove the existence of periodic orbits of collision when $\mu \neq 0$.

Proceeding as in (Bhatnagar, 1969), in our case, the condition of collision should be of the form

$$G + \mu F(l, L, g, G, h, \bar{H}) = 0 \tag{57}$$

We, again, consider the case when $e = 1$. In that case the orbit starts as an ejection from the origin and returns to it after time $\tau/4$.

Levi-Civita's condition for collision is

$$\dot{\theta} + 1 = \rho f(\rho, 0) \tag{58}$$

where $\tan \theta = \dfrac{x_2}{x_1 - \mu}$; $\rho = \sqrt{r_1}$

The condition (58) in our case, becomes

$$2\dot{\phi} + 1 = \sqrt{r_1}\, f(\sqrt{r_1}, \theta)$$

or
$$2 \frac{d\phi}{d\tau} \frac{d\tau}{dt} + 1 = \sqrt{r_1}\, f(\sqrt{r_1}, \theta)$$

But
$$2\phi' = \frac{G}{\xi^2} - \xi^2$$

$$\therefore \quad G - \xi^4 + r_1 \xi^2 = \xi^2 r_1^{3/2} f(r_1^{1/2}, 2\phi). \tag{59}$$

This corresponds to (57). Obviously this is satisfied since at $\tau = 0$,

$$G = 0,\ \xi = 0,\ \xi_3 = 0\ (i.e.\ r_1 = 0)$$

Since condition (59) is satisfied along the entire orbit, the infinitesional body will approach the origin with characteristics of a collision orbit. The proof of the existence of such periodic orbits in the collision is, therefore, fully established.

REFERENCES

1. K. B. Bhatnagar, Periodic orbits of collision in the plane elliptic restricted problem of three bodies, Proc. Nat. Inst. Sci. India Part A 35 (1969), 829-844.

2. R. K. Choudhry, Existence of periodic orbits of the third kind in the elliptic restricted problem of three bodies and the stability of the generating solution, Proc. Nat. Acad. Sci. India Sect. A 36 (1966), 249-264.

3. G. N. Duboslim, Celestial Mechanics (Analytical and Qualitative Methods) (Russian, 1964), 178-184.

4. T. Levi-Civita, Traiettorie singolari ed urti nel problema ristretto dei tre corpi, Ann. Mat. Pura Appl. (3) 9 (1904), 1.

5. T. Levi-Civita, Sur la re'solution qualitative du proble'm restreint des trois corps, Acta Math. 30 (1906), 305-327.

6. V. Szebehely, Theory of orbits (the restricted problem of three bodies), Academic Press Inc., New York, 1967.

Jñānabha Sect. A, Vol. 6, 1976

INTEGRALS INVOLVING THE H-FUNCTION OF TWO VARIABLES

By **Y. N. Prasad and S. N. Singh**

Applied Mathematics Section, Institute of Technology, Banaras Hindu University, Varanasi–221005, *India*

(Received : August 17, 1976 ; in revised form : April 12, 1977)

Abstract. In the present paper we have evaluated an integral in which the integrand is a product of two H-functions of two variables and an algebraic function. The integral obtained in this paper generalizes the recent result obtained by Gupta and Mittal [4]. This integral is believed to be among the most general integrals evaluated so far and includes interesting integrals (which are themselves quite general in nature) as its particular cases. This integral, in turn, yields a large number of new and known integrals (obtained by earlier authors) as its special cases including Laplace's integral of the class of double Barnes integral which may prove useful in solving certain boundary value problems.

1. Introduction. The double Barnes integral occurring in this paper will be referred to as the H-function of two variables throughout our present study and will be defined and represented as follows :

$$(1.1)\quad H^{0,\,0\,:\,(m,\,n)\,;\,(u,\,v)}_{p,\,q\,[P,\,Q]\,;\,[X,\,Y]}\left[\begin{matrix}\{(a_p\,;\,\alpha_p,\,A_p)\}:\{(c_P,\,\gamma_P)\}\,;\,\{(e_X,\,E_X)\}\,;\\ \{(b_q\,;\,\beta_q,\,B_q)\}:\{(d_Q,\,\delta_Q)\}\,;\,\{(f_Y,\,F_Y)\}\,;\end{matrix}\;x,\,y\right]$$

$$=-\frac{1}{4\pi^2}\int_{L_1}\int_{L_2}\phi(s,t)\,\theta_1(s)\,\theta_2(t)\,x^s\,y^t\,ds\,dt,$$

where

$$(1.2)\quad \phi(s,t)=\left[\prod_{j=1}^{q}\Gamma(1-b_j+\beta_j s+B_j t)\prod_{j=1}^{p}\Gamma(a_j-\alpha_j s-A_j t)\right]^{-1}$$

$$(1.3)\quad \theta_1(s)=\frac{\prod_{j=1}^{m}\Gamma(d_j-\delta_j s)\prod_{j=1}^{n}\Gamma(1-c_j+\gamma_j s)}{\prod_{j=m+1}^{Q}\Gamma(1-d_j+\delta_j s)\prod_{j=n+1}^{P}\Gamma(c_j-\gamma_j s)},$$

$$(1.4)\quad \theta_2(t)=\frac{\prod_{j=1}^{u}\Gamma(f_j-F_j t)\prod_{j=1}^{v}\Gamma(1-e_j+E_j t)}{\prod_{j=u+1}^{Y}\Gamma(1-f_j+F_j t)\prod_{j=v+1}^{X}\Gamma(e_j-E_j t)},$$

where x, y are not equal to zero and an empty product is interpreted as unity ; $\{(a_p, \alpha_p)\}$ and $\{(a_p ; \alpha_p, A_p)\}$ abbreviate to p-parameter sequences $(a_1, \alpha_1), \ldots, (a_p, \alpha_p)$ and $(a_1 ; \alpha_1, A_1), \ldots, (a_p ; \alpha_p, A_p)$ respectively. For the relations between the integers and parameters, and for the conditions of convergence and validity of the integral defined in (1.1), the reader will be referred to the earlier works by Mittal and Gupta [5], and Srivastava and Panda (cf. [2] and [3]).

Throughout the present study, where there is no possibility of being misunderstood, we shall denote the left member of (1.1) by

$$(1.5)\qquad H^{0,\,0\,:\,(m,\,n)\,;\,(u,\,v)}_{p,\,q\,:\,[P,\,Q]\,;\,[X,\,Y]}\left(x, y\right).$$

In the subsequent discussions, we have used a result on expansion of the H-function of two variables due to Prasad and Gupta [6, p. 40, Eq. (2.1)]

$$(1.6)\qquad H^{0,\,0\,:\,(m,\,n)\,;\,(u,\,v)}_{p,\,q\,:\,[P,\,Q]\,;\,[X,\,Y]}\left(bx^{\sigma}, cx^{\mu}\right)$$

$$=\frac{1}{F_1}\sum_{r=0}^{\infty}\frac{(-1)^r}{r\,!}\,\psi(\rho_r)(cx^{\mu})^{\rho_r}$$

$$\times\; H^{m,\,n}_{p+P,\,q+Q}\left[bx^{\sigma}\,\middle|\,\begin{matrix}\{(c_P, \gamma_P)\}, \{(a_p - A_p\,\rho_r, \alpha_p)\}\\ \{(d_Q, \delta_Q)\}, \{(b_q - B_q\,\rho_r, \beta_q)\}\end{matrix}\right]$$

where

$$(1.7)\qquad \rho_r=\frac{f_1+r}{F_1},$$

$$(1.8)\quad \psi(\rho_r)=\frac{\prod_{j=2}^{u}\Gamma(f_j-F_j\,\rho_r)\prod_{j=1}^{v}\Gamma(1-e_j+E_j\,\rho_r)}{\prod_{i=u+1}^{Y}\Gamma(1-f_j-F_j\,\rho_r)\prod_{j=v+1}^{X}\Gamma(e_j-E_j\rho_r)},$$

$\delta' < R(f_1/F_1) < \beta'$; $|\arg(b)| < \frac{1}{2} u'\pi, u' > 0$; $|\arg(c)| < \frac{1}{2} v'\pi, v' > 0$;

(1.9) $$\delta' = \max R\left(\frac{e_i - 1}{E_i}\right), i = 1, \ldots, v ;$$

$$\beta' = \min R(f_j/F_j), \quad j = 1, \ldots, u ;$$

$$u' = -\sum_{j=1}^{p} \alpha_j - \sum_{j=1}^{q} \beta_j + \sum_{j=1}^{m} \gamma_j - \sum_{j=n+1}^{P} \gamma_j + \sum_{j=1}^{m} \delta_j - \sum_{j=m+1}^{Q} \delta_j ;$$

$$v' = -\sum_{j=1}^{p} A_j - \sum_{j=1}^{q} B_j + \sum_{j=1}^{u} F_j - \sum_{j=u+1}^{Y} F_j + \sum_{j=1}^{v} E_j - \sum_{j=v+1}^{X} E_j,$$

and the Gamma functions involved in (1.8) exist and the H-function involved in (1.6) is the H-function due to Fox [1].

2. Main Integral. We establish our main integral as follows :

(2.1) $$\int_0^\infty x^{\rho-1} (1+ax^{\xi})^{-\lambda}$$

$$\times H^{0,0:(m,n);(u,v)}_{p,q:[P,Q];[X,Y]} \left[\begin{matrix} \{(a_p ; \alpha_p, A_p)\} ; \{(c_P, \gamma_P)\} ; \{(e_X, E_X)\} ; \\ \{(b_q ; \beta_q, B_q)\} ; \{(d_Q, \delta_Q)\} ; \{(f_Y, F_Y)\} ; \end{matrix} \right.$$

$$\left. bx^{\sigma}, cx^{\mu}(1+ax^{\xi})^{\nu} \right] . H^{0,0:(m',n');(u',v')}_{p',q':[P',Q'];[X',Y']} \left[\begin{matrix} \{(a'_{p'} ; \alpha'_{p'}, A'_{p'})\} ; \\ \{(b'_{q'} ; \beta'_{q'}, B'_{q'})\} ; \end{matrix} \right.$$

$$\left. \begin{matrix} \{(c'_{P'}, \gamma'_{P'})\} ; \{(e'_{X'}, E'_{X'})\} ; \\ \{(d'_{Q'}, \delta'_{Q'})\} ; \{(f'_{Y'}, F'_{Y'})\} ; \end{matrix} \frac{x^h}{(1+ax^{\xi})^h}, \frac{x^k}{(1+ax^{\xi})^k} \right] dx$$

$$= \frac{b^{-\rho/\sigma}}{\sigma F_1} \sum_{r=0}^{\infty} \sum_{N=0}^{\infty} \frac{(-1)^r a^N c^{\rho_r} b^{-(\mu\rho_r + \xi N)/\sigma}}{r!\, N!} \psi(\rho_r).$$

$$\times H^{n+1,m}_{p'+q+Q+1,\, q'+p+P+1} :$$

$$\begin{matrix} (m',n') ; (u',v') \\ [P',Q'] ; [X',Y'] \end{matrix} \left[\begin{matrix} C ; \{(c'_{P'}, \gamma'_{P'})\} ; \{(e'_{X'}, E'_{X'})\} ; \\ D ; \{(d'_{Q'}, \delta'_{Q'})\} ; \{(f'_{Y'}, F'_{Y'})\} ; \end{matrix} \frac{\beta}{b^{h/\rho}}, \frac{\delta}{b^{k/\sigma}} \right]$$

where

$$C \equiv \left\{ \left(1 - d_Q - \frac{1}{\sigma} d_Q E ; \frac{h}{\sigma} \delta_Q, \frac{k}{\sigma} \delta_Q \right) \right\}, \left\{ \left(1 - b_q + B_q \rho_r \right. \right.$$

$$\left. \left. - \frac{1}{\sigma} \beta_q E, \frac{h}{\sigma} \beta_q, \frac{k}{\sigma} \beta_q \right) \right\}, \left\{ (a'_{p'}, \alpha'_{p'}, A'_{p'}) \right\},$$

$$(1 - \nu \rho_r - \lambda - N, h, k) ;$$

$$D \equiv (1-\lambda+\nu\, \rho_r, h, k), \left\{\left(1-c_P-\frac{1}{\sigma}\gamma_P E;\ \frac{h}{\sigma}\gamma_P, \frac{k}{\sigma}\gamma_P\right)\right\},$$

$$\left\{(b'_{q'};\beta'_{q'}, B'_{q'}\right\}, \left\{(1-a_p-A_p\rho_r-\frac{1}{\sigma}\alpha_p E;\ \frac{h}{\sigma}\alpha_p, \frac{k}{\sigma}\alpha_p)\right\}$$

$$E \equiv \rho+\mu\rho_r+\xi N,$$

and the conditions given below are satisfied :

$\sigma, \mu, \nu, h, k>0$; $R(\rho+\sigma\alpha'+\mu\beta'+h\alpha''+k\beta'')>0$;

$R[\rho-\xi\lambda+\sigma\gamma'+(\mu+\xi\nu)\delta'+(1-\xi)h\gamma''+(1-\xi)k\delta'']<0$;

$|\arg b|<\frac{1}{2}u'\pi, u'>0$; $|\arg c|<\frac{1}{2}v'\pi, v'>0$;

$|\arg\beta|<\frac{1}{2}u''\pi, u''>0$; $|\arg\delta|<\frac{1}{2}v''\pi, v''>0$;

$\delta'<R(f_1/F_1)<\beta'$; δ', β', u', v' are given by equation (1.9) ;

$$\alpha'=\min R(d_j/\delta_j), \qquad j=1,\dots,m ;$$

$$\gamma'=\max R\left(\frac{c_i-1}{\gamma_i}\right), \quad i=1,\dots,n ;$$

and $\alpha'', \beta'', \gamma'', \delta'', u''$ and v'' are given by the expressions for $\alpha', \beta', \gamma', \delta', u', v'$, respectively, by putting one more dash on the parameters involved.

Proof. Expanding the first H-function of two variables in the integrand of (2.1) by the expansion formula (1.6) and interchanging the order of integration and summation which is justifiable under the given conditions, the left hand side of (2.1) reduces to

$$(2.2)\ \frac{1}{F_1}\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}\psi(\rho_r)\int_0^\infty x^{\rho-1}(1+ax^\xi)^{-\lambda}\{cx^\mu(1+ax^\xi)^\nu\}^{\rho_r}$$

$$\times H^{m,n}_{p+P,q+Q}\left[bx^\sigma\left|\begin{matrix}\{(c_P,\gamma_P)\},\{(a_p-A_p\rho_r,\alpha_p)\}\\ \{(d_Q,\delta_Q)\},\{(b_q-B_q\rho_r,\beta_q)\}\end{matrix}\right.\right]$$

$$\times H^{0,0:(m',n');(u',v')}_{p',q':[P',Q'];[X,Y]}\left(\frac{\beta x^h}{(1+ax^\xi)^h}, \frac{\delta x^k}{(1+ax^\xi)^k}\right)dx,$$

where the second H-function in (2.2) is the same as the second H-function in (2.1).

Now, interpreting the second H-function in (2.2) in the form of double Barnes integral, we arrive at

$$(2.3)\quad -\frac{1}{4\pi^2 F_1}\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}\psi(\rho_r)\, c^{\rho_r}\int_{L_1'}\int_{L_2'}\phi'(s,t)\,\theta_1'(s)\,\theta_2'(t)\beta^s\delta^t\, ds dt$$

$$\times\int_0^\infty x^{\rho+\mu\rho_r+hs+kt-1}(1+ax^{\xi})^{-\lambda-hs-kt+\nu\rho_r}$$

$$\times H_{p+P,\,q+Q}^{m,\,n}\left[bx^{\sigma}\,\middle|\,\begin{matrix}\{(c_P,\gamma_P)\},\{(a_p, A_p\,\rho_r,\alpha_p)\}\\ \{(d_Q,\delta_Q)\},\{(b_q-B_q\,\rho_r,\beta_q)\}\end{matrix}\right]dx,$$

where L_1', L_2', $\phi'(s,t)$, $\theta_1'(s)$, $\theta_2'(t)$ correspond to the second H-function in (2.1).

The binomial expansion of $(1+ax^{\xi})^{-\lambda-hs-kt+\nu\rho_r}$ reduces (2.3) in the form

$$(2.4)\quad -\frac{1}{4\pi^2 F_1}\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}\psi(\rho_r)\, c^{\rho_r}\int_{L_1'}\cdots\int_{L_2'}\phi'(s,t)\,\theta_1'(s)\,\theta_2'(t)$$

$$\times\beta^s\delta^t\, dsdt\sum_{N=0}^{\infty}\frac{a^N}{N!}\,\frac{\Gamma(1-\lambda-hs-kt+\nu\rho_r)}{\Gamma(-\lambda-hs-kt+\nu\rho_r+1-N)}$$

$$\times\int_0^\infty x^{\rho+\mu\rho_r+hs+kt+\xi N-1}$$

$$\times H_{p+P,\,q+Q}^{m,\,n}\left[bx^{\sigma}\,\middle|\,\begin{matrix}\{(c_P,\gamma_P)\},\{(a_p-A_p\rho_r,\alpha_p)\}\\ \{(d_Q,\delta_Q)\},\{(b_q-B_q\,\rho_r,\beta_q)\}\end{matrix}\right]dx.$$

Now, evaluating the inner integral in (2.4) with the help of the known result

$$(2.5)\quad \int_0^\infty x^{\rho-1} H_{p,\,q}^{m,\,n}\left[zx^{-\sigma}\,\middle|\,\begin{matrix}\{(a_p,\alpha_p)\}\\ \{(b_q,\beta_q)\}\end{matrix}\right]dx$$

$$=\frac{1}{\sigma}z^{\rho/\sigma}\frac{\prod_{j=1}^{m}\Gamma(b_j-\beta_j\,\rho/\sigma)\prod_{j=1}^{n}\Gamma(1-a_j+\alpha_j\,\rho/\sigma)}{\prod_{j=m+1}^{q}\Gamma(1-b_j+\beta_j\,\rho/\sigma)\prod_{j=n+1}^{p}\Gamma(a_j-\alpha_j\,\rho/\sigma)}$$

provided that $\sigma>0$; $\beta_0<R(\rho/\sigma)<\delta_0$; $|\arg z|<\frac{1}{2}\lambda_0\pi$, $\lambda_0>0$ and $A>0$, where

$$\delta_0=\min R(b_j/\beta_j), j=1,\dots,m\ ;\ \beta_0=\max R\left(\frac{a_i-1}{\alpha_i}\right), i=1,\dots,n\ ;$$

$$\lambda_0=\sum_{j=1}^{n}\alpha_j-\sum_{j=n+1}^{p}\alpha_j+\sum_{j=1}^{m}\beta_j-\sum_{j=m+1}^{q}\beta_j\ ;\ A=\sum_{j=1}^{q}\beta_j-\sum_{j=1}^{p}\alpha_j\ ;$$

and finally, interpreting the above double Barnes contour integral with the help of (1.1), we obtain the right-hand side of (2.1), provided that the conditions stated with (2.1) are satisfied.

3. Interesting Particular Cases. (i) Taking $p=q=0=a$ in (2.1), we obtain the following interesting result :

$$(3.1)\quad \int_0^\infty x^{\rho-1} H^{m,n}_{P,Q}\left[bx^\sigma \left|\begin{matrix}\{(c_P,\gamma_P)\}\\ \{(d_Q,\delta_Q)\}\end{matrix}\right.\right] H^{u,v}_{X,Y}\left[cx^\mu \left|\begin{matrix}\{(e_X,E_X)\}\\ \{(f_Y,F_Y)\}\end{matrix}\right.\right]$$

$$\times H^{0,0\ :\ (m',n')\ ;\ (u',v')}_{p',q'\ :\ [P',Q']\ ;\ [X',Y']}\left(\beta x^h, \delta x^k\right)dx$$

$$=\frac{b^{-\rho/\sigma}}{\sigma F_1}\sum_{r=0}^{\infty}\frac{(-1)^r}{r\,!}b^{-\mu\rho_r/\sigma}\psi(\rho_r).$$

$$\times H^{n+1,m+1\ :\ (m',n')\ ;\ (u',v')}_{p'+Q+1,q'+P+1\ :\ [P',Q']\ ;\ [X',Y']}$$

$$\left[\begin{matrix}C^*\ ;\ \{(c'_{P'},\gamma'_{P'})\}\ ;\ \{(e'_{X'},E'_{X'})\}\ ;\ \dfrac{\beta}{b^{h/\sigma}},\dfrac{\delta}{b^{k/\sigma}}\\ D^*\ ;\ \{(d'_{Q'},\delta'_{Q'})\}\ ;\ \{(f'_{Y'},F'_{Y'})\}\ :\end{matrix}\right],$$

where

$$C^*=\left\{\left(1-d_Q-\frac{1}{\sigma}(\rho+\mu\rho_r)\delta_Q\ ;\ \frac{h}{\sigma}\delta_Q,\ \frac{k}{\sigma}\delta_Q\right)\right\}$$

$$\{(a'_{p'}\ ;\ \alpha'_{p'},A'_{p'})\},\ (1+\nu\rho_r-\lambda\ ;\ h,k)\ ;$$

$$D^*=(1-\lambda+\nu\rho_r;h,k),\ \left\{\left(1-c_P-\frac{1}{\sigma}(\rho+\mu\rho_r)\gamma_P\ ;\ \frac{h}{\sigma}\gamma_P,\ \frac{k}{\sigma}\gamma_P\right)\right\}$$

$$\{(b'_{q'}\ ;\ \beta'_{q'},B'_{q'})\},$$

and the conditions given with (2.1) with $p=q=0=a$ are satisfied.

(ii) Taking $\nu=X=0=p=q=f_1=n=\nu$; $F_1=\sigma=1=\mu=\xi$ and taking $h\to0$ in (2.1), we obtain the recent result due to Gupta and Mittal [4, p. 435, Eq. (2.1)]

(iii) Putting $u=Y=1=\mu=\sigma=F_1$; $v=X=0=n=a=f_1$ and taking $c\to 0$ in (3.1), we arrive at the recently obtained result due to Gupta and Mittal [4, p. 435, Eq. (3.1)].

(iv) Putting $n=P=0=v=X=a=d_1=f_1$; $u=Y=1=\mu=\sigma=F_1=m$ $=Q=\delta_1$, and taking $c\to 0$ in (3.1), we arrive at the result recently obtained by Gupta and Mittal [4, p. 436, Eq. (3.2)].

ACKNOWLEDGEMENT. The authors are greatly indebted to Prof. H. M. Srivastava, Department of Mathematics, University of Victoria, B. C., Canada, for his valuable guidance during the preparation of this paper.

REFERENCES

1. C. Fox, The G and H functions as symmetrical Fourier kernels, Trans. Amer. Math. Soc. 98 (1961), 395–429.

2. H. M. Srivastava and Rekha Panda, Some bilateral generating functions for a class of generalized hypergeometric polynomials, J. Reine Angew. Math. 283/284 (1976), 265–274.

3. H. M. Srivastava and Rekha Panda, Expansion theorems for the H-function of several complex variables, J. Reine Angew. Math. 288 (1976) 129–145.

4. K. C. Gupta and P. K. Mittal, Integrals involving a generalized function of two variables, Indian J. Pure and Appl. Math. 5 (1974), No. 5, 430–437.

5. P. K. Mittal and K. C. Gupta, An integral involving generalized function of two variables, Proc. Indian Acad. Sci. Sect. A 75 (1972), 117–123.

6. Y. N. Prasad and R. K. Gupta, An expansion formula for the H-function of two variables and its applications, Vijnana Parishad Anusandhan Patrika 19 (1976), 39–45.

Jñānabha Sect A, Vol. 6, 1976

PROJECTIVE CURVATURE COLLINEATION IN A FINSLER SPACE

By **H. D. Pande and A. Kumar†**

Department of Mathematics, University of Gorakhpur,

Gorakhpur–273001, *U. P., India*

(Received : December 6, 1975 ; in revised form : March 12, 1977)

1. Introduction. Let F_n be an n-dimensional Finsler space [1] equipped with the $2n$ line elements $(x^i, \dot{x}^i)$ and the fundamental metric function $F(x^i, \dot{x}^i)$. The metric function $F(x, \dot{x})$ is positively homogeneous of degree one in $\dot{x}^i$ and satisfies the condition imposed upon it. The metric tensor $g_{ij}(x, \dot{x})$ of F_x is given by

$$(1.1)\,a \qquad g_{ij}(x, \dot{x}) \stackrel{def.}{=} \tfrac{1}{2}\dot{\partial}_i \dot{\partial}_j F^2(x, \dot{x})$$

$$(1.1)\,b \qquad g_{ij}\, g^{jk} = \delta^k_i{}_j\,.$$

where δ^k_i is the Kronecker delta,

With the help of the above metric tensor, we have

$$(1.2) \qquad C_{ijk} = \tfrac{1}{2}\dot{\partial}_k\, g_{ij},$$

which satisfies the following relations :

$$(1.3) \qquad C^i_{jk} = g^{ih}\, C_{hjk},\; C_{ijk}\, \dot{x}^i = C_{ijk}\, \dot{x}^j = C_{ijk}\, \dot{x}^k = 0.$$

Let $X^i(x, \dot{x})$ be a vector field depending upon the directional as well as positional coordinates. The projective covariant derivative [2] of a vector field $X^i(x, \dot{x})$ is given by

$$(1.4) \qquad X^i_{((k))} = \partial_k X^i - (\dot{\partial}_m X^i)\, \Pi^m_{hk}\, \dot{x}^h + X^h\, \Pi^i_{hk}.$$

where

$$(1.5) \qquad \Pi^i_{hj}(x, \dot{x}) \stackrel{def.}{=} \left\{ G^i_{hj} - \frac{1}{(n+1)}\left(2\delta^i_{(h} G^r_{j)r} + \dot{x}^i\; G^r_{rhj} \right) \right\}.$$

† Present address : Department of Applied Sciences, M.M.M. Engineering College, Gorakhpur–273010, U. P., India.

The functions $\Pi^{\,i}_{hj}$ are called the coefficients of projective connection. They are symmetric in their lower indices and are positively homogeneous of degree zero in their directional arguments. They also satisfy the following identities [2] :

(1.6) $(a)\ \Pi^{\,i}_{hjk}\,\dot{x}^h=0,\quad (b)\ \dot{\partial}_k \Pi^{\,i}_{jk} \overset{\text{def.}}{=} \Pi^{\,i}_{hjk}.$

The projective covariant of $\dot{x}^i$ vanishes, *i.e.*, $\dot{x}^i_{((k))}=0$.

Let us consider an infinitesimal point transformation

(1.7) $$\bar{x}^i = x^i + v^i(x)\,dt.$$

With the help of the above point change we can obtain the Lie-derivatives of a tensor field $T^i_j\ (x, \dot{x})$ and a projective connection coefficient $\Pi^{\,i}_{jk}\ (x, \dot{x})$ as follows [4].

(1.8) $$\pounds_v \Pi^{\,i}_{jk}(x, \dot{x}) = v^i_{((j))((k))} + Q^i_{jkh}\ v^h + \Pi^{\,i}_{jkh}\ v^h_{((r))}\,\dot{x}^r$$

and

(1.9) $$\pounds_v T^i_j\ (x, \dot{x}) = T^i_{j((h))} v^h - (\dot{\partial}_m T^i_j)\, v^m_{((r))}\ \dot{x}^r - T^h_j\, v^i_{((h))} + T^i_h\ v^h_{((j))},$$

where

(1.10) $$Q^i_{hjk}\ (x, \dot{x}) \overset{\text{def.}}{=} 2\ \{\partial_{[k} \Pi^{\,i}_{j]h} - \Pi^{\,i}_{rh[j} \Pi^{\,r}_{k]j} + \Pi^{\,r}_{h[j} \Pi^{\,i}_{k]r}\}$$

are called the projective entity [2]. The commutation formulas involving the operators of Lie-projective covariant derivative and partial derivatives, are given by

(1.11) $$\dot{\partial}_j\left(\pounds_v T^i_k\right) - L_v\left(\dot{\partial}_j\ T^i_k\right) = 0,$$

(1.12) $$L_v\left(T^i_{j((r))}\right) - \left(\pounds_v T^i_j\right)_{((r))} = T^s_j\ \pounds_v \Pi^{\,i}_{rs} - T^i_s\ \pounds_v \Pi^{\,s}_{rj} - \left(\pounds_v \Pi^{\,s}_{rm}\right)\left(\dot{\partial}_s\ T^i_j\right)\dot{x}^m,$$

(1.13) $$\left(\pounds_v \Pi^{\,i}_{jh}\right)_{((k))} - \left(\pounds_v\ \Pi^{\,i}_{kh}\right)_{((j))} = \pounds_v\ Q^i_{hjk} + 2\,\dot{x}^s\,\Pi^{\,i}_{rh[s}\ \pounds_v\ \Pi^{\,r}_{k]s}.$$

2. Projective Motion

Definition 1. (Rund [1]). An F_n is said to admit a motion if there exists a vector $v^i(x)$ such that

$$M_{ij}(x, \dot{x}) \underline{\underline{def}}. \pounds_v g_{ij} = 0 \tag{2.1}$$

Definition 2. (Pande and Kumar [3]). An F_n is said to admit projective affine motion if there exists a vector $v^i(x)$ such that

$$\pounds_v \Pi^i_{jk} = 0. \tag{2.2}$$

Definition 3. (Hiramatu [5]). An F_n is said to admit a homothetic motion if there exists a vector $v^i(x)$ such that

$$\pounds_v g_{ij} = M_{ij} = 2\sigma g_{ij}. \tag{2.3}$$

Applying the commutation formula (1.12) to the metric tensor $g_{ij}(x, \dot{x})$ and noting the fact that projective covariant derivative of g_{ij} does not vanish, we have

$$\left(\pounds_v g_{ij}\right)_{((k))} - \pounds_v \left(g_{ij((k))}\right) = g_{rj} \pounds_v \Pi^r_{ik} + g_{ir} \pounds_v \Pi^r_{jk} + 2\, C_{ijr} \pounds_v \Pi^r_{sk} \dot{x}^s \tag{2.4}$$

With the help of Equations (2.1) and (2.4) we can deduce that

$$\pounds_v \Pi^i_{js} = \frac{1}{2} g^{ik} \Big[\left\{ M_{jk((s))} - M_{js((k))} + M_{ks((j))} \right\} - \pounds_v \left(g_{jk((s))}\right) + \left(\pounds_v g_{js((k))}\right) - \left(\pounds_v g_{ks((j))}\right) \Big] - \left\{ C^i_{ja} \pounds_v \Pi^a_{bs} + C^i_{sa} \pounds_v \Pi^a_{bj} - g^{ik} C_{jsa} \pounds_v \Pi^a_{bk} \right\} \dot{x}^b \tag{2.5}$$

Transvecting (2.5) by $\dot{x}^j \dot{x}^l$ and noting Equation (1.3), we get

$$\pounds_v \Pi^i_{jl} \dot{x}^j \dot{x}^l = \tfrac{1}{2} g^{ik} \left[M_{jk((l))} - M_{jl((k))} + M_{kl((j))} + S_{jkl} \right] \dot{x}^j \dot{x}^l, \tag{2.6}$$

where

$$S_{jkl}(x, \dot{x}) \underline{\underline{def.}} \left\{ \left(\pounds_v g_{jl((k))}\right) - \left(\pounds_v g_{kl((j))}\right) - \left(\pounds_v g_{jk((l))}\right) \right\}. \tag{2.7}$$

If F_n admits motion (*i.e.*, $M_{jk} = 0$), then Equation (2.6) reduces to

$$2\, \pounds_v \Pi^i_{jl} \dot{x}^j \dot{x}^l = S^i_{jl} \dot{x}^j \dot{x}^l. \tag{2.8}$$

Thus, we have

Theorem 1. In a Finsler space F_n, every motion is a projective affine motion provided that $S^i_{jl} \dot{x}^j \dot{x}^l$ vanishes.

3. Projective Curvature Collineation

Definition 4. The infinitesimal point transformation (1.7) is said to define a projective curvature collineation if there exists a vector $v^i(x)$ such that

$$\pounds_v Q^i_{hjk}=0. \tag{3.1}$$

With the help of Equations (2.5), (2.7) and (3.1), Equation (1.13) takes the form

$$\begin{aligned}(3.2)\quad & g^{is}_{[((k))}\Big\{\Big(M_{j]s((h))}-M_{j]h((s))}-M_{<sh>((j))]}\Big)-S_{j]sh}\Big\}\\ & +g^{is}\Big\{\Big(M_{[j<s><((h))>((k))}-M_{[j<h><((s))>((k))}\Big)\\ & -S_{[j<sh>((k))]}\Big\}-2\Big\{C^i_{r[j((k))]}\pounds_v\Pi^r_h-\Big(\pounds_v\Pi^r_h\Big)_{[((k))}C^i_{j]r}\\ & +C^i_{hr[((k))}\pounds_v\Pi^r_{j]}+C^i_{hr}\Big(\pounds_v\Pi^r_{[j}\Big)_{((k))]}-g^{is}_{[((k))}C_{j]hr}\pounds_v\Pi^r_s\\ & -g^{is}C_{hr[j((k))]}\pounds_v\Pi^r_s-g^{is}\Big(\pounds_v\Pi^r_s\Big)_{[((k))}C_{j]hr}\Big\}\\ & -\dot\partial_r\Pi^i_{h[j}\Big[g^{rs}\Big\{\Big(M_{k]s((h))}-M_{k]h((s))}+M_{<sh>((k))}\Big)\dot x^h\\ & -S_{k]sh}\dot x^h-C^r_{k]p}\pounds_v\Pi^p_h\dot x^h\Big]=0.\end{aligned}$$

Multiplying (3.2) by $2g_{ja}$ and subtracting it from the equation obtained by interchanging the indices a and j in it, we get

$$\begin{aligned}(3.3)\quad & g_{i[a}\Big[g^{is}_{<((k))>}\Big\{\Big(M_{j]s((h))}-M_{j]h((s))}-M_{<sh>((j))}\Big)\\ & -S_{j]sh}\Big\}-g^{is}_{((j))]}\Big\{\Big(M_{ks((h))}-M_{kh((s))}-M_{sh((k))}\Big)-S_{skh}\Big\}\Big]\\ & +\Big\{M_{kh[((a))((j))}-M_{[j<h>((a))((k))}-M_{k[a<((h))>((j))]}\\ & -\Big(S_{[ja]h((k))}-S_{k[a<h>((j))]}\Big)\Big\}-\Big\{g_{i[a}C^i_{j]r((k))}\\ & -g_{i]a}C^i_{<kr>((j))]}\Big\}\pounds_v\Pi^r_h-\dot\partial_r\Pi^i_{h[j}\Big[g_{a]i}g^{rs}\Big\{\Big(M_{ks((h))}\\ & -M_{kh((s))}+M_{sh((k))}\Big)\dot x^h-S_{ksh}\dot x^h\Big\}-C^r_{ks}\pounds_v\Pi^s_h\dot x^h\Big]\\ & +\dot\partial_r\Pi^i_{hk}\Big[g_{i[a}g^{rs}\Big\{\Big(M_{j]s((h))}-M_{j]h((s))}-M_{<sh>((j))]}\Big)\dot x^h\\ & -S_{j]sh}\dot x^h\Big\}-C^r_{j]s}\pounds_v\Pi^s_h\dot x^h\Big]=0.\end{aligned}$$

Thus, we have

Theorem 2. A Finsler space F_n admits a projective curvature collineation provided there exists a vector $v^i(x)$ such that Equation (3.3) holds.

REFERENCES

1. H. Rund, The differential geometry of Finsler spaces (Springer-Verlag, Berlin, 1959).
2. R.B. Misra, The projective transformation in a Finsler space, Ann. Soc. Sci. Bruxelles Ser. I **80** (1966), 227–239.
3. H.D. Pande and A. Kumar, Lie-derivative, projective affine motion and projective curvature collineation in Finsler space, Yokohama Math. J. (Communicated).
4. K. Yano, The theory of Lie derivatives and its applications (North-Holland, Amsterdam, 1957).
5. H. Hiramatu, Groups of homothetic transformations in a Finsler space, Tensor (N.S.) **3** (1954), 131–143.
6. B.N. Prasad, Curvature collineations in Finsler space, Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8) **49** (1970), 194–197.

Jñānabha Sect. A, Vol. 6, 1976

CURVATURE PROPERTIES IN THE AREAL SPACES OF THE SUBMETRIC CLASS

By **Pushya Mitra Chauhan**

Department of Mathematics, R. B. S. College,

Agra–282002, *U. P., India*

(Received : February 16, 1976 ; in revised form : February 24, 1977)

1. Introduction. In the areal spaces of general type, the properties of the curvature tensors have been studied by K. Tandai [1], [2]. Subsequently, M. Gama [3] and S. Kikuchi [4] also studied these properties, but the spaces considered by them were the areal spaces of the submetric class.

In the present paper, we discuss some properties of curvature tensors in the areal space of the submetric class by using different methods. Throughout this paper, a bar (|) followed by an index denotes the covariant derivative and a comma (,) or semi-colon (;) denotes the partial derivative. Also, Latin indices run from 1 to n and Greek indices run from 1 to m.

2. Properties of curvature tensors. Let us consider the metric tensor g_{ij}, which satisfies the following relations [4]

$$g_{ij}\, p^i_\alpha\, p^j_\beta = g_{\alpha\beta},\quad g_{ij}\,\gamma^i_h\, p^j_\alpha = 0,\quad \gamma^i_h = \delta^i_h - \beta^i_h,\quad \beta^i_h = p^i_\alpha\, p^\alpha_h$$

and let its covariant derivative be defined by

$$g_{ij\,|\,k} = g_{ij,\,k} - g_{ij\,;\,l}^{\;\alpha}\,\overset{*}{\Gamma}{}^l_{\alpha k} - g_{lj}\,\overset{*}{\Gamma}{}^l_{ik} - g_{il}\,\overset{*}{\Gamma}{}^l_{jk} = 0. \tag{1}$$

where $\qquad \overset{*}{\Gamma}{}^l_{\alpha k} = p^s_\alpha\,\overset{*}{\Gamma}{}^l_{sk}$ and $\overset{*}{\Gamma}{}^l_{sk} = \overset{*}{T}{}^l_{ks}$.

Also, let us define the covariant derivative of a vector X^i by

$$X^i_{|k} = X^i_{,\,k} - X^{i;\,\alpha}_{\;\;l}\,\overset{*}{\Gamma}{}^l_{\alpha k} + X^j\,\overset{*}{\Gamma}{}^i_{jk}\,;$$

therefore from (1), we find that

$$g_{lj}\,R^{l}_{ikm}+g_{il}R^{l}_{jkm}+g_{ij\,;\,l}\,\overset{\alpha}{}\,R^{l}_{tkm}\,p^{t}_{\alpha}=0,$$

which reduces to

(2) $$R_{ijkm}+R_{jikm}=-g_{ij\,;\,l}^{\;\alpha}\,R^{l}_{tkm}\,p^{t}_{\alpha},$$

where $\qquad R_{ijkm}=g_{lj}R^{l}_{ikm}$ and

(3) $$R^{l}_{ikm}=\overset{*}{\Gamma}{}^{l}_{ik,\,m}-\overset{*}{\Gamma}{}^{l}_{im,\,k}-\overset{*}{\Gamma}{}^{l}_{ik\,;\,n}{}^{\alpha}\,\overset{*}{\Gamma}{}^{n}_{\alpha m}$$
$$+\overset{*}{\Gamma}{}^{l}_{im\,;\,n}{}^{\alpha}\,\overset{*}{\Gamma}{}^{n}_{\alpha k}+\overset{*}{\Gamma}{}^{l}_{nm}\,\overset{*}{\Gamma}{}^{n}_{ik}-\overset{*}{\Gamma}{}^{l}_{nk}\,\overset{*}{\Gamma}{}^{n}_{im}.$$

This tensor R^{l}_{ikm} is the curvature tensor of our space. From (3) we have directly the following result :

Theorem 1. The curvature tensor R^{l}_{ikm} satisfies the following relations :

(4) $$R^{l}_{ikm}+R^{l}_{imk}=0\,;\,R_{ilkm}+R_{ilmk}=0,$$

(5) $$R^{l}_{ikm}+R^{l}_{kmi}+R^{l}_{mik}=0\,;\,R_{ilkm}+R_{klmi}+R_{mlik}=0,$$

where $R_{ilkm}=g_{jl}\,R^{j}_{ikm}$.

Now the covariant differentiation of (3) with respect to x^{h} gives

$$R^{l}_{ikm\,|h}=R^{l}_{ikm,\,h}-\left(R^{l}_{ikm}\right)_{;\,t}^{\;\alpha}\,\overset{*}{\Gamma}{}^{t}_{\alpha h}+R^{t}_{ikm}\,\overset{*}{\Gamma}{}^{l}_{th}-R^{l}_{tkm}\,\overset{*}{\Gamma}{}^{t}_{ih}$$
$$-R^{l}_{itm}\,\overset{*}{\Gamma}{}^{t}_{kh}-R^{l}_{ikt}\,\overset{*}{\Gamma}{}^{t}_{mh}.$$

Two similar equations are obtained by cyclic interchanges of the indices k, m and h. These relations are added and rearranged ; by virtue of (4) this gives

$$R^{l}_{ikm\,|h}+R^{l}_{imh|k}+R^{l}_{ihk|m}=[R^{l}_{imh,\,k}+R^{l}_{imh,\,k}+R^{l}_{ihk,\,m}]$$
$$-[(R^{l}_{ikm})_{;\,t}^{\;\alpha}\,\overset{*}{\Gamma}{}^{t}_{h}+(R^{l}_{imh})_{;\,t}^{\;\alpha}\,\overset{*}{\Gamma}{}^{t}_{\alpha k}+(R^{l}_{ihk})_{;\,t}^{\;\alpha}\,\overset{*}{\Gamma}{}^{t}_{\alpha m}]$$
$$+\left[R^{t}_{ikm}\,\overset{*}{\Gamma}{}^{l}_{th}+R^{t}_{imh}\,\overset{*}{\Gamma}{}^{l}_{tk}+R^{t}_{ihk}\,\overset{*}{\Gamma}{}^{l}_{tm}\right]-\left[R^{l}_{tkm}\,\overset{*}{\Gamma}{}^{i}_{ih}\right.$$
$$\left.+R^{l}_{tmh}\,\overset{*}{\Gamma}{}^{it}_{k}+R^{l}_{thk}\,\overset{*}{\Gamma}{}^{t}_{im}\right].$$

This identity, on substituting the value of R^{l}_{ikm} from (3) and then after some calculations, reduces to

$$(6)\quad R^{l}_{ikm|h}+R^{l}_{imh|k}+R^{l}_{ihk|m}+\overset{*}{\Gamma}{}^{l}_{ik}\,{}^{\alpha}_{;\,t}\,R^{t}_{\alpha mh}+\overset{*}{\Gamma}{}^{l}_{im}\,{}^{\alpha}_{;\,t}\,R^{t}_{\alpha hk}+\overset{*}{\Gamma}{}^{l}_{ih}\,{}^{\alpha}_{;\,t}\,R^{t}_{\alpha km}=0$$

and this identity is known as the first form of the Bianchi identity. Hence we have

Theorem 2. In the areal spaces of the submetric class, the curvature tensor R^{l}_{ikm} satisfies the Bianchi identity of the first kind.

Again, we have K^{i}_{hkl} and $P^{i}_{hk;}{}^{\lambda}_{r}$, two other types of curvature tensors [3] of our space, defined as

$$(7)\qquad K^{i}_{hkl}\equiv R^{i}_{hkl}+C^{i}_{h,}{}^{\lambda}_{r}\,R^{r}_{skl}\,p^{s}_{\lambda}$$

and

$$(8)\qquad P^{i}_{hk,}{}^{\lambda}_{r}\equiv\overset{*}{\Gamma}{}^{i}_{hk}\,{}^{\lambda}_{;\,r}-C^{i}_{h,}{}^{\lambda}_{r|k}+C^{i}_{h,}{}^{\delta}_{n}\,p^{s}_{\delta}\,\overset{*}{\Gamma}{}^{n}_{sk}\,{}^{r}_{;\,\lambda}$$

where R^{i}_{hkl} is defined by (3), $C_{ih,}{}^{\lambda}_{r}+C_{hi,}{}^{\lambda}_{r}=\overset{*}{C}_{ih,}{}^{\lambda}_{r}=g_{ih}{}^{\lambda}_{;\,r}$ and $C_{ih,}{}^{\lambda}_{r}=g_{jh}C^{j}_{i,}{}^{\lambda}_{r}$. From (4), (5) and (7) directly we, have

Theorem 3. The curvature tensor K^{i}_{hkl} satisfies the following relations :

$$K^{i}_{hkl}+K^{i}_{hlk}=0$$

and

$$K^{i}_{hkl}+K^{i}_{klh}+K^{i}_{lhk}=\left(C^{i}_{h,}{}^{\lambda}_{r}\,R^{r}_{skl}+C^{i}_{k,}{}^{\lambda}_{r}\,R^{r}_{slh}+C^{i}_{l,}{}^{\lambda}_{r}\,R^{r}_{shk}\right)p^{s}_{\lambda}.$$

Now we have, from (2),

$$(9)\qquad R_{hikl}=-R_{ihkl}-\overset{*}{C}_{ih,}{}^{\lambda}_{r}\,R^{r}_{skl}\,p^{s}_{\lambda}.$$

and, from (7),

(10) $$K_{hikl}+K_{ihkl}=(R_{hikl}+R_{ihkl}) + \left(C_{hi,\ r}^{\ \ \lambda}+C_{ih,\ r}^{\ \ \lambda}\right)R^{r}_{skl}p^{s}_{\alpha}.$$

So, we conclude from (9) and (10) :

Theorem 4. The curvature tensor K_{hikl} is skew-symmetric in the indices i and h where $K_{hikl}=g_{si}K^{s}_{hkl}$.

Again, we can write (7) in the form

$$R^{i}_{hkl}=K^{i}_{hkl}-C^{i\ \ \lambda}_{h,\ r}R^{r}_{skl}p^{s}_{\lambda}$$

On differentiating this relation with respect to x^{j}, we get

$$R^{i}_{hkl|j}=K^{i}_{hkl|j}-C^{i\ \ \lambda}_{h,\ r|j}R^{r}_{skl}p^{s}_{\lambda}-C^{i\ \ \lambda}_{h,\ r}R^{r}_{skl|j}p^{s}_{\lambda},\ \left(\because\ p^{s}_{\lambda|j}=0\right).$$

Two similar relations result from cyclic interchanges of the indices k, l and j. On adding the three relations thus obtained, we find

$$R^{i}_{hkl|j}+R^{i}_{hlj|k}+R^{i}_{hjk|l}=\left(K^{i}_{hkl|j}+K^{i}_{hlj|k}+K^{i}_{hjk|l}\right) -\left(C^{i\ \ \lambda}_{h,\ r|j}R^{r}_{skl}+C^{i\ \ \lambda}_{h,\ r|k}R^{r}_{slj}+C^{i\ \ \lambda}_{h,\ r|l}R^{r}_{sjk}\right)p^{s}_{\lambda} -C^{i\ \ \lambda}_{h,\ r}\left(R^{r}_{skl|j}+R^{r}_{slj|k}+R^{r}_{sjk|l}\right)p^{s}_{\lambda}.$$

The indentity (6) is now applied to both sides of this relation, and we get

$$-\left(\overset{*}{\Gamma}{}^{i}_{hk\ ;\ m}{}^{\alpha}R^{m}_{\alpha lj}+\overset{*}{\Gamma}{}^{i}_{hl\ ;\ m}{}^{\alpha}R^{m}_{\alpha jk}+\overset{*}{\Gamma}{}^{i}_{hj\ ;\ m}{}^{\alpha}R^{m}_{\alpha kl}\right)=\left(K^{i}_{hkl|j}+K^{i}_{hlj|k} +K^{i}_{hjk|l}\right)-\left(C^{i\ \ \lambda}_{h,\ r|j}R^{r}_{skl}+C^{i\ \ \lambda}_{h,\ r|k}R^{r}_{slj}+C^{i\ \ \lambda}_{h,\ r|k}R^{r}_{sjk}\right) +C^{i\ \ \lambda}_{h,\ r}\left(\overset{*}{\Gamma}{}^{r}_{sk\ ;\ m}{}^{\alpha}R^{m}_{\alpha lj}+\overset{*}{\Gamma}{}^{r}_{sk\ ;\ m}{}^{\alpha}R^{m}_{\alpha jk}+\overset{*}{\Gamma}{}^{r}_{sj\ ;\ m}{}^{\alpha}R^{m}_{\alpha kl}\right)p^{s}_{\lambda}.$$

Collecting terms, we see that this may be written as

$$K^{i}_{hkl|j}+K^{i}_{hlj|k}+K^{i}_{hjk|l}+R^{m}_{\alpha lj}(\overset{*}{\Gamma}{}^{i}_{hk\ ;\ m}{}^{\alpha}-C^{i\ \ \alpha}_{h,\ m|k} +C^{i\ \ \lambda}_{h,\ r}p^{s}_{\lambda}\overset{*}{\Gamma}{}^{r}_{sk\ ;\ m}{}^{\alpha})+R^{m}_{\alpha jk}(\overset{*}{\Gamma}{}^{i}_{hl\ ;\ m}{}^{\alpha}-C^{i\ \ \alpha}_{h,\ m|l}+C^{i\ \ \lambda}_{h,\ r}p^{s}_{r}\overset{*}{\Gamma}{}^{r}_{sl\ ;\ m}{}^{\alpha}) +R^{m}_{\alpha kl}(\overset{*}{\Gamma}{}^{i}_{hj\ ;\ m}{}^{\alpha}-C^{i\ \ \alpha}_{h,\ m|k}+C^{i\ \ \lambda}_{h,\ r}p^{s}_{\lambda}\overset{*}{\Gamma}{}^{r}_{sj\ ;\ m}{}^{\alpha})=0,$$

and by virtue of (8), we have

$$K^{i}_{hkl\|j}+K^{i}_{hlj\|k}+K^{i}_{hjk\|l}+R^{m}_{\alpha lj}\,P^{i}_{hk,}{}^{\alpha}_{m}$$

$$+R^{m}_{\alpha jk}\,P^{i}_{hl,}{}^{\alpha}_{m}+R^{m}_{\alpha kl}\,P^{i}_{hj,}{}^{\alpha}_{m}=0. \tag{11}$$

This indentity is known as the Bianchi indentity of the second kind. Therefore, from (3), (7), (8) and (11), we conclude

Theorem. 5. The curvature tensors R^{i}_{jkl}, K^{i}_{jkl} and $P^{i}_{hk,}{}^{\alpha}_{m}$ of the areal spaces of the submetric class satisfy the Bianchi indetity of the second kind.

ACKNOWLEDGEMENTS. I wish to record my deepest sense of gratitude to Dr. R. P. Goyal and Dr. H. D. Singh for their valuable suggestions. I am also grateful to Dr. Roshan Singh, Principal, R.B.S. College, Agra, for encouragement and facilities.

REFERENCES

1. K. Tandai, On areal spaces. VI. On the characterization of metric areal spaces, *Tensor* (*N. S*) **3** (1953), 40–45.
2. K. Tandai, On general connections in an areal space. III. General connections on the tangent *m*-frame bundle, *Tensor* (*N.S.*) **14** (1963), 26-46.
3. M. Gama, On areal spaces of the submetric class. III, *Tensor* (*N.S.*) **17** (1966), 79-85.
4. S. Kikuchi, Some properties of the curvature tensor in an areal space of the submetric class, *Tensor* (*N.S.*) **19** (1968), 179-182.

Jñānabha Sect. A, Vol. 6, 1976

FLOW OF A COMPRESSIBLE FLUID THROUGH A RECTANGULAR PIPE

By **S. S. Seth**

Department of Mathematics, Agra College,
Agra–282001, *U. P., India*

(*Received* : *October* 14, 1975)

Abstract. In this paper the problem of the steady flow of a perfect gas through a rectangular pipe has been studied, conductivity and viscosity being taken into account. It has been found that the adiabatic flow is possible but Stokes's condition has to be modified.

1. Introduction. The problem of the steady flow of a perfect gas through a circular pipe was discussed by Ray [3]. Kapur [1] studied the same problem again and has shown that the adiabatic flow is not possible under the assumptions $u=0$, $v=0$, $\lambda=$const., $\mu=$const., $\sigma=1$ made by Ray and for the assumptions and the equations to be consistent for a compressible fluid, either μ has to be assumed variable or Stokes's condition has to be modified. Kumar and Warsi [2] have solved the same problem by the method of Hankel transform without assuming the form of w as has been done by Ray [3] and Kapur [1]. The analysis of the steady flow of a perfect gas through elliptic and equilateral triangular pipes has been reported by Sharma ([4] and [5]). In the present note, the problem of the flow of a perfect gas through a rectangular pipe bounded by the walls $x=0$, $x=a$, $y=0$, $y=b$ has been studied.

2. Equations of Motion. Let the mouth of the pipe be the section $z=0$ and the flow direction be parallel to the z-axis. Then the components u and v of velocity parallel to x and y axes vanish. Hence, in the absence of extraneous forces, the three basic equations of motion for steady viscous compressible flow and the equation of continuity reduce to (cf. [4])

$$0=-\frac{\partial p}{\partial x}+(\lambda+\mu)\frac{\partial^2 w}{\partial x\partial z} \tag{1}$$

$$0=-\frac{\partial p}{\partial y}+(\lambda+\mu)\frac{\partial^2 w}{\partial y\partial z} \tag{2}$$

$$\rho w \frac{\partial w}{\partial z}=-\frac{\partial p}{\partial z}+\mu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right)+(\lambda+2\mu)\frac{\partial^2 w}{\partial z^2} \tag{3}$$

and
$$\frac{\partial(\rho w)}{\partial z}=0 \tag{4}$$

where p is the pressure, μ the ordinary coefficient of viscosity, λ the second coefficient of viscosity, w the velocity parallel to z-axis and ρ the density.

Equations (1) and (2) give

$$p=p_1+(\lambda+\mu)\frac{\partial w}{\partial z} \tag{5}$$

where p_1, a constant, is the hydrostatic pressure.

Assuming ρw to be independent of x and y, equation (4) gives

$$\rho w=\rho_0 w_0 \tag{6}$$

where ρ_0 and w_0 are the density and velocity at the mouth of the pipe.

The assumption that ρw is independent of x and y implies that there is a layer of fluid of infinite density in contact with the surface of the pipe. At the surface, where $w=0$, gas sticks to the boundary and accumulates and therefore ρ becomes large.

Substituting (5) in (3) and using (6), we obtain

$$\mu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right)=\rho_0 w_0 \frac{\partial w}{\partial z} \tag{7}$$

The boundary conditions are :

$$\left.\begin{array}{lll}\text{(i)} & w=0, & \text{when } x=0,\ x=a,\ y=0,\ y=b,\ z\geqslant 0\\ \text{(ii)} & w=w_0, & \text{when } 0<x<a,\ 0<y<b,\qquad z=0\\ \text{(iii)} & w=0, & \text{when } z\to\infty.\end{array}\right\} \tag{8}$$

3. Solution of the Problem. To solve the equation (7) under the boundary conditions (8), we assume

$$w=w_1 e^{-\alpha z} \tag{9}$$

where w_1 is a function of x and y and α is some constant to be determined.

Substituting (9) in (7), we obtain

$$\frac{\partial^2 w_1}{\partial x^2}+\frac{\partial^2 w_1}{\partial y^2}+\left(\alpha^2+\frac{\alpha}{\mu}\rho_0 w_0\right)w_1=0 \tag{10}$$

The solution of this differential equation satisfying the conditions $w_1=0$ when $x=0$, $x=a$, $y=0$, $y=b$ is

$$w_1=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{m,n}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \tag{11}$$

provided

$$\alpha^2+\frac{\alpha}{\mu}\ \rho_0 w_0-\pi^2\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)=0 \tag{12}$$

Hence from (9), we obtain

$$w=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} e^{-\alpha z} A_{m,n} \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \tag{13}$$

where α is given by (12).

The boundary conditions (i) of (8) are satisfied by (13). For the condition (iii) of (8), α given by (12) must be positive, so that

$$\alpha=\frac{-\rho_0 w_0+\sqrt{[\rho_0^2 w_0^2+4\pi^2\mu^2(m^2/a^2+n^2/b^2)]}}{2\mu} \tag{14}$$

In order that the condition (ii) of (8) may also be satisfied, we must have

$$w_0=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{m,n}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \tag{15}$$

Now from the theory of Fourier's series, we have

$$w_0=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} w_0\, A'_{m,n}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \tag{16}$$

where
$$A'_{m,n}=\frac{4}{ab}\int_0^a\int_0^b \sin\frac{m\pi x'}{a}\sin\frac{n\pi y'}{b}\,dx'dy'$$
$$=\frac{4}{mn\pi^2}(1-\cos m\pi)(1-\cos n\pi).$$

Substituting this in (16) and comparing the result with (15) we see that $A_{m,n}$ must be zero unless m and n are both odd, and in that case it is equal to $16w_0/mn\pi^2$. Thus finally from (13),

$$w=\sum_m\sum_n\frac{16w_0}{mn\pi^2}e^{-\alpha z}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \tag{17}$$

where m and n are both odd and α is given by (14).

From this and (5), we obtain

$$p=p_1+(\lambda+\mu)\frac{\partial w}{\partial z}$$
$$=p_1-\frac{16w_0}{\pi^2}(\lambda+\mu)\sum_m\sum_n\frac{\alpha}{mn}e^{-\alpha z}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \tag{18}$$

Also, since $\rho w=\rho_0 w_0$, we have

$$\rho=\frac{\pi^2\rho_0}{16}\left[\sum_m\sum_n\frac{e^{-\alpha z}}{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\right]^{-1} \tag{19}$$

Evidently, each term of the series in (17) and also its sum satisfy the equations of motion and the equation of continuity and hence are solutions.

4. **Energy Equation.** We shall now find out if the equation of energy is compatible with the derived form of velocity, pressure and density.

Assuming μ to be constant and $\sigma=1$, the energy equation becomes (cf. [2] and [3])

$$\rho w\frac{\partial i}{\partial z}-w\frac{\partial p}{\partial z}=\mu\left(\frac{\partial^2 i}{\partial x^2}+\frac{\partial^2 i}{\partial y^2}+\frac{\partial^2 i}{\partial z^2}\right)+\frac{6\lambda+8\mu}{3}\left(\frac{\partial w}{\partial z}\right)^2 + \mu\left\{\left(\frac{\partial w}{\partial x}\right)^2+\left(\frac{\partial w}{\partial y}\right)^2\right\} \tag{20}$$

where i, the enthalpy, is the heat constant, σ the Prandtl number.

The energy equation obtained by Ray has been modified by introducing an additional term $\left(\lambda+\frac{2}{3}\mu\right)\left(\frac{\partial w}{\partial z}\right)^2$, since in our case the Stokes's condition $3\lambda+2\mu=0$ is not true.

The equation of state for a perfect gas is

$$i\rho=\frac{\gamma}{\gamma-1}\,p \tag{21}$$

where γ is the ratio of specific heats at constant pressure and at constant volume.

From (21) and (5), we obtain

$$i=\frac{\gamma}{\gamma-1}\cdot\frac{p}{\rho}=\frac{\gamma}{\gamma-1}\cdot\frac{1}{\rho_0 w_0}\left\{wp_1+(\lambda+\mu)w\frac{\partial w}{\partial z}\right\} \tag{22}$$

and hence the energy equation (20) gives

$$\begin{aligned}\frac{\gamma p_1}{\gamma-1}&\left\{\frac{\partial w}{\partial z}-\frac{\mu}{\rho_0 w_0}\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right)\right\}\\ &=-\frac{\gamma(\lambda+\mu)}{\gamma-1}\left\{\left(\frac{\partial w}{\partial z}\right)^2+w\frac{\partial^2 w}{\partial z^2}\right\}+(\lambda+\mu)w\frac{\partial^2 w}{\partial z^2}\\ &\quad+\frac{\gamma\lambda(\lambda+\mu)}{(\gamma-1)\rho_0 w_0}\left[w\frac{\partial}{\partial z}\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right)+\frac{\partial w}{\partial z}\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right)\right.\\ &\qquad\left.+2\mu\left(\frac{\partial w}{\partial x}\cdot\frac{\partial^2 w}{\partial x\partial z}+\frac{\partial w}{\partial y}\cdot\frac{\partial^2 w}{\partial y\partial z}+\frac{\partial w}{\partial z}\cdot\frac{\partial^2 w}{\partial z^2}\right)\right]\\ &\quad+\frac{6\lambda+8\mu}{3}\left(\frac{\partial w}{\partial z}\right)^2+\mu\left\{\left(\frac{\partial w}{\partial x}\right)^2+\left(\frac{\partial w}{\partial y}\right)^2\right\}\end{aligned} \tag{23}$$

Using (7), this reduces to

$$(\lambda+\mu)w_i\frac{\partial^2 w}{\partial z^2}+\frac{\gamma\mu(\lambda+\mu)}{(\gamma-1)\rho_0 w_0}\left[\frac{\partial}{\partial z}\left\{\left(\frac{\partial w}{\partial x}\right)^2+\left(\frac{\partial w}{\partial y}\right)^2+\left(\frac{\partial w}{\partial z}\right)^2\right\}\right]$$
$$+\frac{6\lambda+8\mu}{3}\left(\frac{\partial w}{\partial z}\right)^2+\mu\left\{\left(\frac{\partial w}{\partial x}\right)^2+\left(\frac{\partial w}{\partial y}\right)^2\right\}. \tag{24}$$

Since each term of (17) is a solution of (1), (2), (3) and (4), let us substitute, say, the k th term of (17) in (24). We then have the following equations

$$(\lambda+\mu)\alpha_k^2-\frac{2\gamma\mu(\lambda+\mu)}{(\gamma-1)\rho_0 w_0}\alpha_k^3+\frac{6\lambda+8\mu}{3}\alpha_k^2=0 \tag{25}$$

and

$$-\frac{2\gamma\mu(\lambda+\mu)}{(\gamma-1)\rho_0 w_0}\alpha_k+\mu=0. \tag{26}$$

Eliminating α_k between (25) and (26), we obtain

$$9\lambda+8\mu=0. \tag{27}$$

Then, from (26), we get

$$\alpha_k=\frac{9\rho_0 w_0}{2\mu}\cdot\frac{\gamma-1}{\gamma} \tag{28}$$

Thus it is evident that although each term of the series (17) is a solution of (1), (2), (3) and (4), only one term for which α_k is given by (28) satisfies the energy equation (20). Hence finally we get the expressions for velocity, pressure and density as given by (17), (18) and (19), provided

$$\alpha_k=\frac{9\rho_0 w_0}{2\mu}\cdot\frac{\gamma-1}{\gamma}$$

REFERENCES

1. J. N. Kapur, Proc. of the Third Congress on Theoretical and Appl. Mech. (1957), 243–250.
2. Ram Kumar and Z. U. A. Warsi, Bull. Calcutta Math. Soc. 55 (1963), 131–137.
3. M. Ray, Proc. Nat. Inst. Sci. India Part A, 22 (1956), 408–413
4. H. S. Sharma, Proc. Nat. Inst. Sci. India Part A, 31 (1965) 502–511
5. H. S. Sharma, Agra Univ. J. Res. 21 (1972), 1–8.

Jñānabha Sect. A, Vol 6, 1976

SOME INFINITE INTEGRALS INVOLVING SPHEROIDAL FUNCTION AND FOX'S H-FUNCTION

By **R. K. Gupta**

Department of Mathematics, M. R. Engineering College, Jaipur–302004, *India*

AND

S. D. Sharma*

Department of Mathematics, Meerut College, Meerut–250001, *India*

(Received : February 8, 1976 ; in final form : May 11, 1977)

Abstract. The object of this paper is to evaluate six infinite integrals involving spheroidal function [10], and H-function of C. Fox [4] The results established here are of general character. A number of new and interesting particular cases involving spheroidal wave functions [11], Mathieu functions [8] and H-functions have been derived.

1. Introduction. Let us abbreviate, for convenience, the parameter sequences $(a_1, \alpha_1),\ldots, (a_p, \alpha_p)$ and $(b_1, \beta_1),\ldots,(b_q, \beta_q)$ by $[(a_p, \alpha_p)]$ and $[(b_q, \beta_q)]$, respectively.

We start by recalling the definition of Fox's H-function in the form (cf. [4], p. 408) :

$$H_{p,q}^{m,n}\left(z\left|\begin{matrix}[(a_p, \alpha_p)]\\ [(b_q, \beta_q)]\end{matrix}\right.\right)=\frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{m}\Gamma(b_j-\beta_j s)\prod_{j=1}^{n}\Gamma(1-a_j+\alpha_j s)}{\prod_{j=1+m}^{q}\Gamma(1-b_j+\beta_j s)\prod_{j=1+n}^{p}\Gamma(a_j-\alpha_j s)} \cdot z^s\, ds, \quad (1.1)$$

where L is a suitable contour.

*Present address : Department of Mathematics, M.R. Engineering College, Jaipur-302004, India.

The H-function of two complex variables has been analogously defined by Mittal and Gupta [9, p. 117]. Here we have used its contracted representation as given by Srivastava and Panda [12, p. 130] for the multivariate H-function (in two and more complex variables).

The known results ([10, eq. (14)]; [6, p. 226]; [3, p. 331, eqs. (26), (28); p. 371 eq. (51)]; [7, p. 381]; [2]), required in the sequel, may be recalled as follows :

(i) The spheroidal function $\psi_{\alpha n}(c, \eta)$ of general order $(\alpha > -1)$ can be expanded as

$$\psi_{\alpha n}(c, \eta) = \frac{i^n \sqrt{2\pi}}{\nu_{\alpha n}(c)} \sum_{k=0 \text{ or } 1}^{\infty}{}^{*} a_k(c \mid \alpha n)(c\eta)^{-(\alpha+\frac{1}{2})} J_{k+\alpha+\frac{1}{2}}(c\eta), \tag{1.2}$$

which represents the function uniformly on $(-\infty, \infty)$, where the coefficients $a_k(c \mid \alpha n)$ satisfy the recursion formula [11, eq. (67)] and the asterisk (*) over the summation sign indicates that the sum is taken over only even or odd values of k according as n is even or odd.

(ii) $$z^u J_\nu(z) = 2^u H^{1,0}_{0,2}\left(\frac{z^2}{4}\middle|\overline{[\frac{1}{2}(u+\nu), 1], [\frac{1}{2}(u-\nu), 1]}\right); \tag{1.3}$$

(iii) $$\int_0^\infty x^{\lambda-1} K_\nu(ax)dx = a^{-\lambda} 2^{\lambda-2} \Gamma(\tfrac{1}{2}(\lambda \pm \nu)), \tag{1.4}$$

provided $Re(a)>0$, $Re(\lambda \pm \nu)>0$.

(iv) $$\int_0^\infty x^{\lambda-1} e^{-ax} K_\nu(ax)\, dx = \frac{\sqrt{\pi}\, \Gamma(\lambda \pm \nu)}{(2a)^\lambda\, \Gamma(\lambda+\frac{1}{2})}, \tag{1.5}$$

provided $Re(a)>0$, $Re(\lambda) > | Re(\nu) |$.

(v) $$\int_0^\infty x^{\lambda-1} K_u(ax) K_\nu(ax)\, dx = \frac{2^{\lambda-3}\, \Gamma(\frac{1}{2}(\lambda \pm u \pm \nu))}{a^\lambda\, \Gamma(\lambda)}, \tag{1.6}$$

provided $Re(a)>0$, $Re(\lambda) > | Re(u) | + | Re(\nu) |$.

(vi) $$\int_0^\infty x^{\lambda-1} J_\nu^u(ax)\, dx = a^{-\lambda}\, \Gamma(\lambda)/\Gamma(1+\nu-\lambda u), \tag{1.7}$$

provided $Re(\lambda)>0$, $| \arg a | < (1-u).\frac{1}{2}\pi$, $u<1$.

(vii) $$\int_0^\infty x^{\lambda-1} e^{-ax} W_{k,u}(2ax)\, dx = \frac{\Gamma(\frac{1}{2} \pm u + \lambda)}{(2a)^\lambda \Gamma(1-k+\lambda)}, \tag{1.8}$$

provided $Re(a)>0$, $Re(\lambda+\frac{1}{2} \pm u)>0$.

(viii) $$\int_0^\infty x^{\lambda-1} e^{-ax} E(\alpha, \beta :: ax)dx$$

$$= \frac{\Gamma(\alpha)\, \Gamma(\beta)\, \Gamma(\alpha+\lambda)\, \Gamma(\beta+\lambda)}{a^\lambda\, \Gamma(\alpha+\beta+\lambda)}, \tag{1.9}$$

provided $Re(\alpha+\lambda)>0$, $Re(\beta+\lambda)>0$, $Re(a)>0$.

2. **Main Integrals.** Main results to be proved here are the following :

(i) $\int_0^\infty x^{\lambda-1} K_v(ax)\,\psi_{\alpha n}(c, 2x^{\sigma/2}) H_{p,\,q}^{m,\,n'}\left(zx^{s} \middle| \begin{matrix} [(a_p, \alpha_p)] \\ [(b_q, \beta_q)] \end{matrix} \right) dx$

$$=\tfrac{1}{4} M\, 2^{-N} \sum_{r=0 \text{ or } 1}^{\infty}{}^{*} a_r(c \mid \alpha n)$$

$$.H_{2,\,0\,:\,[p,\,q]\,;\,[0,\,2]}^{0,\,2\,:\,(m,\,n')\,;\,(1,\,0)}\left(\begin{matrix} 1-\frac{1}{2}(\lambda \pm v)+\frac{1}{2}N : s/2, \sigma/2 : \\ \overline{} \quad : \end{matrix} \right.$$

$$\left. \begin{matrix} [(a_p, \alpha_p)] ; \overline{} ; \\ [(b_q, \beta_q)] ; [\pm\frac{1}{2}(r+\alpha+\frac{1}{2}), 1] ; \end{matrix} (2/a)^s z,\, (2/a)^{\sigma} c^2 \right), \qquad (2.1)$$

where, for convenience, $M=[i^n \sqrt{2\pi}/a^{\lambda-N} v_{\alpha n}(c)\,(2c)^{\alpha+\frac{1}{2}}]$,

$N=[\frac{1}{2}\sigma(\alpha+\frac{1}{2})]$; $|\arg(z)| < (s+\sum_1^m \beta_j - \sum_{1+m}^q \beta_j + \sum_1^n \alpha_j - \sum_{1+n}^p \alpha_j)\frac{\pi}{2} > 0$;

c^2 (a real constant) $<\sigma\frac{\pi}{2}$, $0<\sigma<2$; $s+\sum_1^p \alpha_j < \sum_1^q \beta_j$;

$p \geqslant n \geqslant 0$. $q \geqslant m \geqslant 0$, $q \geqslant 2+p$; $Re(a)>0$;
$Re\,[\lambda \pm v + s\, b_j/\beta_j + r\,\sigma/2] > 0$, $j=1, \ldots, m$.

(ii) $\int_0^\infty x^{\lambda-1} e^{-ax} K_v(ax)\,\psi_{\alpha n}(c, 2x^{\sigma/2}) H_{p,\,q}^{m,\,n'}\left(zx^{s} \middle| \begin{matrix} (a_p, \alpha_p) \\ (b_q, \beta_q) \end{matrix} \right) dx$

$$=M\sqrt{\pi}\, 2^{N} \sum_{r=0 \text{ or } 1}^{\infty}{}^{*} a_r(c \mid \alpha n)$$

$$.H_{2,\,1\,:\,[p,\,q]\,;\,[0,\,2]}^{0,\,2\,:\,(m,\,n')\,;\,(1,\,0)}\left(\begin{matrix} [1-\lambda \pm v+N : s, \sigma] : \\ [\frac{1}{2}-\lambda \quad +N : s, \sigma] : \end{matrix} \right.$$

$$\left. \begin{matrix} [(a_p, \alpha_p)] ; \overline{} ; \\ [(b_q, \beta_q)] ; [\pm\frac{1}{2}(r+\alpha+\frac{1}{2}), 1] ; \end{matrix} z/(2a)^{s},\, c^2/(2a)^{\sigma} \right), \qquad (2.2)$$

valid under the same conditions as given in (2.1) ;

(iii) $\int_0^\infty x^{\lambda-1} K_u(ax)\, K_v(ax)\,\psi_{\alpha n}(c, 2x^{\sigma/2})$

$$.H_{p,\,q}^{m,\,n'}\left(zx^{s} \middle| \begin{matrix} [(a_p, \alpha_p)] \\ [(b_q, \beta_q)] \end{matrix} \right) dX$$

$$=\frac{1}{8} M\, 2^{\lambda-N} \sum_{r=0 \text{ or } 1}^{\infty}{}^{*} a_r(c \mid \alpha n)\, H_{4,\,1\,:\,[p,\,q]\,;\,[0,\,2]}^{0,\,4\,:\,(m,\,n')\,;\,(1,\,0)}$$

$$\left(\begin{matrix} [1-\frac{1}{2}(\lambda \pm u \pm v)+\frac{1}{2}N : \frac{1}{2}s, \frac{1}{2}\sigma] : \\ [1-\lambda+N : s, \sigma] \quad : \end{matrix} \right.$$

$$\left. \begin{matrix} [(a_p, \alpha_p)] ; \overline{} ; \\ [(b_q, \beta_q)] ; [\pm\frac{1}{2}(r+\alpha+\frac{1}{2}), 1] ; \end{matrix} (2/a)^{s} z,\, (2/a)^{\sigma} c^2 \right), \qquad (2.3)$$

where $Re\,[\lambda \pm u + \nu + s\,(b_j/\beta_j) + r\sigma/2] > 0$; $j = 1, \ldots, m$; and the remaining conditions are the same as given in (2.1).

(iv) $$\int_0^\infty x^{\lambda-1} J_\nu^u\,(ax)\,\psi_{\alpha n}\,(c, 2x^{\sigma/2})\,H_{p,q}^{m,n'}\left(zx^s \,\middle|\, \begin{matrix}[(a_p, \alpha_p)]\\ [(b_q, \beta_q)]\end{matrix}\right)dx$$

$$= M \sum_{r=0 \text{ or } 1}^{\infty *} a_r\,(c \mid \alpha n)\, H^{1,\,0\,:\,(m,\,n')\,;\,(1,\,0)}_{0,\,2\,:\,[p,\,q]\,;\,[0,\,2]}$$

$$\left(\begin{matrix}[1-\lambda+N : s, \sigma] & : [(a_p, \alpha_p)] ; & \overline{\qquad\qquad} ; \\ [1+\nu-(\lambda+N)u : s\,u, \sigma\,u] & : [(b_q, \beta_q)] ; & [\pm\tfrac{1}{2}(r+\alpha+\tfrac{1}{2}), 1] ;\end{matrix}\; z/a^s,\; c^2/a^\sigma\right). \qquad (2.4)$$

where, $|\arg(z)| < \left[(1-u)\, s + \sum_1^m \beta_j - \sum_{1+m}^q \beta_j + \sum_1^n \alpha_j + \sum_{1+n}^1 \alpha_j\right] . \tfrac{1}{2}\pi > 0$;

$c^2 < (1-u)\,\sigma\,\dfrac{\pi}{2} > 0$; $(1-u)\,s + \sum_1^p \alpha_j < \sum_1^q \beta_j$, $s > 0$;

$0 < (1+u)\,\sigma < 2$; $Re\,(a) > 0$, $Re\,[\lambda + s\,(b_j/\beta_j) + r\,\sigma/2] > 0$,

$j = 1, \ldots, m$.

(v) $$\int_0^\infty x^{\lambda-1} e^{-ax}\, W_{k,u}\,(2ax)\,\psi_{\alpha n}\,(c, 2x^{\sigma/2})\, H_{p,q}^{m,n'}\left(zx^s \,\middle|\, \begin{matrix}[(a_p, \alpha_p)]\\ [(b_q, \beta_q)]\end{matrix}\right)dx$$

$$= \frac{M}{2^{\lambda-N}} \sum_{r=0 \text{ or } 1}^{\infty *} a_r\,(c \mid \alpha n)\, H^{0,\,2\,:\,(m,\,n')\,;\,(1,\,0)}_{2,\,1\,:\,[p,\,q]\,;\,[0,\,2)}$$

$$\left(\begin{matrix}[\tfrac{1}{2} \pm u - \lambda + N : s, \sigma] : [(a_p, \alpha_p)] ; & \overline{\qquad\qquad} ; \\ [K - \lambda + N : s, \sigma] : [(b_q, \beta_q)] ; & [\pm\tfrac{1}{2}(r+\alpha+\tfrac{1}{2}), 1] ;\end{matrix}\; z/(2a)^s,\; c^2/(2a)^\sigma\right), \qquad (2.5)$$

where

$|\arg(z)| < \left(\sum_1^m \beta_j - \sum_{1+m}^q \beta_j + \sum_1^n \alpha_j - \sum_{1+n}^p \alpha_j - s\right)\dfrac{\pi}{2} > 0$, $s > 0$;

$Re\,[\lambda + \tfrac{1}{2} \pm u + s\,(b_j/\beta_j) + r\,\sigma/2] > 0$, $j = 1, \ldots, m$; with the remaining conditions being the same as given in (2.1).

$$\text{(vi)} \int_0^\infty x^{\lambda-1} e^{-ax} E(\alpha', \beta :: ax] \psi_{\alpha n}(c, 2x^{\sigma/2}) H_{p,q}^{m,n'}\left(zx^s \middle| \begin{matrix}[(a_p, \alpha_p)]\\ [(b_q, \beta_q)]\end{matrix}\right) dx = M\, \Gamma(\alpha')(\beta) \sum_{r=0 \text{ or } 1}^{\infty}{}^{*} a_r(c \mid \alpha n)$$

$$H_{2,1:\ [p,q];\ [0,2]}^{0,2:\ (m,n');\ (1,0)}\left(\begin{matrix}[1-\alpha'-\lambda+N : s, \sigma], [1-\beta-\lambda+N : s, \sigma] : [(a_p, \alpha_p)] ; \text{———} ;\\ [1-\lambda-\alpha'-\beta+N : s, \sigma] : [(b_q, \beta_q)] ; [\pm\tfrac{1}{2}(r+\alpha+\tfrac{1}{2}), 1] ;\end{matrix}\ z/a^s, c^2/a^\sigma\right), \quad (2.6)$$

where $Re[sb_j/\beta_j + r\ \sigma/2 - \alpha' - \beta - \lambda] > 0$, $j = 1, \ldots, m$, with remaining conditions being the same as given for (2.1).

Proofs. In order to prove the results (2.1) to (2.6), we first express the spheroidal function in its expansion form (1.2), change the order of integration and summation. Then, after using the result (1.3), we express both the H-functions in contour integral forms (1.1), and again change the order of integrals. Evaluating inner the integrals by virtue of results (1.4) to (1.9), respectively, and interpreting the resulting double contour integrals by the definition of the H-function of two variables [9. p. 117], we arrive at our main results.

The change of the order of integration is justified [1] due to absolute convergence of the integrals involved in each process.

3. Special Cases. The spheroidal wave functions $S_{mn}(c, 2x^{\sigma/2})$ [11] and the periodic Mathieu functions $ce_n(\cos^{-1} x^{\sigma/2}, c^2)$, $se_{n+1}(\cos^{-1} x^{\sigma/2}, c^2)$ are special cases related to $\psi_{\alpha n}(c, 2x^{\sigma/2})$ as follows: [10, eqs. (18), (23), and (25)] :

$$\psi_{\alpha n}(c, 2x^{\sigma/2}) = \begin{cases} (1-4x^\sigma)^{-m/2} . S_{mn}(c, 2x^{\sigma/2}), \ \alpha = m = 0, 1, \ldots, & (3.1)\\ ce_n(\cos^{-1} x^{\sigma/2}, c^2), \ \alpha = -\tfrac{1}{2}, & (3.2)\\ (1-4x^\sigma)^{-\frac{1}{2}} . se_{n+1}(\cos^{-1} x^{\sigma/2}, c^2), \ \alpha = \tfrac{1}{2}. & (3.3)\end{cases}$$

Thus, by virtue of the above properties of $\psi_{\alpha n}(c, 2x^{\sigma/2})$, new results corresponding to the results (2.1) to (2.6) can be easily deduced. However, we mention here only a few of them due to lack of space.

(i) In (2.1), if we take $\alpha=m=0, 1, 2, \ldots$, it reduces to the following result :

$$\int_0^\infty x^{\lambda-1}(1-4x^{\sigma})^{-m/2} K_\nu(ax)\ S_{mn}(c, 2x^{\sigma/2})\ H_{p,q}^{m',n'}\left(zx^s \middle| \begin{matrix}[(a_p, \alpha_p)]\\ [(b_q, \beta_q)]\end{matrix}\right) dx$$

$$=\tfrac{1}{4}M'\, 2^{\lambda-N'} \sum_{r=0 \text{ or } 1}^{\infty}{}^{*}\ a_r(c/mn)\ H^{0,2:(m',n');(1,0)}_{2,0:[p,q]\ ;[0,2]}\left(\begin{matrix}[1-\tfrac{1}{2}(\lambda\pm\nu)+\tfrac{1}{2}N' : s/2, \sigma/2] : [(a_p, \alpha_p)] ; \text{———} ;\\ \text{———} : [(b_q, \beta_q)] ; [\pm\tfrac{1}{2}(r+m+\tfrac{1}{2}), 1] ;\end{matrix}\ (2/a)^s z, (2/a)^{\sigma} c^2\right), \quad (3.4)$$

where, for convenience,

$$M'=[i^n\sqrt{2\pi}/a^{\lambda-N'}\ V_{mn}(c)\ (2c)^{m+\frac{1}{2}}],\ N'=[\tfrac{1}{2}\sigma(m+\tfrac{1}{2})],$$

and the validity conditions are the same as given for (2.1) for $\alpha=m=0, 1, 2, \ldots$

A similar set of results can also be obtained from (2.2) to (2.6).

(ii) In (2.1), if we put $\alpha=-\frac{1}{2}$, it reduces to the following result :

$$\int_0^\infty x^{\lambda-1}\ K_\nu(ax)\ ce_n(\cos^{-1} x^{\sigma/2}, c^2)\ H_{p,q}^{m,n'}\left(zx^s \middle| \begin{matrix}[(a_p, \alpha_p)]\\ [(b_q, \beta_q)]\end{matrix}\right) dx$$

$$=\frac{ce_n(\pi/2, c^2)\, 2^{\lambda}}{4A_0^{(n)}(c^2)\, a^{\lambda}} \sum_{r=0 \text{ or } 1}^{\infty}{}^{*}\ i^r A_r^{(n)}(c^2)\ H^{0,2:(m,n');(1,0)}_{2,0:[p,q]\ ;[0,2]}$$

$$\left(\begin{matrix}[1-\tfrac{1}{2}(\lambda\pm\nu) : s/2, \sigma/2] : [(a_p, \alpha_p)] ; \text{———} ;\\ \text{———} : [(b_q, \beta_q)] ; [\pm r/2, 1] ;\end{matrix}\ (2/a)^s z,\ (2/a)^{\sigma} c^2\right), \quad (3.5)$$

where the validity conditions are the same as given for (2.1) with $\alpha=-\frac{1}{2}$.

A similar set of new integrals can be easily obtained by setting $\alpha=-\frac{1}{2}$ in the results (2.2) to (2.6) and also $\alpha=\frac{1}{2}$ in the results (2.1) to (2.6).

Also, on further specializing the parameters, our results (2.4), (2.5) and (2.6) reduce to those recently obtaired by Gupta and Jain [5].

ACKNOWLEDGEMENTS. The authors wish to express their sincere thanks to Prof. R. P. Agarwal and the worthy referee for their useful suggestions. The authors are also grateful to Prof. H. M. Srivastava of the University of Victoria, Canada, for his valuable suggestions for improvement and in the preparation of the fina lfrom of this peper.

REFERENCES

1. T. J. I. A. Bromwich, An introduction to the theory of infinite series, MacMillan, London, 1956.
2. A. Erdelyi et al., Higher transcendental functions, Vol. I, Mc Graw-Hill, New York, 1953.
3. A. Erdelyi et al., Tables of integral transforms, Vols. I and II, Mc-Graw-Hill, New York, 1954.
4. C. Fox, The G and H-functions as symmetrical Fourier kerneis, Trans. Amer. Math. Soc. **98** (1961), 395–429.
5. K. C. Gupta and U.C. Jain, The H-function. II, Proc. Nat. Acad. Sci. India Sect. A 36 (1966), 594-609.
6. Y. L. Luke, The special functions and their approximations, Academic Press, New York and London, 1969.
7. T. M. Mac-Robert, Functions of a complex variable, Pergamon Press, Oxford and London, 1954.
8. N. W. Mc-Lachlan, Theory and applications of Mathieu functions, Dover Publications, Inc., New York, 1964.
9. P. K. Mittal and K. C. Gupta, An integral involving generalized function of two variables, Proc. Indian Acad. Sci. Sect. A **75** (1972), 117-123.
10. D. K. Rhodes, On the spheroidal functions, J. Res. Nat. Bur. Standards Sect. B. **74** (1970), 187–209.
11. J. A. Stratton et al., Spheroidal wave functions, The M. I. T. Press and John Wiley and Sons, 1956.
12. H. M. Srivastava and R. Panda, Expansion theorems for the H function of several complex variables, J. Reine Angew. Math. **288** (1976). 129–145.

Jñānabha Sect. A, Vol 6, 1976

UNSTEADY AXI-SYMMETRIC PERIODIC BOUNDARY-LAYER FLOW

By **D. Atchuta Rao,**

Department of Mathematics, College of Engineering,
Osmania University, Hyderabad–500007, *India*

(Received : September 20, 1974 ; in revised form : August 29, 1975)

Abstract. The boundary-layer equations of an axi-symmetric body of revolution for the unsteady periodic external flow have been worked out. The velocity components near the wall have been obtained for the first and second approximations with the assumptions of corresponding stream functions. The velocity component u_0 of the first approximation lags behind by a certain angle α_1, whereas the velocity component v_0 leads by an angle α_2 over the wall fluctuations. In the second approximation, the velocity component u_1 leads by angles β_1, β_2 over the wall fluctuations. All the amplitudes of the velocity components in both the approximations are found to increase with the increase of the distance from the wall.

1. Introduction. Blasius [1] used the method of approximation for solving the non-steady boundary-layer equations. Boltze [2] has performed the calculations on the steady boundary-layer equations of an axi-symmetric body of revolution. Later on, Schlichting [3] found a solution for the non steady periodic boundary-layers. In the present steady, the unsteady periodic boundary layer equations are considered taking the external flow of the form $u_0(x)e^{i\omega t}$ considered by Roy [4]. The stream functions for the first and second approximations have been expressed in terms of certain functions $X_0(\eta)$, $X_{11}(\eta)$, $X_{12}(\eta)$, $X_{21}(\eta)$ and $X_{22}(\eta)$, and are expanded for small values of η. The velocity components near the solid boundary for the first and second approximations have been calculated. The variations in these approximations have been represented graphically.

Present address : Department of Mathematics, Sardar Patel College, Secunderabad–500025, India.

2. Basic Equations. The boundary-layer equations of an axially symmetric body of revolution are given by

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\frac{\partial U}{\partial t}+U\frac{\partial U}{\partial x}+\nu\frac{\partial^2 u}{\partial y^2} \tag{1}$$

$$\frac{\partial}{\partial x}(ru)+\frac{\partial}{\partial y}(rv)=0. \tag{2}$$

where $r(x)$ is the radius of the cross-section of the body of revolution and $U(x, t)$ is the velocity outside the boundary-layer. The boundary conditions are

$$\left.\begin{array}{lll} y=0 & : & u=v=0 \\ y=\infty & : & u=U(x, t) \end{array}\right\} \tag{3}$$

To apply the general method of approximations, we write

$$u(x, y, t)=u_0(x, y, t)+u_1(x, y, t) \tag{4}$$

and obtain the first and second approximation equations from (1) and (4) as

$$\frac{\partial u_0}{\partial t}-\nu\frac{\partial^2 u_0}{\partial y^2}=\frac{\partial U}{\partial t} \tag{5}$$

$$\frac{\partial u_1}{\partial t}-\nu\frac{\partial^2 u_1}{\partial y^2}=U\frac{\partial U}{\partial x}-u_0\frac{\partial u_0}{\partial y}-v_0\frac{\partial u_0}{\partial y} \tag{6}$$

with the boundary conditions

$$\left.\begin{array}{lll} y=0 & : & u_0=u_1=0 \\ y=\infty & : & u_0=U(x, t),\ u_1=0. \end{array}\right\} \tag{7}$$

We shall take the potential flow to be periodic in time as

$$U(x, t)=U_0(x)\, e^{i\omega t}. \tag{8}$$

where

$$\eta=y\sqrt{\frac{\omega}{\nu}} \tag{9}$$

Thus, we get

$$u_0=U_0X_0^1\, e^{i\omega t}. \tag{10}$$

$$v_0=-\frac{dU_0}{dx}\sqrt{\frac{\nu}{\omega}}X_0e^{i\omega t}-\sqrt{\frac{\nu}{\omega}}\ \frac{U_0}{r}\ \frac{dr}{dx}\ X_0e^{i\omega t} \tag{11}$$

$$u_1=U_0\frac{dU_0}{dx}\ \frac{1}{\omega}\left\{X_{11}^1e^{2i\omega t}+X_{12}^1\right\}+\frac{u_0^2}{\omega r}\ \frac{dr}{dx}\left\{X_{21}^1e^{2i\omega t}+X_{22}^1\right\} \tag{12}$$

$$v_1=-\sqrt{\frac{\nu}{\omega}}\ \frac{1}{\omega}\left\{U_0\frac{d^2U_0}{dx^2}+\left(\frac{dU_0}{dx}\right)^2+U_0\frac{dU_0}{dx}\right\}\left\{X_{11}e^{2i\omega t}+X_{12}\right\}$$
$$+\left\{U_0^2\frac{d^2r}{dx^2}+2U_0\frac{dU_0}{dx}\ \frac{dr}{dx}\right\}\left\{X_{21}e^{2i\omega t}+X_{22}\right\} \tag{13}$$

3. Discussion

The functions $X_0(\eta)$, $X_{11}(\eta)$, $X_{12}(\eta)$, $X_{21}(\eta)$ and $X_{22}(\eta)$ can be obtained for small values of η as follows.

$$X_0 \simeq \left(\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{4}\eta^2-\frac{\sqrt{2}}{48}\eta^4\right) +i\left(\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{4}\eta^2+\frac{1}{6}\eta^3-\frac{\sqrt{2}}{48}\eta^4\right) \tag{14}$$

$$X_{11} \simeq \left(\frac{1}{4}-\frac{1}{4}\eta^2+\frac{1}{12}\eta^3+\frac{\sqrt{2}-1}{24}\eta^4\right) +i\left(\frac{1-2\sqrt{2}}{4}+\frac{1-\sqrt{2}}{4}\eta^2-\frac{1}{3}\eta^3-\frac{2+\sqrt{2}}{48}\eta^4\right) \tag{15}$$

$$X_{12} \simeq \frac{\sqrt{2}}{8}\eta^2-\frac{1}{12}\eta^3+\frac{\sqrt{2}}{32}\eta^4 \tag{16}$$

$$X_{21} \simeq \left(\frac{7-5\sqrt{2}}{8}\eta^2-\frac{1}{8}\eta^3+\frac{7-4\sqrt{2}}{48}\eta^4\right) +i\left(\frac{-7+5\sqrt{2}}{8}\eta^2-\frac{1}{8}\eta^3+\frac{7-4\sqrt{2}}{48}\eta^4\right) \tag{17}$$

$$X_{22} \simeq \frac{7\sqrt{2}}{4}-\frac{\sqrt{2}}{16}\eta^2+\frac{1}{4}\eta^3-\frac{\sqrt{2}}{32}\eta^4. \tag{18}$$

The small value of η implies from (9) that the normal distauce y from the body is small.

The variations of the functions X_{12}, X_{22} with η have been graphically shown in figure 1.

Thus for small distance y from the wall, neglectings the terms containing powers of $y \geqslant 4$, we get the velocity components for the first approximation as

$$u_0 \simeq U_0 \mid B_1 \mid e^{i(\omega t+\alpha_1)} \tag{19}$$

$$v_0 \simeq \left(\frac{dU_0}{dx}+\frac{U_0}{n}\right) \mid B_2 \mid e^{i(\omega t+\alpha_2)} \tag{20}$$

where $\mid B_1 \mid$, $\mid B_2 \mid$, α_1 and α_2 are functions of ν, ω and y.

VARIATION OF STEADY STATE CONTRIBUTION OF VELOCITY

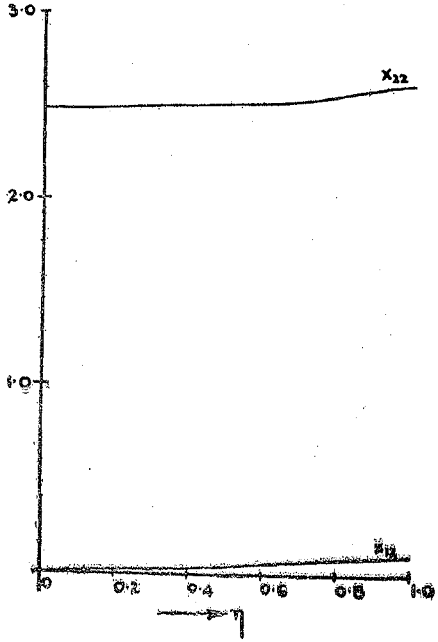

Fig. 1

VARIATION OF AMPLITUDES OF FIRST APPROXIMATION

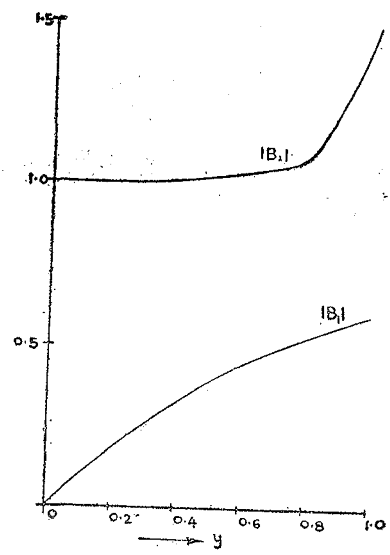

Fig. 2

The variations of the amplitudes $|B_1|$ and $|B_2|$ with the distance y from the wall have been shown in figure 2 for $\omega=0{\cdot}01$. $\nu=0{\cdot}01$. The amplitudes $|B_1|$ and $|B_2|$ are found to increase with y. In figure 3, the variation of the phase angles α_1 and α_2 are represented for $\nu=\omega=0{\cdot}01$.

GRAPH OF PHASE ANGLES OF FIRST APPROXIMATION

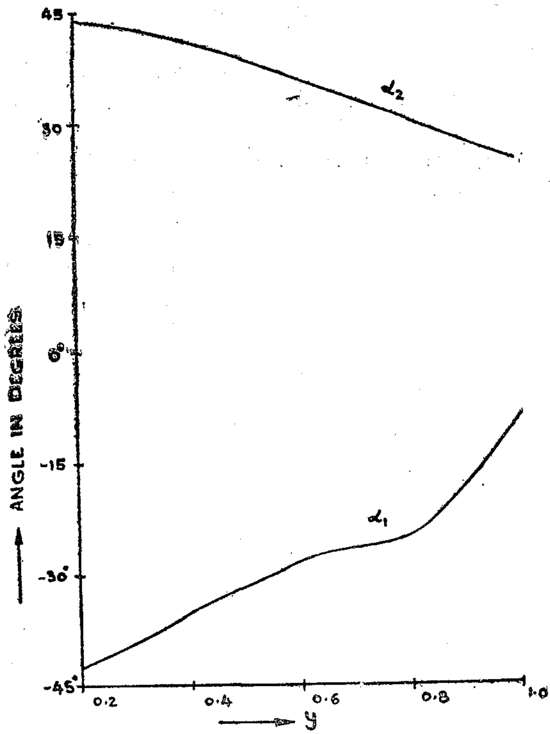

Fig 3

We observe here that the velocity component u_0 lags by an angle α_1, and v_0 leads by an angle α_2 over the wall fluctuations.

Similary, in the case of second approximation for small distances y, the velocity components u_1, v_1 are obtained as

$$u_1 \simeq U_0 \frac{dU_0}{dx} \mid M_1 \mid e^{i\,(2\omega t+\beta_1)} + \frac{U_0^2}{r} \frac{dr}{dx} \mid M_2 \mid e^{i\,(2\omega t+\beta_2)}$$

$$+ U_0 \frac{dU_0}{dx} \frac{X_{12}^1}{\omega} + \frac{U_0^2}{r} \frac{dr}{dx} \frac{X_{22}^1}{\omega} \tag{21}$$

$$v_1 \simeq \left[\frac{d}{dx} \left(U_0 \frac{dU_0}{dx} \right) + U_0 \frac{dU_0}{dx} \right] \left[|N_1| e^{i(2\omega t+\gamma_1)} - \sqrt{\frac{\nu}{\omega}} \frac{X_{12}}{\omega} \right]$$

$$+ \left[\frac{d}{dx} \left(U_0^2 \frac{dr}{dx} \right) \right] \left[|N_2| e^{i\,(2\omega t+\gamma_2)} - \sqrt{\frac{\nu}{\omega}} \frac{X_{22}}{\omega} \right] \tag{22}$$

where $|M_1|$, $|M_2|$, $|N_1|$, $|N_2|$, $|B_1|$, $|B_2|$, γ_1 and γ_2 are all functions of ν, ω and y.

GRAPH OF AMPLITUDES OF SECOND APPROXIMATION

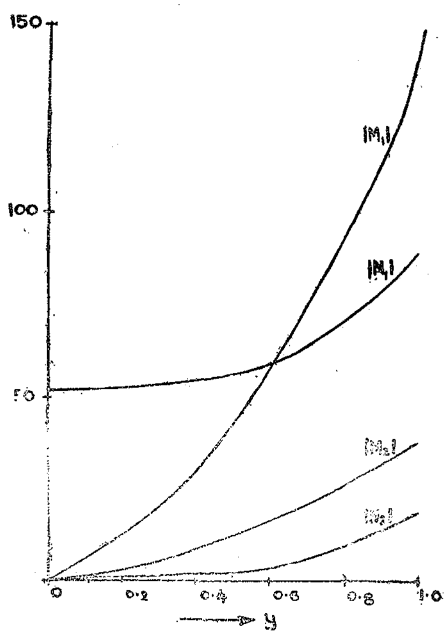

Fig. 4

REFERENCES

1. H. Blasius, Grenzschichten in Flussigkeitten mit kleiner Reibung, ZAMP, 56 (1908), 1–37.
2. E. Boltze, Grenzschichten an Rotation skorepern in Flussing keiten mit Kleiner Reibung. Thesis, Gottington, 1908.
3. H. Schlichting, Berechung ebener periodicscher Grenzschichstrimungen, Phys. Z., **33** (1932) 327–335.
4. D. Roy, Non-steady periodic Boundary Layer. J. Appl. Math. Phys. **12** (1961) 363–366.

Jñānabha Sect. A, Vol. 6, 1976

ON HEAT TRANSFER IN MAGNETOHYDRODYNAMIC CHANNEL FLOW UNDER TRANSVERSE MAGNETIC FIELD

By **T. Prasad and R. S. Pathak**

Department of Mathematics, Banaras Hindu University,

Varanasi–221005, *India*

(Received : December 15, 1975)

Abstract. The influnce of magnetic field on heat transfer in an electrically conducting viscous fluid flow along a rectangular channel with non-conducting walls is considered. The magnetic field is transversely applied to the fluid flow and wall temperature is assumed to vary linearly in the direction of flow. Expressions are derived for temperature and Nusselt number. Temperature profiles are shown graphically for different values of Hartmann number M, magnetic Eckert number E_c, and a non-dimensional number S^*.

1. Introduction. The problem on heat transfer in magnetohydrodynamic channel flow have been discussed by many authors. Erickson, Wang and Fan (1961) studied the effect of magnetic field, electric field and viscous dissipation on heat transfer in the entrance region of a channel with non-conducting walls. Heat flux at the channel walls was not considered in their paper. Soundalgekar (1968) discussed heat transfer in a fully developed channel flow with conducting as well as non-conducting walls in the absence of externally imposed heat flux at the channel walls.

In the present paper heat transfer in magnetohydrodynamic channel flow under transverse magnetic field with non-conducting walls is discussed. The expression for the velocity profile derived by Shercliff (1953) in the case of steady motion of conducting fluid in a rectangular pipe under transverse magnetic field is untilised to find expressions for temperature and Nusselt number. Numerical calculations for temperature and Nusselt number are carried out under different conditions. The temperature profiles are shown graphically and numerical results for Nusselt number are entered in a table for different values of Hartmann number M, magnetic Eckert number E_c and a non-dimensional number S^*.

2. Mathematical Analysis. Take right-handed axes such that the z-axis is parallel to the fluid velocity V_z and the x-axis is parallel to the imposed uniform magnetic field H_0 existing outside the fluid.

Let the origin be on the centre line of the channel. We take $2a$ as the channel width, V_0 as the mean velocity and σ and η as the fluid conductivity and viscosity respectively ; μ, the permeability of the fluid and the walls, is assumed to be unity and the magnetic field is then continuous at the fluid boundary ; ρ, the fluid density, is of no concern as the flow is unaccelerated.

It has been shown by Shercliff (1953) that when M is large compared with unity, the solutions of the magnetohydrodynamic equations governing the present problem take a boundary layer character as in simple Hartmann case and then assuming that

$$\frac{\partial p}{\partial x}=-K_1,$$

we have

$$V_z=\frac{K_1\,a^2}{\eta M}\left\{1-2e^{-M}\cosh\frac{Mx}{a}\right\} \tag{1}$$

and
$$H_z=4\pi\left(\frac{\sigma}{\eta}\right)^{\frac{1}{2}}\frac{K_1\,a^2}{M}\left\{-\frac{x}{a}+2e^{-M}\sinh\frac{Mx}{a}\right\}. \tag{2}$$

Introducing the non-dimensional variables

$$V=\frac{V_0}{b},\ \xi=\frac{x}{a},\ H_z=H\,H_0,$$

we get

$$V=\frac{K_1\,a^2}{\eta bM}\left\{1-2e^{-M}\cosh M\xi\right\} \tag{3}$$

and
$$H=4\pi\left(\frac{\sigma}{\eta}\right)^{\frac{1}{2}}\frac{K_1 a^2}{H_0\,M}\left\{-\xi+2e^{-M}\sinh M\xi\right\}, \tag{4}$$

where $b=(a\mu H_0^2/4\pi\rho)^{\frac{1}{2}}$, Alfven wave velocity ; $M=abc(\lambda\nu)^{\frac{1}{2}}$, Hartmann number ; ν, the viscous diffusivity ; $\lambda=c^2/4\pi\mu\sigma$, the magnetic diffusivity ; c, the velocity of light assumed here unity.

Now, for steady one-dimensional flow of an incompressible, viscous, electrically conducting fluid with constant properties, the energy equation is given by

$$\rho c_p\,V_z\,\frac{\partial T}{\partial z}=\chi\,\frac{d^2T}{dx^2}+\eta\left(\frac{dV_z}{dx}\right)^2+\frac{1}{16\pi^2\sigma}\left(\frac{dH_z}{dx}\right)^2. \tag{5}$$

Also, for linearly varying wall temperature, the temperature distribution in the fluid, following Siegel (1960) is assumed as

$$T = \tau z + \bar{\theta}\,(x). \tag{6}$$

Substituting (3), (4) and (6) in equation (5), we get

$$\frac{d^2\theta^*}{d\xi^2} = -\frac{p_r\,E_c}{M^2}\left\{4M^2\, e^{-2M}\cosh 2M\,\xi + 1 - 4Me^{-M}\cosh M\xi\right\} + \frac{P_r\,R\,S^*}{M}\left\{1 - 2e^{-M}\cosh M\xi\right\} \tag{7}$$

where

$$P_r = \frac{\eta C_p}{\lambda},\ \text{Prandtl number,}$$

$$E_c = \frac{b^2}{C_p\,\theta_1},\ \text{Magnetic Eckert number,}$$

$$R = \frac{ab}{\nu},\ \text{Reynolds number,}$$

$$S^* = \frac{S}{K},\quad \theta^* = \frac{\theta}{k},\ S = \frac{\tau a}{\theta_1},\ \theta = \frac{\bar{\theta}}{\theta_1}$$

$$K = \frac{K_1\,a^2}{\eta b},\ \nu = \frac{\eta}{\rho}.$$

Now, integrating equation (7), we get

$$\theta^* = -\frac{P_r E_c}{M^2}\left\{e^{-2M}\cosh 2M\xi + \tfrac{1}{2}\xi^2 - \frac{4}{M}\,e^{-M}\cosh M\xi\right\} + \frac{P_r\,R S^*}{M}\left\{\tfrac{1}{2}\xi^2 - \frac{2}{M^2}\,e^{-M}\cosh M\xi\right\} + A\xi + B, \tag{8}$$

where A and B are constants of integration.

Applying the boundary condition

$$\theta^* = 0 \quad \text{at} \quad \xi = \pm 1$$

we get the values of constants as $A = 0$,

$$B = \frac{P_r E_c}{M^2}\left\{e^{-2M}\cosh 2M + \frac{1}{2} - \frac{4}{M}\,e^{-M}\cosh M\right\} - \frac{P_r\,R\,S^*}{M}\left\{\frac{1}{2} - \frac{2}{M^2}\,e^{-M}\cosh M\right\}.$$

The Nusselt number N_u is defined as

$$N_u = -\frac{1}{\theta^*_{(0)}}\left(\frac{d\theta^*}{d\xi}\right)_{\xi=1}. \tag{10}$$

From equations (8) and (10), we get

$$N_u = \frac{-\frac{P_r E_c}{M^2}\left\{2M\, e^{-2M} \sinh 2M + 1 - 4e^{-M} \sinh M\right\} + \frac{P_r R\, S^*}{M}\left\{1 - \frac{2}{M}\, e^{-M} \sinh M\right\}}{\frac{P_r E_c}{M^2}\left\{-2M - \frac{4}{M}\, e^{-M}\right\} + \frac{2P_r R\, S^*}{M^3}\, e^{-M} - B} \tag{11}$$

3. Numerical Calculations and Discussion. Numerical calculations for θ^* are carried out for $P_r=1$, $R=0{\cdot}1$, $S^*=0, 1, 2, 3$; $M=2, 4, 5$; $E_c=0{\cdot}3, 0{\cdot}6, 0{\cdot}9$. The results for θ^* are shown graphically in Figs. 1 through 3.

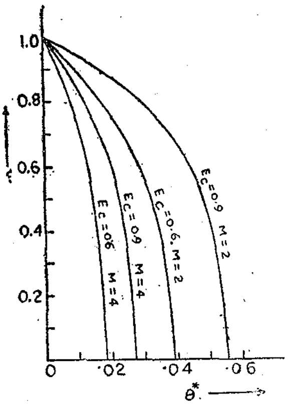

Fig. 1. Tempreature Profile for $S^*=0$, $M=2, 4$, $E_c=0{\cdot}6$, $0{\cdot}9$

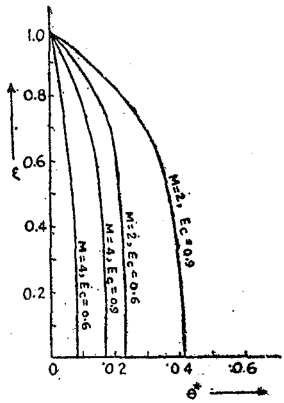

Fig. 2. Temperature Profile for $S^*=1$, $M=2, 4$, $E_c=0{\cdot}6$, $0{\cdot}9$

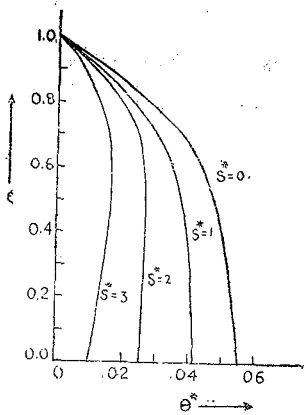

Fig. 3. Variation of θ^* with S^*, $M=2$, $E_c=0{\cdot}9$

Fig. 1 shows the temperature profiles in the case of channel with non-conducting walls for $M=2, 4$; $E_c=0{\cdot}6$, $0{\cdot}9$ and $S^*=0$. In Fig. 2 temperature profiles are drawn for $M=2, 4$; $E_c=0{\cdot}6$, $0{\cdot}9$ and $S^*=1$. It is interesting to see from these figures that the temperature decreases in magnitude with increase in the Hartmann number. Fig 3 shows the variation of θ^* with S^* for $M=2$ and $E_c=0{\cdot}9$. We see that temperature decreases as S^* increases.

Nusselt Number. The numerical results for Nusselt number are entered in the following table for $P_r=1{\cdot}0$ and $R=0{\cdot}1$.

S^*	M	E_c	N_u
0	2	0·3	3·646
		0·6	3·646
		0·9	3·646
	4	0·3	5·524
		0·6	5·524
		0·9	5·524
	5	0·3	6·909
		0·6	6·909
		0·9	6·909
1	2	0·3	24·500
		0·6	6·200
		0·9	5·528
	4	0·3	−15·500
		0·6	11·429
		0·9	8·125
	5	0·3	−37·000
		0·6	15·000
		0·9	10·214

Influence of E_c. For $S=0$, $M=2, 4, 5$ the value N_u remains unchanged when E_c increases. Thus the rate of heat transfer at the wall is not affected by the heat due to viscous dissipation. But for $S^*=1$, the value of N_u is found to decrease with increase in E_c at $M=2$ and at $M\geqslant 4$ it attains negative value for $E_c=0{\cdot}3$ and after that it attians positive value in dereasing order as E_c increases.

REFERENCES

1. L. E Erickson, C. S. Wang, C. L. Hwang and L. Fan, Heat transfer to magnetohy drodynamic flow in a flat duct, Z. Angew. Math. Mech. 15 (1961), 408-418.

2. J. A. Shercliff, Steady motion of conducting fluids in pipes under transverse magnetic fields, Proc. Cambridge Philos. Soc 49 (1953), 136-144.

3. R. Siegel, Heat transfer for laminar flow in ducts with arbitrary time variations in wall temperature, J. Appl. Mech, 27 1960) 241-249.

4. V. M. Soundlagekar, On heat transfer in Electromagneto-hydrodynamic channel flow under transverse magnetic fieid, Progr. Math. (Allahabad) 2 (1968), 73-93.

Jñānabha Sect. A, Vol 1976

MHD SLIP FLOW IN RECTANGULAR POROUS CHANNELS OF DIFFERENT PERMEABILLITY

By **V. Vidyanidhi, V. V. Ramana Rao and V. Bala Prasad**

Department of Applied Mathematics, Andhra University,

Waltair-530003, *India*

(Received : November 1, 1975)

Abstract. The steady laminar flow of a viscous incompressible fluid in the presence of a transverse magnetic field in a channel of rectangular cross section with two porous walls of different permeability has been examined, using the slip flow conditions at the walls. A numerical estimate for the velocity profile, pressure rise and the skin-friction at the walls has been made and conclusions drawn.

1. Introduction. The steady laminar flow of a viscous incompressible fluid between two parallel and porous walls of different permeability has been studied by Terrill and Shrestha [1]. This problem has been extended in the frame work of MHD by Reddy [2]. In the present analysis, the effect of slip velocity at the walls is taken into consideration, following the notation of Reddy [2].

2. Basic Equations and Solutions. We seek the solutions of eqs. (3.1) and (3.2) of Reddy's work subject to the boundary conditions :

$$\lambda=-1 : u(x, \lambda)=\epsilon\frac{\partial u}{\partial \lambda},\ V(x, \lambda)=V_1,$$

$$\lambda=1 : u(x, \lambda)=-\epsilon\frac{\partial u}{\partial \lambda},\ V(x, \lambda)=V_2, \tag{2.1}$$

where $\epsilon=\xi_u/h$, the non-dimensional first order slip as in Schaff and Chambre [3].
when $|V_2|\geqslant|V_1|$. following Reddy's analysis we solve eq. (3.8) of his paper, *i.e.*,

$$f'''+R_2\,(f'^2-ff'')-M^2 f'=K, \tag{2.2}$$

subject to the above boundary conditions, *i.e.*

$$f(-1)=1-\alpha_2,\ f(1)=1,\ f'(\pm 1)=\mp\epsilon f''\ (\pm 1). \tag{2.3}$$

When R_2 is small, we solve eqs. (4.2) of his paper subject to :

$$f_0(-1)=1-\alpha_2, f_0(1)=1, f_n(\pm 1)=0, n\geqslant 1 ;$$
$$f_n'(\pm 1)=\mp\epsilon f_n''(\pm 1) ; n\geqslant 0. \tag{2.4}$$

The first-order perturbation solution of eq. (2.2) subject to eq. (2.4) is obtained as :

$$f^{(1)}(\lambda)=f_0(\lambda)+R_2 f_1(\lambda),$$
$$K^{(1)}=K_0+R_2 K_1, \tag{2.5}$$

where $f_0(\lambda)$, $f_1(\lambda)$, K_0 and K_1 are given by

$$f_0(\lambda)=1-\frac{\alpha_2}{2}+2A_1(\sinh M\lambda-A_2\lambda), \tag{2.6}$$

$$f_1(\lambda)=2\beta_1\{\sinh M\lambda-\lambda\sinh M\}+\frac{A_1(2-\alpha_2)}{M}\left\{1-e^{-M\lambda}\right\}$$
$$-\lambda\left\{\frac{A_1}{M}(2-\alpha_2)(1-e^{-M})+\frac{7A_1^2A_2\cosh M}{M}\right.$$
$$\left.+\left(\frac{2-\alpha_2}{2}-A_1A_2\right)A_1\sinh M\right\}+\frac{7A_1^2A_2}{M}\lambda\cosh M\lambda$$
$$+A_1\left\{\frac{(2-\alpha_2)\lambda}{2}-A_1A_2\lambda^2\right\}\sinh M\lambda, \tag{2.7}$$

$$A_1=\frac{\alpha_2}{4\{\sinh M-M(\cosh M+\epsilon M\sinh M)\}}, \tag{2.8}$$

$$A_2=M(\cosh M+\epsilon M\sinh M). \tag{2.9}$$

$$B_1=\{M\cosh M+(\epsilon M^2-1)\sinh M\}^{-1}\left[\left\{\frac{A_1^2A_2M}{2}-\frac{(2-\alpha_2)}{4}A_1M\right\}\right.$$
$$\cosh M-3A_1^2A_2\sinh M+\frac{A_1(2-\alpha_2)}{2}\left\{\frac{1-e^{-M}}{M}-e^{-M}\right\}$$
$$+\frac{\epsilon}{2}\Big\{(3MA_1^2A_2+2A_1M(2-\alpha_2))\cosh M-(12A_1^2A_2+$$
$$\left.A_1M(2-\alpha_2)+\frac{A_1M^2}{2}(2-\alpha_2)-A_1^2A_2M^2)\sinh M\Big\}\right], \tag{2.10}$$

$$K_0=2A_1M^3(\cosh M+\epsilon M\sinh M), \tag{2.11}$$

$$K_1=4A_1^2(M^2+A_2^2)+7A_1^2A_2M\cosh M$$
$$+A_1M^2\sinh M\left(\frac{2-\alpha_2}{2}-A_1A_2\right)+M^2(1-\alpha_2)$$
$$+A_1M(2-\alpha_2)(e^{-M}-1)+2B_1M^2\sinh M. \tag{2.12}$$

In the absence of magnetic field, the above reduce to :

$$f_0(\lambda)=1-\frac{\alpha_2}{2}+\frac{3\alpha_2(1+2\epsilon)\lambda}{4(1+3\epsilon)}-\frac{\alpha_2\lambda^3}{4(1+3\epsilon)} \tag{2.13}$$

$$f_1(\lambda)=-\frac{\alpha_2^2(1+9\epsilon)\lambda}{560(1+3\epsilon)^3}+\frac{3\alpha_2^2(1+7\epsilon)\lambda^3}{1120(1+3\epsilon)^3}-\frac{\alpha_2^2\lambda^7}{1120(1+3\epsilon)^2}, \tag{2.14}$$

$$K_0=-\frac{3\alpha_2}{2(1+3\epsilon)}, \tag{2.15}$$

$$K_1=\frac{9(36+252\epsilon+560\epsilon^2+420\epsilon^3)\alpha_2^2}{560(1+3\epsilon)^3}-\frac{3\alpha_2(2-\alpha_2)}{4(1+3\epsilon)}. \tag{2.16}$$

When $\alpha_2=2$ and $\epsilon=0$, we recover the solutions of Berman [4] from equations (2.13 through 2.16).

When $|V_1|\geqslant|V_2|$, a perturbation solution for small R_1 can be obtained in a similar manner but this will not present any new physical features.

3. Discussion

(i) Velocity distribution. The velocity components in the axial and perpendicular directions have been obtained from equations (3.5) of Reddy's work, they are

$$\frac{u}{U(0)}=\left[\frac{1}{\alpha_2}-\frac{4R_2}{R^*}\frac{x}{h}\right]f'(\lambda)\ ;\ \frac{V}{U(0)}=\frac{4R_2}{R^*}f(\lambda), \tag{3.1}$$

where $f(\lambda)$ is given by eq. (2.5), $R^*=4h\,U(0)/\nu$ is the entrance Reynolds number and

$$\begin{aligned} f'(\lambda)=\;&2A_1(M\cosh M\lambda-A_2)+R_2\Big[2B_1(M\cosh M\lambda-\sinh M)\\ &+A_1(2-\alpha_2)e^{-M\lambda}-\Big\{\frac{A_1}{M}(2-\alpha_2)(1-e^{-M})+\frac{7A_1^2A_2\cosh M}{M}\\ &+\left(\frac{2-\alpha_2}{2}-A_1A_2\right)A_1\sinh M\Big\}+\frac{7A_1^2A_2}{M}\{\lambda M\sinh M\lambda+\cosh M\lambda\}+A_1\\ &\Big\{\left(\frac{(2-\alpha_2)\lambda}{2}-A_1A_2\lambda^2\right)M\cosh M\lambda+\left(\frac{2-\alpha_2}{2}-2A_1A_2\lambda\right)\sinh M\lambda\Big\}\Big] \end{aligned} \tag{3.2}$$

The function $f'(\lambda)$ has been plotted in Fig. 1 for $R_2=0{\cdot}5$ and for various values of M, α_2 and ϵ. Increase in the value of α_2 increases the velocity distribution in the channel. It is seen that the dimensionless slip velocity $f'(\pm 1)$ increases as M increases. We note, when $\alpha_2=2$, $R_2=0$ and $M=0$, the effect of slip decreases the centre line velocity $f'(0)$ from its maximum value of 1·5 (continuum flow) towards a value of 1·0. It is further observed that the flow at the centre of the channel gets retarded in the presence of the slip as well as the magnetic field.

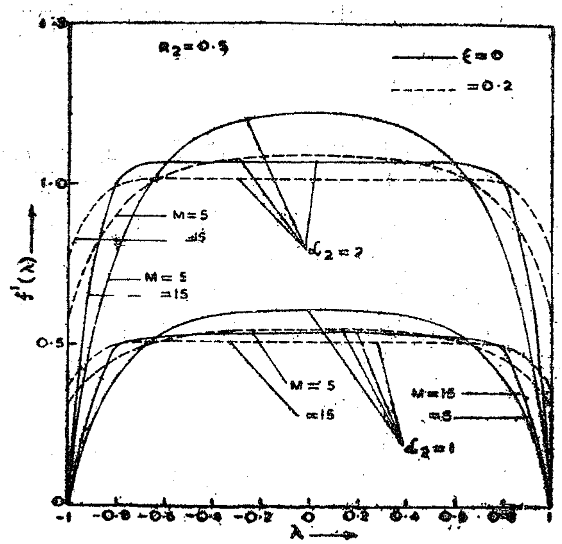

Fig. 1. Plot of velocity distribution in the axial direction

(ii) Pressure distribution. The non-dimensional pressure drop in the axial direction is given by eq. (5·10) of his paper, *i.e.*

$$P_x = \frac{1}{R^*}\left(\frac{x}{h}\right)\left[\frac{1}{\alpha_2} - \frac{2R_2}{R^*}\left(\frac{x}{h}\right)\right](-8\,K), \tag{3.3}$$

where K is given by eq. (2.5)

The variation of the pressure drop in this direction for $R^*=1000$, $R_2=0{\cdot}5$ has been plotted in Fig. 2 for various values of M, α_2 and ϵ. As in Reddy's work, the pressure drop increases with increase in the Hartmann number M. By increasing the values of ϵ and α_2, it decreases rapidly, more so along the axial direction.

(iii) Friction-coefficient : From eq. (5.13) of his paper, the wall friction in non-dimensional form is given by

$$c_f^{(\mp 1)} = \frac{8}{R^*}\left(\frac{1}{\alpha_2} - \frac{4R_2}{R^*}\,\frac{x}{h}\right)\Big[f''(\lambda)\Big]_{\lambda=-1 \text{ and } 1} \tag{3.4}$$

where

$$f''(-1)=-[2A_1 M^2 \sinh M+R_2\{(2B_1-A_1^2 A_2)M^2 \sinh M$$

$$+\frac{A_1 M(2-\alpha_2)}{2}(2-M)\sinh M$$

$$+A_1^2 A_2(12\sinh M+3M\cosh M)\}], \qquad (3.5)$$

$$f''(1)=2A_1M^2\sinh M+R_2\{(2B_1-A_1^2 A_2)M^2\sinh M$$

$$+\frac{A_1M(2-\alpha_2)}{2}(2+M)\sinh M$$

$$+A_1^2 A_2(12\sinh M+3M\cosh M)\} \qquad (3.6)$$

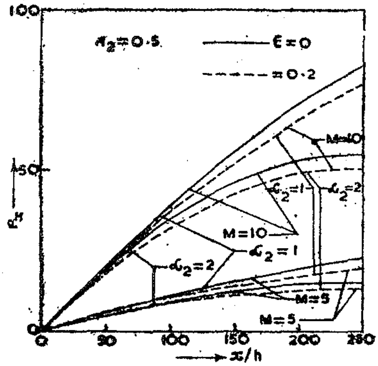

Fig. 2. Plot of pressure drop in the axial direction.

From eqs. (3.5) and (3·6) we note, for $\alpha_2=2$, the skin friction at the walls is equal and opposite. From Fig 3, we observe that this coefficient at the lower wall increases with an increase in the Hartmann number in agreement with Terrill and Shrestha [5]. It is also observed that the skin friction increases with increase in α_2 only upto a certain value of x/h, beyond which it decreases. The effect of the slip coefficient is found, in general, to decrease the skin-friction at the lower wall. From Fig. 4, similar features are also found to hold good for the skin-friction at the upper wall.

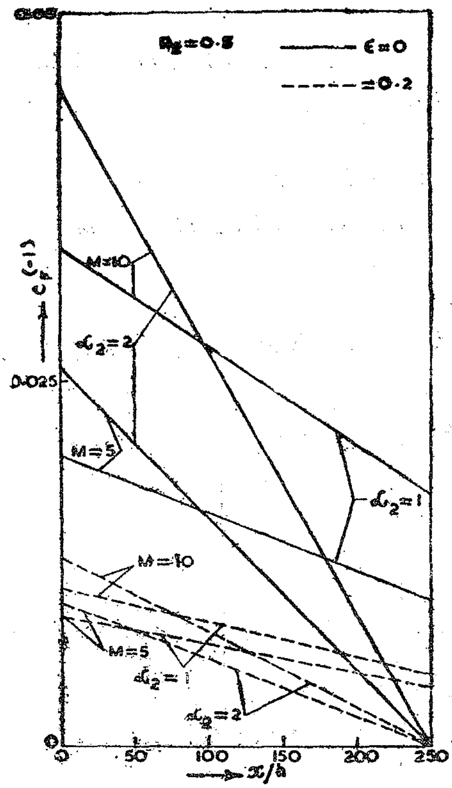

Fig 3. Plot of skin-friction at the lower wall.

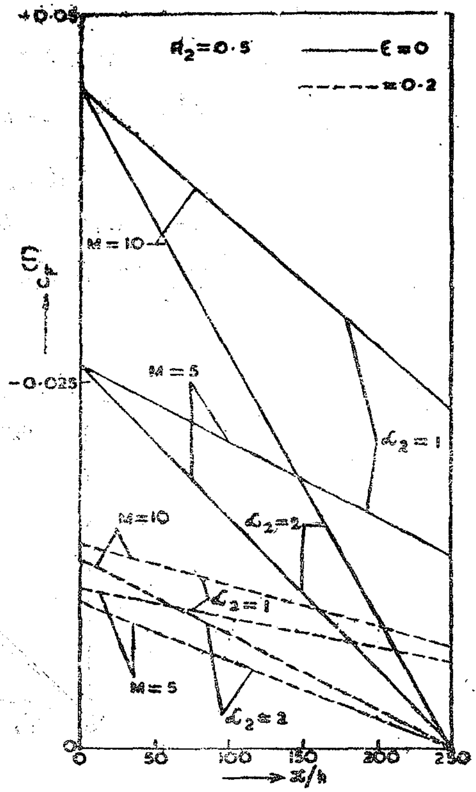

Fig. 4. Plot of skin-friction at the upper wall.

REFERENCES

1. R. M. Terrill and G. M. Shrestha, ZAMP 16 (1965), 470.
2. P. G. Reddy, Arch. Mech. Stosowanej (3) 19 (1967), 385.
3. S. A. Schaff and P. L. Chambre, Flow of Rarefied Gases, Princeton University Press, 1961.
4. A. S. Berman, J. Appl. Phys. 24 (1953), 1232.
5. R. M. Terrill and G. M. Shrestha, Appl. Sci. Res. (B) 11 (1964), 134

Jñānabha Sect. A, Vol. 6, 1976

FREE AMINO ACID ANALYSIS OF SOME LEGUMES

By **G. S. Niranjan**

Department of Organic Chemistry, D. V. Post-graduate College,

Orai–285001, *U. P., India*

(Received : July 31, 1976)

Amino acids of leguminous seeds, irrespective of their occurrence in free or protein-bound state are essential factors in nutrition, particularly those assigned as the 'essential amino acid.' The present communication deals the analysis of free amino acid content of fine seeds ; Cassia tora, C. occidentalis, Cassia fistula, Mucuna prurita and M. capitata of Indian origin. The seeds have been collected locally in the Bundelkhand region, particularly the forest of Jhansi district. The total free amino acid content of the seeds in terms of glycine (basic unit) was also estimated by employing the method of Rosen [1].

The seeds of Mucuna capitata was only obtained from the department of chemistry, University of Allahabad, Allahabad. All the powdered seeds were defatted in a Soxhlet apparatus with petroleum ether (60-80° C) and preserved them in air tight bottles. One gram of each of the defatted seed powders was well stirried with ethanol (8-10 ml, 70% v/v) for half an hour. After centrifugation the residue was reextracted with 70% ethanol, spun and the two supernatants combined. This process was repeated 8-10 times till the supernatant was negative to Ninhydrin test. The pooled supernatant was evaporated to dryness in Vacuo, dissolved in distilled water (0·5-1·0 ml), centrifuged and the clear supernatant (2-10 ml) was employed for qualitative and quantitative analysis of free amino acids.

The two-dimentional chromatographic analysis of Datta, Dent and Harris [2] was employed using phenol (80% w/v)– NH_3 and n-Butanol : acetic acid : water (4 : 1 : 5) as the developing solvents. The different amino acids present were confirmed by special spray reagent by Sakaguchi [3], Smith [4], Pant [5], Block etal [6] and Dent [7].

For estimating the total free amino acid content an accurately measured volume of seed extract (2–10 μl) was diluted with glass distilled water (1 ml) followed by the addition of acetate-cyanide buffer (0·5 ml) and ninhydrin solution (0·5 ml) in methyl cellosolve. The tubes were heated in a water bath at 100°C for 20 minutes and quickly diluted by the addition of isopropanol (5 ml) with constant shaking. The solution was cooled to room temperature and colour density read in a Calorimeter at 570 nm with a reagent, black and standard glycine solution.

Table 1. **Qualitative Pattern and Total Content of Free Amino Acids in Leguminous Seeds.**

Amino Acid	Cassia tora/	C. occidentalis/	C. fistula/	M. prurita/	M. capitata
£–alanine	+	+	+	+	+
Arginine	+	+	+	+	+
Aspartic acid	—	+	+	+	—
Cysteic acid	+	—	—	+	—
Glutamic acid	+	+	+	+	+
Glycine	+	+	+	+	+
Histidine	—	—	—	+	+
Leucine-Isoleucine	+	+	+	+	+
Lysine	+	+	+	+	+
Methionine	—	+	—	—	—
Proline	+	+	+	+	+
Phenylalanine	+	+	+	—	—
Serine	+	+	+	+	+
Threonine	+	+	+	+	+
Tryptophan	—	—	+	—	—
Tyrosine	+	+	+	+	+
Valine	+	+	+	—	—
Total free amino acids per 100 gm. seeds	·38	·36	·28	·16	·19

Table 1 represents the qualitative pattern and total content of free amino acid composition of the leguminous seeds analysed. Each seed has its own pattern of amino acids and no single seed was found to be complete with respect to essential amino acids. All the seeds contain 14–17 amino acid in the free state. Table 1 revealed that the various leguminous seeds contain free amino acid in the order of ·13–0·32 g (in terms of glycine) per 100 g of dry seed powder.

ACKNOWLEDGEMENT. The author expresses his gratitude to the State Council of Science & Technology, Lucknow, U. P., India, for financial assistance, and to Dr. B. B. Lal and Dr. S. P. Saxena, of this College for providing necessary laboratory facilities.

REFERENCES

1. Rosen, H., Arch. Biochem. Biophys., **67**, 10 (1957).
2. Datta, S. P., Dent, C. E. and Harris, H., Science, **112**, 620 (1950)
3. Sakaguchi, S., J. Biol. Chem. (Tokyo), **5**, 25 (1925).
4. Smith, I., Nature (London), **43**, 171 (1953).
5. Pant, R. and Agarwal, H. C., Naturwissen schaften, **50**, 530 (1963).
6. Block, R. J., Winegard, H. M. and Toennies, G., Science, **108**, 506 (1948).
7. Dent, C. E., Biochem. J., **43**, 169 (1948).

INFORMATION FOR CONTRIBUTORS / SUBSCRIBERS

Jñānabha is published annually. It is an interdisciplinary journal devoted primarily to research articles in *all* areas of the mathematical, physical and biological sciences; it does, however, encourage original mathematical works which are motivated by and relevant to applications in the social, management, biomedical or engineering sciences. Papers intended for publication in this journal should be in typed form or offset-reproduced, (*not dittoed*), double spaced with generous margins, and they may be written in Hindi or English. Manuscripts in duplicate, if possible, may be submitted to *either* of the Editors

The submission of a paper implies the author's assurance that the paper has not been widely circulated, copyrighted, published or submitted for publication elsewhere.

Authors are advised to submit *only* neatly (and carefully) typewritten and thoroughly checked manuscripts employing accepted conventions of references, notations, displays, etc. ; *all* typescripts *not* in a format suitable for publication in this journal will be returned *unrefereed.*

25 reprints are provided free for each article; additional reprints, in multiples of 25, may be obtained at cost price, if the authors give prior intimation while returning proofs.

Effective with Volume 2 the price per volume is Rs. 20·00 (or U.S. $ 10·00). Individual members of the Vijnana Parishad are entitled to free subscriptions to the current issues of this journal. Individual membership : Rs. 15·00 (or U.S. $ 5·00) per calendar year ; Life membership : Rs. 150·00 (or U.S $ 50·00). Back volumes are available at special price.

[By a reciprocity agreement with the American Mathematical Society, an individual/life member of the Parishad residing outside the North American continent may join the Society by submitting an application for membership on the form that may be obtained from the office of the Society (P. O. Box 6248, Providence, Rhode Island 02940, U.S.A.) and by paying U S. $ 18·00 per calendar year, which is only half of the current rate for annual dues for members whose annual professional income is below $ 20,000·00 ; the usual requirements that the applicant be endorsed by two members of the Society and that the candidate be elected by the Council are waived, but this reduction in dues to the Society does not apply to those individual/life members of the Parishad who reside, even on a temporary basis, in the North American area.

The mathematical content of this journal is indexed in the Society's biweekly **Current Mathematical Publications** and is reviewed, among others, by **Mathematical Reviews** (U.S.A.), **Zentralblatt für Mathematik** (W. Germany) and **Applied Mechanics Reviews** (U.S.A.).]

All communications regarding subscriptions, orders for back volumes, membership of the Vijnana Parishad, change of address, etc., and all books for review, should be addressed to :

The Secretary, **Vijnana Parishad of India,**
D. V. Postgraduate College,
Orai–285 001, U.P., India.

CONTENTS