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Generalized Gauss-Codazzi Equations for the Curvature Tensors R^{i}_{jhk} $(x, \dot{x})$ in a Hypersurface of a Finsler Space

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Summary. Rund [1] derived Gauss-Codazzi equations for the curvature tensor $K_{\alpha\beta\gamma\epsilon}$ in a Finsler space. Srivastava and subsequent authors (cf. [3], [4] and [5]) have derived them for Cartan's first and second curvature tensors in a subspace of a Finsler space. The object of the present paper is to derive these equations for the curves of congruences associated with a hypersurface of an n-dimensional Finsler space.

1. Introduction. Let us consider an n-dimensional Finsler space F_n with the fundamental metric tensor $F(x, \dot{x})$ homogeneous of degree 1 in $\dot{x}^i$. The metric tensor is given by $g_{ij}\ (x, \dot{x}) \overset{\text{def}}{=} \frac{1}{2}\dot{\partial}_i\, \dot{\partial}_j\ F^2(x, \dot{x})$ and is symmetric in i and j. The symbols ∂_i, $\dot{\partial}_i$ and $\dot{\partial}_{ij}^{2}$ denote the partial derivatives $\partial/\partial x^i$, $\partial/\partial \dot{x}^i$ and $\partial^2/\partial \dot{x}^i\, \partial \dot{x}_j$, respectively.

The covariant derivatives of $T^i_j\ (x, \dot{x})$ with respect to x^k in the sense of Cartan are given by

$$(1.1)a \qquad T^i_{j|k} = \partial_k T^i_j - \left(\dot{\partial}_m\, T^i_j\right) G^m_k + T^h_j\, \Gamma^{*i}_{hk} T - {}^{i}_{h}\Gamma^{*h}_{jk}$$

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and

(1.1)b $$T^i_j\big|_k = F\dot{\partial}_k T^i_j + T^h_j A^i_{hk} - T^i_h A^h_{jk},$$

where the connection coefficient $\overset{*}{\Gamma}{}^i_{hk}(x, \dot{x})$ and tensor field $A^i_{hk}(x, \dot{x})$ have their usual meanings. For details see Rund [2].

A hypersurface F_{n-1} of F_n may be represented parametrically in the form $x^i = x^i(u^\alpha)$, $\alpha = 1, \ldots\ldots, n-1$, where the $n-1$ parameters u^α form the coordinate system of F_{n-1} and it will be assumed that the matrix of the projection factors $B^i = \partial_\alpha x^i$ has the rank $n-1$. Throughout this discussion we shall denote $B^i_{\alpha\beta} = \partial^2_{\alpha\beta} x^i$ and $B^{ij\ldots k}_{\alpha\beta\ldots\gamma} = B^i_\alpha B^j_\beta B^k_\gamma$.

It is easy to verify that the vector $\dot{u}^\alpha$, the induced metric tensor $g_{\alpha\beta}(u, \dot{u})$ and the induced symmetric connection defined in the hypersurface F_{n-1} are related with the corresponding quantities in the space F_n by the following relations.

(1.2) $$\dot{x}^i = \dot{u}^\alpha B^i_\alpha,$$

(1.3) $$g_{\alpha\beta} = g_{ij} B^{ij}_{\alpha\beta},$$

(1.4)a $$\overset{*}{\Gamma}{}^\alpha_{\beta\gamma}(u, \dot{u}) = B^\alpha_i \left(B^i_{\beta\gamma} + \overset{*}{\Gamma}{}^i_{hk} B^{hk}_{\beta\gamma} \right).$$

Furthermore,

$$B^\alpha_i = g^{\alpha\epsilon} B^s_\epsilon g_{is}, \quad B^\alpha_i B^i_\beta = \delta^\alpha_\beta.$$

If $N^i(x, \dot{x})$ denotes the unit normal at each point P of F_{n-1} with respect to the tangential direction $\dot{x}^i$ at P, we then have

(1.4)b $$g^{ij} N_i N_j = N_i N^i = 1, \quad N_i B^i_\alpha = g_{ij} N^j B^i_\alpha = 0,$$

$$N^i B^\alpha_i = 0, \quad \text{and} \quad g_{ij} N^i N^j = 1.$$

The induced mixed covariant derivative of T^i_α as denoted by $T^i_{\alpha\,\|\,\gamma}$ is given by

(1.5) $$T^i_{\alpha \| \gamma} = \partial_\gamma T^i_\alpha - \left(\dot{\partial}_\beta T^i_\alpha\right) \overset{*}{\Gamma}{}^{\beta}_{\rho\gamma} \dot{u}^\rho + T^s_\alpha \overset{*}{\Gamma}{}^{i}_{sh} B^h_\gamma - T^i_\epsilon \overset{*}{\Gamma}{}^{\epsilon}_{\alpha\gamma}.$$

With the help of mixed covariant derivative $T^i_{\alpha \| \gamma}$ we can construct a mixed tensor [5]

(1.6) $$I^i_{\alpha\beta} = B^i_{\alpha \| \beta} = B^i_{\alpha\beta} - B^i_\epsilon \overset{*}{\Gamma}{}^{\epsilon}_{\alpha\beta} + \overset{*}{\Gamma}{}^{i}_{hk} B^{hk}_{\alpha\beta},$$

which are regarded as a vector of imbeding space F_{n-1} and are also normal to F_{n-1}. Thus, we can write

(1.7) $$I^i_{\alpha\beta} = N^i \,\overline{\Omega}_{\alpha\beta},$$

where $\overline{\Omega}_{\alpha\beta}$ is the second fundamental form symmetric in α and β.

Verma and Sinha [5] obtained the induced derivative $N^i_{\| \beta}$ in the form

(1.8) $$N^i_{\| \beta} = -\overline{\Omega}_{\alpha\beta}\, g^{\alpha\delta} B^i_\delta + E^i_m \, I^m_{\rho\beta} \,\dot{u}^\rho,$$

where

(1.9) $$E^i_m = N^i M_m - 2M^i_m, \; M^i_m = C^i_{mp} N^p,$$

$$M_m = C_{pkm} N^p N^k = M_{km} N^k.$$

2. Generalized Gauss-Codazzi Equations. We consider congruence of curves such that each of them passes through each point of the hypersurface. Let λ^i be the contravariant components of the unit vector in the direction of the curve of congruences. This vector may be expressed linearly in terms of B^i_α and normal vectors N^i of F_{n-1}. Thus, we write

(2.1) $$\lambda^i = t^\alpha B^i_\alpha + d N^i,$$

where d and t are the parameters. By taking the induced covariant derivative of type (1.5) of the equation (2.1), we have

(2.2) $$\lambda^i_{\| \beta} = B^i_\alpha \, t^\alpha_{\| \beta} + t^\alpha B^i_{\alpha \| \beta} + d_{\| \beta} N^i + d N^i_{\| \beta}.$$

Now differentiating (2.2) covariantly with respect to u^γ in the sense of (1.5), we obtain

(2.3) $$\lambda^i_{\|[\beta\gamma]}=B^i_\alpha t^\alpha_{\|[\beta\gamma]}+t^\alpha B^i_{\alpha\|[\beta\gamma]}+dN^i_{\|[\beta\gamma]}+N^i d_{\|[\beta\gamma]}.$$

Also from equations (1.6), we can write

(2.4) $$B^i_{\alpha\|[\beta\gamma]}=I^i_{\alpha[\beta\|\gamma]}.$$

On substituting the value of $I^i_{\alpha\beta}$ and $N^i_{\|\gamma}$ from (1.7) and (1.8), we obtain

(2.5) $$B^i_{\epsilon\|[\beta\gamma]}=N^i\bar{\Omega}_{\epsilon[\beta\|\gamma]}+\bar{\Omega}_{\epsilon[\beta}\bar{\Omega}_{\gamma]\alpha}g^{\alpha\delta}B^i_\delta +E^i_m I^m_{\rho[\gamma}\bar{\Omega}_{\beta]\epsilon}\dot{u}^\rho.$$

Similarly, with the help of equations (1.7) and (1.8), we obtain

(2.6) $$N^i_{\|[\beta\gamma]}=B^i_\delta\bar{\Omega}_{\alpha[\gamma\|\beta]}g^{\alpha\delta}+E^i_m(I^m_{\rho[\beta}\dot{u}^\rho)_{\|\gamma]} +I^m_{\rho[\beta}E_{<m>\gamma]}\dot{u}^\rho.$$

On differentiating the equation $\lambda^i_{\|\beta}=\lambda^i_{|h}B^h_\beta$ with respect to u^γ and using the commutation formula

$$2\lambda^i_{|[hk]}=R^i_{jhk}\lambda^j-K^j_{mhk}\,e^m\,\lambda^i_{|j},$$

we get

(2.7) $$2\lambda^i_{\|[\beta\gamma]}=\left(R^i_{jhk}\lambda^j-K^j_{mhk}\,l^m\,\lambda^i\big|_j\right)B^{hk}_{\beta\gamma}.$$

Similarly, we have

(2.8) $$2t^\delta_{\|[\beta\gamma]}=R^\delta_{\alpha\beta\gamma}t^\alpha-K^\alpha_{\epsilon\beta\gamma}l^\epsilon\,t^\delta_{,\alpha}$$

where a comma denotes the induced covariant derivative of (1.1)b and

(2.9) $$R^\delta_{\alpha\beta\gamma}(u,\dot{u})=2\Big\{\partial_{[\gamma}\overset{*}{\Gamma}{}^\delta_{\beta]\alpha}-\left(\dot{\partial}_{[\rho}\overset{*}{\Gamma}{}^\delta_{\alpha[\beta}\right)G^\alpha_{\gamma]} +C^\delta_{\alpha\epsilon}(\partial_{[\gamma}\dot{\partial}_{\beta]}G^\epsilon-G^\epsilon_{\rho[\beta}G^\rho_{\gamma]} +\overset{*}{\Gamma}{}^\delta_{\epsilon[\gamma}\overset{*}{\Gamma}{}^\delta_{\beta]\alpha}\Big\},$$

(2.10) $$K^\delta_{\alpha\beta\gamma}(u,\dot{u})=2\left\{\partial_{[\gamma}\overset{*}{\Gamma}{}^\delta_{\beta]\alpha}-\left(\dot{\partial}_\rho\overset{*}{\Gamma}{}^\delta_{\alpha[\beta}\right)G^\rho_{\gamma]}+\overset{*}{\Gamma}{}^\delta_{\epsilon[\gamma}\overset{*}{\Gamma}{}^\epsilon_{\beta]\alpha}\right\}$$

We know that

(2.11) $$d_{\|\beta}=\partial_\beta d-(\dot{\partial}_\alpha d)G^\alpha_\beta.$$

Differentiating (2.11) covariantly with respect to u^γ in the sense of (1.5) again and subtracting the equation thus obtained by interchanging the indices β and γ, we obtain

$$d_{\,\|\,[\beta\gamma]} = \dot{\partial}_\alpha\, d\left(\partial_{[\beta} dG^\alpha_{\gamma]} + G^\alpha_{\rho[\beta} G^\rho_{\gamma]}\right). \tag{2.12}$$

By substituting equations (2.5), (2.6), (2.7), (2.8) and (2.12) in (2.3) and after arranging the terms, we get

$$\begin{aligned}(2.13)\quad R^i_{jhk}(x,\dot{x})\lambda^j B^{hk}_{\beta\gamma} &= R^\delta_{\alpha\beta\gamma}(u,\dot{u})t^\alpha B^i_\delta + K^j_{mhk} l^m \lambda^i\Big|_j B^{hk}_{\beta\gamma} \\ &- B^i_\delta K^\alpha_{\epsilon\beta\gamma} l^\epsilon t^\delta_{\ ,\alpha} + 2\Big[B^i_\delta \left\{ \alpha\, \overline{\Omega}_{\alpha[\gamma\,\|\,\beta]} + t^\epsilon\, \overline{\Omega}_{\epsilon[\beta}\, \overline{\Omega}_{\gamma]\alpha} \right\} g^{\alpha\delta} \\ &+ N^i \left\{ \dot{\partial}_\epsilon\, d\left(\partial_{[\beta} dG^\epsilon_{\gamma]} + G^\epsilon_{\rho[\beta} G^\rho_{\gamma]}\right) + t^\epsilon\, \overline{\Omega}_{\epsilon[\beta\,\|\,\gamma]} \right\} \\ &+ E^i_m \left\{ t^\epsilon\, \overline{\Omega}_{\epsilon[\beta} I^m_{<\rho>\gamma]}\, \dot{u}^\rho + d\left(I^m_{\rho[\beta} \dot{u}^\rho\right)_{\|\,\gamma]} \right\} \\ &+ d\, I^m_{\rho[\beta} E^i_{<m>\,\|\,\gamma]}\, \dot{u}^\rho \Big].\end{aligned}$$

Multiplying equation (2.13) by B^δ_i and noting (1.9), we have

$$\begin{aligned}(2.14)\quad & R^i_{jhk}(x,\dot{x})\,\lambda^j B^\delta_i B^{hk}_{\beta\gamma} \\ &= R^\delta_{\alpha\beta\gamma}(u,\dot{u})\,t^\alpha + K^j_{mhk}\, l^m \lambda^i\Big|_j B^\delta_i B^{hk}_{\beta\gamma} - K^\alpha_{\epsilon\beta\gamma}\, l^\epsilon t^\delta_{\ ,\alpha} \\ &+ 2\Big[g^{\alpha\delta}\left\{ d\, \overline{\Omega}_{\alpha[\gamma\,\|\,\beta]} + t^\epsilon \overline{\Omega}_{\epsilon[\beta}\, \overline{\Omega}_{\gamma]\alpha} \right\} \\ &- B^\delta_i \Big\{ 2M^i_m\left(t^\epsilon \overline{\Omega}_{\epsilon[\beta}\, I^m_{<\rho>\gamma]}\dot{u}^\rho + d\left(I^m_{\rho[\beta}\dot{u}^\rho\right)_{\|\,\gamma]}\right) \\ &- d\; I^m_{\rho[\beta} E^i_{<m>\,\|\,\gamma]}\dot{u}^\rho \Big\}\Big]\end{aligned}$$

Again multiplying (2.13) by N_i and using (1.4)b, we get

$$\begin{aligned}(2.15)\quad R^i_{jhk}(x,\dot{x})\, N_i \lambda^j B^{hk}_{\beta\gamma} &= K^j_{mhk} N_i l^m \lambda^i|_j B^{hk}_{\beta\gamma} \\ &+ 2\Big[\dot{\partial}_\epsilon d \left\{ \partial_{[\beta} dG^\epsilon_{\gamma]} + G^\epsilon_{\rho[\beta} G^\rho_{\gamma]} \right\} + t^\epsilon \overline{\Omega}_{\epsilon[\beta\,\|\,\gamma]} \\ &+ N_i \Big\{ E^i_m\; t^\epsilon \overline{\Omega}_{\epsilon[\beta} I^m_{<\rho>\gamma]}\, \dot{u}^\rho + d\left(I^m_{\rho[\beta}\, \dot{u}^\rho\right)_{\|\,\gamma]} \\ &+ d\, I^m_{\rho[\beta} E^i_{<m>\,\|\,\gamma]}\, \dot{u}^\rho \Big\}\Big].\end{aligned}$$

Equations (2.14) and (2.15) which are based on a vector λ^i of most general nature can be regarded as generalization of Gauss-Codazzi equations in a hypersurface imbeded in a Finsler space F_n.

3. Particular Cases. Since the vector λ^i is a linear combination of B^i_α and N^i, therefore we can consider a congruence of curves in three different ways. Firstly, it is such that the vector λ^i in the direction of the curves of the congruences is normal to F_{n-1}, *i.e.*, $\lambda^i = dN^i$; secondly, it lies in the space spanned by B^i_α, *i.e.*, $\lambda^i = t^\alpha B^i_\alpha$; and thirdly, it is tangential to the curves $\dot{x}^i = \dot{u}^\alpha B^i_\alpha$.

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