Jñānabha Sect. A, Vol 6, 1976

SOME IDENTITIES AND HYPERGEOMETRIC FUNCTIONAL EXPANSIONS

By **B. M. Singhal**

Department of Mathematics, Government Science College, Gwalior–474001, M.P., India

(Received : September 15, 1975 ; in revised form : March 26, 1976)

1. Introduction. Some years ago, Srivastava [5] gave various classes of expansions for the generalized Appell functions of two variables (cf., *e.g.*, [1], p. 150). Subsequently, Ragab [4] rediscovered an obvious special form of one of Srivastava's expansions ([5], p. 48, Eq. (7)). These expansion formulas seem to have been motivated by similar results for the ordinary Appell functions which were given earlier by Burchnall and Chaundy ([2], [3]). The object of the present note is to derive certain general identities, involving double series with arbitrary terms, from which some of Srivastava's expansions would follow as special cases. Indeed, if we let $\{f(n)\}$, $\{g(n)\}$ and $\{h(n)\}$ be sequences of arbitrary complex numbers, then the identities to be established may be unified formally as :

$$(1.1) \qquad \sum_{r=0}^{\infty} A_r \sum_{m,n=0}^{\infty} B_{r,m,n}\, f(m+n+2r)\, g(m+r)\, h(n+r)$$

$$= \sum_{m,n=0}^{\infty} f(m+n)\, g(m)\, h(n) \sum_{r=0}^{\infty} A_r\, B_{r,m-r,n-r}$$

for arbitrary coefficients A_r and $B_{r,m,n}$.

The following known forms of certain familiar hypergeometric summation theorems will be required in our analysis (cf. [2]) :

$$(1.2) \qquad \sum_{r=0}^{\infty} \frac{(-m)_r(-n)_r}{r!\,(\alpha)_r} = \frac{(\alpha)_{m+n}}{(\alpha)_m\,(\alpha)_n} ;$$

$$(1.3) \qquad \sum_{r=0}^{\infty} \frac{(-m)_r(-n)_r}{r!\,(-\alpha-m-n+1)_r} = \frac{(\alpha)_m\,(\alpha)_n}{(\alpha)_{m+n}} ;$$

Of these (1.2) is valid if m, n are positive integers and (1.3) is valid when one of m, n is an integer. Also the following summation theorems are valid only when one of m, n is a positive integer :

$$(1.4)\qquad \sum_{r=0}^{\infty}(-1)^r\frac{(\alpha)_{2r}\,(-m)_r\,(-n)_r}{r!\,(\alpha+r-1)_r\,(m+\alpha)_r\,(n+\alpha)_r}=\frac{(\alpha)_m\,(\alpha)_n}{(\alpha)_{m+n}}\,;$$

$$(1.5)\qquad \sum_{r=0}^{\infty}\frac{(\gamma-\alpha)_r\,(\gamma)_{2r}\,(-m)_r\,(-n)_r}{r!\,(\gamma+r-1)_r\,(m+\gamma)_r\,(n+\gamma)_r\,(\alpha)_r}=\frac{(\alpha)_{m+n}(\gamma)_m(\gamma)_n}{(\gamma)_{m+n}(\alpha)_m(\alpha)_n}\,;$$

$$(1.6)\qquad \sum_{r=0}^{\infty}\frac{(\alpha-\gamma)_r\,(-m)_r\,(-n)_r}{r!\,(\alpha)_r\,(-\gamma-m-n+1)_r}=\frac{(\alpha)_{m+n}\,(\gamma)_m(\gamma)_n}{(\gamma)_{m+n}\,(\alpha)_m(\alpha)_n}.$$

2. Reducible forms of (1.1). By suitably specializing the coefficients A_r and B_r, m, n, the innermost series on the right-hand side of (1.1) can be summed by one or the other of the known results (1.2) to (1.6), and we thus arrive at the following identities :

$$(2.1)\qquad \sum_{r=0}^{\infty}\frac{(-1)^r\,(\alpha)_r\,(\gamma-\alpha)_r}{r!\,(\gamma)_r}$$

$$\times\sum_{m,\,n=0}^{\infty}\frac{(\alpha+r)_{m+n}}{m!\,n!}f(m+n+2r)\,g(m+r)\,h(n+r)$$

$$=\sum_{m,\,n=0}^{\infty}\frac{(\gamma)_{m+n}\,(\alpha)_m\,(\alpha)_n}{m!\,n!\,(\gamma)_m\,(\gamma)_n}f(m+n)\,g(m)\,h(n)\,;$$

$$(2.2)\qquad \sum_{r,\,m,\,n=0}^{\infty}\frac{f(m+n+2r)\,g(m+r)\,h(n+r)}{r!\,m!\,n!\,(\gamma)_r}$$

$$=\sum_{m,\,n=0}^{\infty}\frac{(\gamma)_{m+n}}{(\gamma)_m\,(\gamma)_n\,m!\,n!}f(m+n)\,g(m)\,h(n)\,;$$

$$(2.3)\qquad \sum_{r=0}^{\infty}\frac{(-1)^r\,(\alpha)_r}{r!}$$

$$\times\sum_{m,\,n=0}^{\infty}\frac{(\alpha+r)_{m+n}}{m!\,n!}f(m+n+2r)\,g(m+r)\,h(n+r)$$

$$=\sum_{m,\,n=0}^{\infty}\frac{(\alpha)_m\,(\alpha)_n}{m!\,n!}f(m+n)\,g(m)\,h(n)\,;$$

$$(2.4)\qquad \sum_{r=0}^{\infty}\frac{(-1)^r}{r!\,(\gamma+r-1)_r\,(\gamma)_{2r}}\sum_{m,\,n=0}^{\infty}\frac{f(m+n+2r)\,g(m+r)\,h(n+r)}{(\gamma+2r)_m\,(\gamma+2r)_n\,m!\,n!}$$

$$=\sum_{m,\,n=0}^{\infty}\frac{f(m+n)\,g(m)\,h(n)}{m!\,n!\,(\gamma)_{m+n}}\,;$$

$$(2.5)\qquad \sum_{r=0}^{\infty} \frac{(\gamma-\alpha)_r}{r!\,(\gamma+r-1)_r\,(\alpha)_r\,(\gamma)_{2r}}$$

$$\times \sum_{m,\,n=0}^{\infty} \frac{f(m+n+2r)\,g(m+r)\,h(n+r)}{(\gamma+2r)_m\,(\gamma+2r)_n\,m!\,n!}$$

$$= \sum_{m,\,n=0}^{\infty} \frac{(\alpha)_{m+n}}{(\gamma)_{m+n}\,(\alpha)_m\,(\alpha)_n} \cdot \frac{f(m+n)\,g(m)\,h(n)}{m!\,n!},$$

provided that the various series involved are absolutely convergent.

3. Applications. By appropriately choosing the arbitrary sequences $\{f(n)\}$, $\{g(n)\}$ and $\{h(n)\}$, the identities (2.1) to (2.5) of the preceding section can be applied to derive some of the expansion formulas given earlier by Srivastava [5]. For example, if we let

$$(3.1)\qquad f(n)=\frac{\prod_{j=1}^{A}(a_j)_n}{(\gamma)_n \prod_{j=1}^{C}(c_j)_n}, \qquad g(n)=\frac{\prod_{j=1}^{B}(b_j)_n}{\prod_{j=1}^{D}(d_j)_n}\,x^n,$$

and

$$(3.2)\qquad h(n)=\frac{\prod_{j=1}^{B'}(b')_{jn}}{\prod_{j=1}^{D'}(d'_j)_n}\,y^n, \qquad n=0,1,2,\ldots,$$

our identity (2.1) will yield the following result due to Srivastava ([5], p. 48, Eq. (7)) :

$$(3.3)\qquad F\left[\begin{matrix}(a):(b),\alpha\,;(b'),\alpha\,;\\(c):(d),\gamma\,;(d'),\gamma\,;\end{matrix}\;x,y\right]$$

$$=\sum_{r=0}^{\infty}\frac{(-1)^r(\alpha)_r(\gamma-\alpha)_r f(2r)\,g(r)\,h(r)}{r!\,(\gamma)_r}$$

$$\times F\left[\begin{matrix}(a)+2r,\ \alpha+r\ :(b)+r\,;(b')+r\,;\\(c)+2r,\ \gamma+2r:(d)+r\,;(d')+r\,;\end{matrix}\;x,y\right],$$

where f, g, h are given by (3.1) and (3.2), (a) abbreviates the sequence of A parameters $a_1, \ldots, a_A$, with similar interpretations for (b), (b'), etc., and the notation used for the double hypergeometric functions occurring on either side of (3.3) is due to Burchnall and Chaundy [3, p. 112].

Remark. An obvious special form of Srivastava's expansion (3.3) above happens to be the main result (12) on page 346 of a latter paper by Ragab [4].

ACKNOWLEDGEMENTS. The author wishes to thank Dr. B. M. Agrawal for his kind supervision and to Professor H. M. Srivastava for some helpful suggestions.

REFERENCES

1. P. Appell and J. Kampe′ de Fe′riet, Fonctions hyperge′ome′triques et hypersphe′riques, Paris, 1926.
2. J. L. Burchnall and T. W. Chaundy, Expansions of Appell's double hypergeometric functions (I) Quart. J. Math. (Oxford), **11** (1940) 249–270.
3. J. L. Burchnall and T. W. Chaundy, Expansions of Appell's double hypergeometric functions (II), Quart. J. Math. (Oxford) **12** (1941), 112–128.
4. F. M. Ragab, Expansions of Kampe′ de Fe′riet's double hypergeometric functions of higher order, J. Natur. Sci. Math. **12** (1972), 343–361.
5. H. M. Srivastava, Certain pairs of inverse series relations, J. Reine Angew. Math. **245** (1970), 47–54.