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On Uniform Summability–Λ of Orthonormal Expansions of Functions of Class L^{∞}

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1. Introduction. Let $\|\lambda_{\nu\mu}\|$ $(\nu, \mu=0, 1, 2, \ldots)$ be an infinite matrix of real numbers. Let $f(x)$ be an integrable function and $\{\phi_\nu(x)\}$, $\nu=0, 1, 2, \ldots$, be an orthonormal system in $[0, 1]$.

Let

$$(1) \qquad f(x) \sim \sum_{\nu=0}^{\infty} c_\nu \phi_\nu(x)$$

where

$$c_\nu = \int_0^1 f(t)\phi_\nu(t)\, dt \quad (\nu=0, 1, 2, \ldots),$$

be an orthonormal expansion of $f(x)$ in the system $\{\phi_\nu(x)\}$.

If

$$\Lambda_{\mu, x}(f) = \sum_{\nu=0}^{\infty} \lambda_{\nu\mu} c_\nu \phi_\nu(x)$$

is uniformly covergent in x, for each $\mu=0, 1, 2, \ldots$,

and
$$\lim_{\mu\to\infty} \Lambda_{\mu, x}(f) = \Lambda_x(f)$$

exists uniformly in x, then (1) is said to be uniformly summable–Λ. The Λ–summation method is not necessarily regular.

Now let $\{\lambda_\nu\}$ be an infinite sequence of real numbers. If the series $\sum_{\nu=0}^{\infty} \lambda_\nu c_\nu \phi_\nu(x)$ converges uniformly in x, then $\{\lambda_\nu\}$ is called a sequence of uniform convergence factors of orthonormal expansions of $f(x)$.

If (1) holds, then

$$S_n(f\,;x)=\sum_{\nu=0}^{n} \lambda_\nu c_\nu \phi_\nu (x)=\int_0^1 f(t)\Big(\sum_{\nu=0}^{n} \lambda_\nu \phi_\nu(x)\phi_\nu (t)\Big)\, dt$$

$$=\int_0^1 f(t)K_n(x,t)\, dt$$

where

$$K_n(x,t)=\sum_{\nu=0}^{n} \lambda_\nu \phi_\nu(x)\phi_\nu(t).$$

In this paper, the orthonormal system $\{\phi_\nu(x)\}$ is considered closed in C, the class of continuous functions on $[0, 1]$, and $|\phi_\nu(x)|<A_\nu$, where the A_ν are independent of x.

In [3] the following theorem on uniform convergence factors is proved.

Theorem 1. The necessary and sufficient conditions that $\{\lambda_\nu\}$ be a sequence of uniform convergence factors of orthonormal expansions of $f(x)\in L^\infty$, are

(2) that there exists an M such that

$$\int_0^1 |K_n(x,t)|\, dt<M$$

for all x and n.

(3) For every $\epsilon>0$, there exist N and $\eta>0$ such that for any measurable subset $H \cup [0, 1]$ of $m(H)<\eta$, we have

$$\left|\int_H (K_m(x,t)-K_n(x,t))\, dt\right|<\epsilon$$

for all $m, n>N$ and for all x.

We note that if (1) holds. then

$$\Lambda_{n,\mu,x}(f)=\sum_{\nu=0}^{n} \lambda_{\nu\mu} c_\nu\phi_\nu(x)=\int_0^1 f(t)\Big(\sum_{\nu=0}^{n} \lambda_{\nu\mu}\phi_\nu(x)\phi_\nu(t)\, dt\Big)$$

$$=\int_0^1 f(t)K_{n,\mu}(x,t)\, dt$$

where

$$K_{n,\mu}(x,t)=\sum_{\nu=0}^{n} \lambda_{\nu\mu}\,\phi_\nu(x)\phi_\nu(t).$$

We also write

$$\wedge_{\mu,\, x}(f)=\lim_{n\to\infty} \wedge_{n,\, \mu,\, x}(f)=\sum_{\nu=0}^{\infty} \lambda_{\nu\mu}\, c_\nu\, \phi_\nu(x)$$

$$=\lim_{n\to\infty} \int_0^1 f(t)\, K_{n,\,\mu}(x,\, t)\, dt.$$

Now if

$$\lim_{n\to\infty} \int_0^1 f(t) K_{n,\,\mu}(x,\, t)\, dt$$

exists for every $f(t) \in L^{\infty}$, then $\{K_{n,\,\mu}(x,\, t)\}$ converges weakly in L (class of summable functions). But since L is weakly complete ([2], p. 240), there exists a $K_{\mu}(x,\, t) \in L$ such that

$$\wedge_{\mu,\, x}(f)=\lim_{n\to\infty} \int_0^1 f(t)K_{n,\,\mu}(x,\, t)\, dt=\int_0^1 f(t)K_{\mu}(x,\, t)\, dt.$$

2. Main Result.

Theorem 2. The necessary and sufficient conditions for uniform summability-$\wedge$ of orthonormal expansions of $f(x) \in L^{\infty}$, are

(4) $\quad \lim_{\mu\to\infty} \lambda_{\nu\mu}$ exists ($\nu=0, 1, 2, \ldots$)

(5) $\quad \int_0^1 \left| K_{n,\,\mu}(x,\, t)\right| dt < M_{\mu}$ ($\mu=0, 1, 2, \ldots$)

for all x and n.

(6) For every $\epsilon>0$, there exists N and $\eta>0$ such that for any measurable subset $H\subset[0,\, 1]$ of $m(H)<\eta$, we have

$$\left|\int_H (K_{m,\,\mu}(x,\, t)-K_{n,\,\mu}(x,\, t))\, dt\right| <\epsilon \quad (\mu=0, 1, 2, \ldots)$$

for all $m, n>N$ and for all x.

(7) $$\int_0^1 \left| K_m(x,\, t)\right| dt<M$$

for all x and μ.

(8) For every $\epsilon>0$, there exists μ and $\eta>0$ such that for any measurable subset $H\subset[0,\, 1]$ of $m(H)<\eta$, we have

$$\left|\int_H (K_{\mu'}(x,\, t)-K_{\mu''}(x,\, t))\, dt\right| < \epsilon$$

for all $\mu',\, \mu''>\mu$ and for all x.

PROOF. In order that $\{\lambda_{\nu\mu}\}$ for each fixed $\mu=0, 1, 2, \ldots$, be a sequence of uniform convergence factors of orthonormal expansions of $f(x) \in L^{\infty}$, by Theorem 1, the conditions (5) and (6) are necessary and sufficient. We first establish the necessity of conditions (4), (7) and (8).

Since

$$\Lambda_{\mu, x}(f)$$

converges uniformly in x, for every $f(t) \in L^{\infty}$, and by hypothesis

$$|\phi_k(t)| < A_k, \text{ independent of } t,$$

therefore,

$$\lim_{\mu\to\infty} \Lambda_{\mu, x}(\phi_k(t)) = \lim_{\mu\to\infty} \int_0^1 \phi_k(t) \left(\sum_{\nu=0}^{\infty} \lambda_{\nu\mu} \phi_\mu(x)\, \phi_\nu(t) \right) dt$$
$$= \lim_{\mu\to\infty} \lambda_{k\mu}\ \phi_k(x)$$

exists uniformly in x, for every k.

This implies that

$$\lim_{\mu\to\infty} \lambda_{k\mu}$$

exists for every k.

The necessity of condition (4) is established.

Now suppose we contradict (7). Then there exists a sequence of indices $\{\mu_i\}$ and a sequence of points $\{x_i\}$ such that

$$\overline{\lim}_{i\to\infty} \int_0^1 \left| K_{\mu_i}(x_i, t) \right| dt = +\infty. \tag{9}$$

But, by Banach-Steinhaus Theorem ([2], p. 80), for every $f(t) \in L^{\infty}$,

$$\overline{\lim}_{i\to\infty} \int_0^1 f(t) K_{\mu_i}(x_i, t) dt = +\infty \tag{10}$$

which contradicts the uniform convergence of $\Lambda_{\mu, x}(f)$.

Hence the necessity of condition (7).

Now we contradict (8). Then for some $\in > 0$, there exist two increasing sequences of positive integers $\{\mu_i'\}$ and $\{\mu_i''\}$, a sequence of points $\{x_i\}$, and a sequence of sets $\{H_i\}$ with $m(H_i) \to 0$ as $i \to \infty$, such that

$$\left| \int_{H_i} (K_{\mu_i'}(x_i, t) - K_{\mu_i''}(x_i, t))\, dt \right| \geqslant \in. \tag{11}$$

Now put $K_{\mu_i'}(x_i, t)-K_{\mu_i''}(x_i, t)=\alpha_i(t)$.

Since

$$\lim_{\mu\to\infty} \wedge_{\mu, x}(f)$$

exists uniformly in x, it follows that

$$\lim_{i\to\infty} \int_0^1 f(t)\, \alpha_i(t) dt=0$$

for all $f(t) \in L^\infty$.

Therefore, by the Theorem of Lebesgue ([2], pp. 7-8), *viz.*, for every $\epsilon>0$, there exists an $\eta>0$ such that for any subset $H\subset[0, 1]$ of $m(H)<\eta$, we have

$$\left|\int_H \alpha_i(t)\, dt\right| <\epsilon \quad (i=1, 2, 3,...)$$

which contradicts (11). Hence the necessity of condition (8).

Now it remains to show the sufficiency of conditions (4), (7) and (8). Let $\epsilon>0$, then by Luzin's Theorem, for every $f(t)\in L^\infty$, there exist $h(t)\in C$ and an $\eta>0$ such that

(12) $$|f(t)-h(t)| <\epsilon/2,$$

except in a set $H\subset[0, 1]$, $m(H)<\eta$.

Since $\{\phi_\nu(t)\}$ is closed in C, there exists $g(t)=\sum_{\rho=0}^{k} \gamma_\rho\, \phi_\rho(t)$ such that

(13) $$|h(t)-g(t)| <\epsilon/2$$

in $[0, 1]$.

Combining (12) and (13), we get

(14) $$|f(t)-g(t)| <\epsilon$$

for all $t \in [0, 1]-H$, where $m(H)<\eta$.

Above we note that

(15) $$\wedge_{\mu, x}(g)=\lim_{n\to\infty} \wedge_{n, \mu, x}(g)$$

$$=\lim_{n\to\infty} \int_0^1 \left(\sum_{\rho=0}^{k} \gamma_\rho\, \phi_\rho(t)\right) \left(\sum_{\nu=0}^{n} \lambda_{\nu\mu}\phi_\nu(x)\phi_\nu(t)\right) dt$$

$$=\sum_{\rho=0}^{k} \gamma_\rho\, \lambda_{\rho\mu}\, \phi_\rho(x).$$

Now consider

$$(16)\quad \left|\Lambda_{\mu',x}(f)-\Lambda_{\mu'',x}(f)\right|$$

$$=\left|\int_0^1 f(t)\,(K_{\mu'}(x,t)-K_{\mu''}(x,t))dt\right|$$

$$\leqslant\left|\int_{[0,1]-H}(f(t)-g(t))\,(K_{\mu'}(x,t)-K_{\mu''}(x,t))\,dt\right|$$

$$+\left|\int_H (f(t)-g(t))(K_{\mu'}(x,t)-K_{\mu''}(x,t))\,dt\right|$$

$$+\left|\int_0^1 g(t)(K_{\mu'}(x,t)-K_{\mu''}(x,t))\,dt\right|=I_1+I_2+I_3.$$

From (14) and (7)

$$(17)\quad I_1\leqslant\int_{[0,1]-H}\left|f(t)-g(t)\right|\left|K_{\mu''}(x,t)-K_{\mu'}(x,t)\right|dt$$

$$\leqslant \in\left(\int_0^1\left|K_{\mu'}(x,t)\right|dt+\int_0^1\left|K_{\mu''}(x,t)\right|dt\right)\leqslant 2M\in.$$

(18) Let $H=H_1+H_2$, such that

$$(f(t)-g(t))\,(K_{\mu'}(x,t)-K_{\mu''}(x,t))\geqslant 0 \text{ on } H_1$$

and

$$(f(t)-g(t))\,(K_{\mu'}(x,t)-K_{\mu''}(x,t))<0 \text{ on } H_2.$$

Obviously, $m(H_1)<\eta$ and $m(H_2)<\eta$.

Now by (18) and (8)

$$(19)\quad I_2\leqslant\left|\int_{H_1}(f(t)-g(t))\,(K_{\mu'}(x,t)-K_{\mu''}(x,t))\,dt\right|$$

$$+\left|\int_{H_2}(f(t)-g(t))\,(K_{\mu'}(x,t)-K_{\mu''}(x,t)\,dt\right|$$

$$\leqslant(\text{ess sup}\,|f(t)|+\sup|g(t)|)$$

$$\times\left\{\left|\int_{H_1}(K_{\mu'}(x,t)-K_{\mu''}(x,t))\,dt\right|\right.$$

$$\left.+\left|\int_{H_2}(K_{\mu'}(x,t)-K_{\mu''}(x,t))\,dt\right|\right\}$$

$$\leqslant(\text{ess sup}\,|f(t)|+\sup|g(t)|)\,2\in$$

for all $\mu', \mu''>\mu_1$, and for all x.

From (15) and (4) it is clear that

$$(20)\qquad I_3=0$$

for all $\mu', \mu''>\mu_2$, and for all x.

Thus (16) in conjunction with (17), (19) and (20) yields

$$\left| \wedge_{\mu', x}(f) - \wedge_{\mu'', x}(f) \right| \leqslant 2M\in + (\text{ess sup}) f \mid (t) \mid + \sup \mid g(t) \mid) \, 2\in$$

for all μ', $\mu'' > \max(\mu_1, \mu_2)$, and for all x.

Hence $\wedge_{\mu, x}(f)$ converges uniformly in x.

This evidently completes the proof of Theorem 2.

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