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PERIODIC ORBITS OF COLLISION IN THE THREE-DIMENSIONAL RESTRICTED PROBLEM OF THREE BODIES

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Abstract. The existence of periodic orbits of collision in the three dimensional elliptic restricted problem have been examined. However, the eccentricity of the ellipses described by the primaries and the z-coordinate of the infinitesimal mass are taken of the order μ, where μ is the mass of the smaller primary and $1-\mu$, the mass of the bigger primary.

1. Equations of motion. Let μ and $1-\mu$ be the masses of two primaries (finite masses) which are moving in elliptic orbits around their centre of mass. We take the origin at the centre of mass and the plane of motion of the primaries as xy-plane and the line joining μ and $1-\mu$ as x-axis. Let the coordinate system rotate with the variable angular velocity $\dot{f}$ about z-axis where f is the true anomaly of one primary around the other. This introduction of a non-uniformly rotating and pulsating coordinate system results in a fixed location of the primaries. Let the coordinates of the infinitesimal mass be (x, y, z).

The equations of motion of the infinitesimal mass under the gravitational field of the two primaries are given by

$$\left.\begin{aligned} \frac{d^2x}{df^2}-2\frac{dy}{df}&=\frac{\partial V}{\partial x}, \\ \frac{d^2y}{df^2}+2\frac{dx}{df}&=\frac{\partial V}{\partial y}, \\ \frac{d^2z}{df^2}&=\frac{\partial V}{\partial z}, \end{aligned}\right\} \tag{1}$$

where $$V=\frac{\Omega}{1+e'\cos f},$$

$$\Omega=\tfrac{1}{2}[(1-\mu)r_1^2+\mu r_2^2]+\frac{1-\mu}{r_1}+\frac{\mu}{r_2}-\frac{1}{2}\,(1+e'\cos f)z^2,$$

with
$$r_1^2=(x-\mu)^2+y^2+z^2$$
$$r_2^2=(x-\mu+1)^2+y^2+z^2$$

and e' is the eccentricity of the elliptic orbit described by the primary bodies.

2. Regularisation of the solution. For regularising the solution, let us first reduce the equation (1) to canonical form. For this let us introduce the variables

$$\left.\begin{aligned} &x_1=x-\mu\ ;\ x_2=y\ ;\ x_3=z\\ &p_1=\dot{x}-y=\dot{x}_1-x_2\\ &p_2=\dot{y}+x-\mu=\dot{x}_2+x_1\\ &p_3=\dot{x}_3 \end{aligned}\right\}\tag{2}$$

where ($\cdot$) denotes differentiation with respect to f.

The equations of motion (1) become

$$\left.\begin{aligned} &\frac{dx_i}{df}=\frac{\partial H}{\partial p_i}\\ &\frac{dp_i}{df}=-\frac{\partial H}{\partial x_i}\quad(i=1,2,3) \end{aligned}\right\}\tag{3}$$

where
$$H=\tfrac{1}{2}[p_1^2+p_2^2+p_3^2+x_1^2+x_2^2+x_3^2]+(p_1x_2-p_2x_1)$$
$$-\frac{1}{1+e'\cos f}\left[\{\tfrac{1}{2}(1-\mu)r_1^2+\mu r_2^2\}+\frac{1-\mu}{r_1}+\frac{\mu}{r_2}\right]-\tfrac{1}{2}x_3^2\tag{4}$$

Now for regularisation of the solution, we shall introduce Livi-Civita's (1906) parabolic transformation which may be defined as canonical transformation generated by

$$\left.\begin{aligned} &S=(\xi_1^2-\xi_2^2)p_1+2\xi_1\xi_2p_2+\xi_3p_3\\ &\text{such that}\quad x_i=\frac{\partial S}{\partial p_i},\ \pi_i=\frac{\partial S}{\partial \xi_i}\ (i=1,2,3) \end{aligned}\right\}\tag{5}$$

where π_i are the momenta associated with the new coordinates ξ_i.

The equations of motion (3) in terms of the new variables become,

$$\left.\begin{aligned} &\frac{d\xi_i}{df}=\frac{\partial H}{\partial \pi_i}\\ &\frac{d\pi_i}{df}=-\frac{\partial H}{\partial \xi_i}(i=1,2,3) \end{aligned}\right\}\tag{6}$$

where H is given by

$$H=\frac{\pi^2}{8\xi^2}+\frac{1}{2}\pi_3^2+\frac{\xi^4}{2}+\frac{1}{2}(\pi_1\xi_2-\pi_2\xi_1)$$
$$-\frac{1}{1+e'\cos f}\left[\frac{1}{2}\left\{\frac{1}{2}(1-\mu)r_1^2+\mu r_2^2\right\}+\frac{1-\mu}{r_1}+\frac{\mu}{r_2}\right]$$

and

$$r_1^2=\xi^4+\xi_3^2,\ \xi^2=\xi_1^2+\xi_2^2$$
$$r_2^2=1+\xi^4+2(\xi_1^2-\xi_2^2)+\xi_3^2$$
$$\pi^2=\pi_1^2+\pi_2^2.$$

Jacobi's integral (Szebehely, 1967) may be written as

$$H+I=C$$

where
$$I=\int_0^f \frac{e'\Omega\sin f}{(1+e'\cos f)^2}\,df \tag{7}$$

Now, we introduce a new independent variable τ instead of f defined by

$$df=r_1\,d\tau,\ f=0 \text{ at } \tau=0. \tag{8}$$

The equations of motion (6) will be transformed to

$$\left.\begin{aligned}\frac{d\xi_i}{d\tau}&=\frac{\partial K}{\partial\pi_i}\\ \frac{d\pi_i}{d\tau}&=-\frac{\partial K}{\partial\xi_i}\qquad (i=1,2,3)\end{aligned}\right\} \tag{9}$$

where K is the new Hamiltonian, given by

$$K=r_1(H-C)+\int_0^{r_1} I\,dr_1$$
$$=r_1\left[\frac{\pi^2}{8\xi^2}+\frac{\pi_3^2}{2}+\frac{\xi^4}{2}+\frac{(\pi_1\xi_2-\pi_2\xi_1)}{2}\right.$$
$$\left.-\frac{1}{1+e'\cos f}\left\{\frac{(1-\mu)r_1^2}{2}+\frac{\mu r_2^2}{2}+\frac{1-\mu}{r_1}+\frac{\mu}{r_2}\right\}-C\right]$$
$$+\int_0^{r_1}\int_0^f\frac{e'\sin f}{(1+e'\cos f)^2}\left[\frac{1}{2}\left\{(1-\mu)r_1^2+\mu r_2^2\right\}+\frac{1-\mu}{r_1}+\frac{\mu}{r_2}\right.$$
$$\left.-\frac{1}{2}(1+e'\cos f)\xi_3^2\right]dr_1\,df$$

Suppose that e' and ξ_3 are small quantities of the $o(\mu)$, we may take

$$r_1=\xi^2+o(\mu)$$

$e'=\mu e_1$, neglecting second order terms, we get

$$K=K_0+\mu K_1,\ \text{where}$$

$$K_0=\frac{\pi^2}{8}+\frac{1}{2}\ \xi^2\pi_3{}^2+\frac{1}{2}\ \xi^2\ (\pi_1\xi_2-\pi_2\,\xi_1-2\,C_0)-1=-\epsilon,\ (\text{say}) \tag{10}$$

$$K_1=\frac{r_1{}^3}{2}-\frac{1}{2}\,r_1\ r_2{}^2+1\ -\frac{r_1}{r_2}-C_1r_1 \tag{11}$$

The form given to K_0 ensures that the orbits, which are analytically continued from the two-body orbit will belong to $K=0$ manifold, that is, are solutions of the regularised equations of the restricted problem (Giacaglia, 1967).

With Giacaglia, we shall assume that K_0 is negative and so the corresponding two-body problem will admit bounded orbits as a solution in a rotating system of coordinates. It is easy to see that $|\epsilon|<1$.

3. Generating Solution. For Generating Solution, we shall choose K_0 to be our Hamiltonian function Since f is not involved explicitly in the equation (10), so the Hamiltonian-Jacobi equation may be written as

$$\frac{1}{8}\left[\left(\frac{\partial\omega}{\partial\xi_1}\right)^2+\left(\frac{\partial\omega}{\partial\xi_2}\right)^2\right]+\frac{1}{2}\xi^2\left(\frac{\partial\omega}{\partial\xi_3}\right)^2$$
$$+\frac{1}{2}\ \xi^2\left[\xi_2\,\frac{\partial\omega}{\partial\xi_1}-\xi_1\frac{\partial\omega}{\partial\xi_2}-2\,C_0\right]-1=-\epsilon, \tag{12}$$

where $$\pi_i=\frac{\partial\omega}{\partial\xi_i}\quad(i=1,\,2,\,3)$$

Putting $\xi_1=\xi\cos\phi$, $\xi_2=\xi\sin\phi$, the equation (12) becomes

$$\frac{1}{8}\left[\left(\frac{\partial\omega}{\partial\xi}\right)^2+\frac{1}{\xi^2}\left(\frac{\partial\omega}{\partial\phi}\right)^2\right]+\frac{1}{2}\,\xi^2\left(\frac{\partial\omega}{\partial\xi_3}\right)^2+\frac{\xi^2}{2}\left[-\frac{\partial\omega}{\partial\phi}-2\,C_0\right]=\alpha, \tag{13}$$

where $$\alpha=1-\epsilon>0.$$

Proceeding as in (Bhatnagar, 1969), the solution of (13) may be written as

$$\omega=u(\xi)+2G\ \phi+\bar{H}\ \xi_3 \tag{14}$$

$$u(z,\,G,\alpha)=[H^2-2(G+C_0)]^{1/2}\int_{z_1}^{z}\sqrt{f(z)}\,\frac{dz}{z} \tag{15}$$

$$z=\xi^2 \tag{16}$$

Here z_1 is the smaller of the roots of the equation

$$f(z)=-\left[z^2-\frac{2\alpha z}{H^2-2(G+C_0)}+\frac{G^2}{H^2-2(G+C_0)}\right] \tag{17}$$

Let the other root be z_2. For a general solution we need only two arbitrary constants and for these constants, we have, α and G. Therefore the solution may be regarded a general solution.

We introduce the parameters a, e, l by the relations

$$z_1=a\,(1-e)\,;\; z_2=a\,(1+e)\,;\; z=a\,(1-e\cos l) \tag{18}$$

and take

$$\alpha=L[\bar{H}^2-2\,(G+C_0)]^{1/2}>0.$$

From equation (18), we get

$$a=\frac{L}{[\bar{H}^2-2(G+C_0)]^{1/2}} \tag{19}$$

$$e=\left[1-\frac{z_1 z_2}{a^2}\right]^{1/2}=\left[1-\frac{G^2}{L^2}\right]^{1/2}\leqslant 1. \tag{20}$$

From (17), (18), (19) and (20), we get

$$f(z)=a^2e^2\,sin^2\,l \tag{21}$$

The equations of motion associated with K_0 are

$$\left.\begin{aligned}\frac{d\xi_1}{d\tau}&=\frac{\partial K_0}{\partial\pi_1}=\frac{\pi_1}{4}+\frac{1}{2}\,\xi^2\xi_2\\ \frac{d\xi_2}{d\tau}&=\frac{\partial K_0}{\partial\pi_2}=\frac{\pi_2}{4}-\frac{1}{2}\,\xi^2\xi_1\\ \text{and}\quad\frac{d\xi_3}{d\tau}&=\frac{\partial K_0}{\partial\pi_3}=\pi_3\xi^2\end{aligned}\right\} \tag{22}$$

It is easy to see that

$$[\bar{H}^2-2(G+C_0)]^{1/2}\,(\tau-\tau_0)=\int_{z_1}^{z}\frac{dz}{\sqrt{f(z)}},$$

where $z=z_1$ at $\tau=\tau_0$.

or

$$[\bar{H}^2-2(G+C_0)]^{1/2}\,(\tau-\tau_0)=\int_0^l\frac{ae\sin l}{ae\sin l}\,dl=l$$

Hence

$$l=[\bar{H}^2-2\,(G+C_0)]^{1/2}\,(\tau-\tau_0) \tag{23}$$

The parameters a, e, l are, therefore, given by (19), (20) and (23). From equations (14) and (15), we can easily show that

$$\left.\begin{aligned}\frac{\partial\omega}{\partial L}&=\frac{\partial U}{\partial L}=\int_{z_1}^{z}\frac{dz}{\sqrt{f(z)}}=l\\ \frac{\partial\omega}{\partial G}&=2\phi+\frac{\partial U}{\partial G}=2\phi-f-\frac{\sqrt{L^2-G^2}}{\bar{H}^2-2(G+C_0)}\sin\,l=g\\ f&=\sqrt{1-e^2}\int\frac{dl}{1-e\cos l}\\ \frac{\partial\omega}{\partial\bar{H}}&=\xi_3+\frac{\bar{H}[L^2-G^2]^{1/2}}{\bar{H}^2-2(G+C_0)}\sin l=h\end{aligned}\right\} \tag{24}$$

Equations (24) establish the canonical set $(l, L\,;\, g, G\,;\, h, \overline{H})$. We may observe that when $e=1$, we have $G=0, f=0$.

Since $K_0=\alpha-1$, it follows that

$$K_0=L[\overline{H}^2-2(G+C_0)]^{1/2}-1>0,$$

and therefore, for the problem generated by this Hamiltonian, we have

$$\left.\begin{aligned}
&\frac{dL}{d\tau}=-\frac{\partial K_0}{\partial L}=0 && L=\text{const.}=L_0\\
&\frac{dG}{d\tau}=-\frac{\partial K_0}{\partial g}=0 && G=\text{const.}=G_0\\
&\frac{d\overline{H}}{d\tau}=-\frac{\partial K_0}{\partial h}=0 && \overline{H}=\text{const}=H_0\\
&\frac{dl}{d\tau}=\frac{\partial K_0}{\partial L}=[\overline{H}^2-2(G+C_0)]^{1/2}=\text{const.}=n_l\\
&\qquad\qquad\therefore\ l=n_l\tau+l_0\\
&\frac{dg}{d\tau}=\frac{\partial K_0}{\partial g}=\frac{-L}{[\overline{H}^2-2(G+C_0)]^{1/2}}=\text{const.}=n_g\\
&\qquad\qquad\therefore\ g=n_g\tau+g_0\\
&\frac{dh}{d\tau}=\frac{\partial K_0}{\partial \overline{H}}=\frac{L\overline{H}}{[\overline{H}^2-2(G+C_0)]^{1/2}}=\text{const.}=n_h\\
&\qquad\qquad\therefore\ h=n_h\tau+h_0,
\end{aligned}\right\}\tag{25}$$

where l_0, g_0, h_0 are the values of l, g, h at $\tau=0$.

The angle ϕ is given by

$$\phi=\tfrac{1}{2}[f+g]+\tfrac{1}{2}\frac{\sqrt{L^2-G^2}}{\overline{H}^2-2(G+C_0)}\sin l,\ e\neq 1 \tag{26}$$

$$\phi=\tfrac{1}{2}g+\frac{L}{2(\overline{H}^2-2C_0)}\sin l \qquad ,\ e=1 \tag{27}$$

We can easily express the variables ξ_i, π_i in terms of the canonical elements $(l, L, g, G, h, \overline{H})$. We have

$$\left.\begin{aligned}
\pm\xi_1&=\sqrt{z}\cos\phi=\sqrt{a(1-e\cos l)}\cos\phi\\
\pm\xi_2&=\sqrt{z}\sin\phi=\sqrt{a(1-e\cos l)}\sin\phi,\\
\xi_3&=h-\frac{\overline{H}(L^2-G^2)^{1/2}}{\overline{H}^2-2(G+C_0)}\sin l,
\end{aligned}\right\}\tag{28}$$

Therefore

$$\left.\begin{aligned}
\pm\pi_1&=\frac{2e\,L\sin l\cos\phi-2G\sin\phi}{\pm\sqrt{a(1-e\cos l)}}\\
\pm\pi_2&=\frac{2e\,L\sin l\sin\phi+2G\cos\phi}{\pm\sqrt{a(1-e\cos l)}}\\
\pi_3&=\overline{H},
\end{aligned}\right\}\tag{29}$$

where ϕ is given by equation (26).

When $e=1$ $(G=0)$, we have

$$\left.\begin{aligned}
\pm\xi_1 &= \sqrt{2a}\sin l/2\cos\phi\\
\pm\xi_2 &= \sqrt{2a}\sin l/2\sin\phi,\\
\xi_3 &= h-\frac{\bar{H}L}{\bar{H}^2-2C_0}\sin l,\\
\pm\pi_1 &= \frac{4L}{\sqrt{29}}\cos l/2\cos\phi,\\
\pm\pi_2 &= \frac{4L}{\sqrt{29}}\cos l/2\sin\phi,\ \pi_3=\bar{H},
\end{aligned}\right\} \tag{30}$$

where ϕ is given by equation (27).

The original synodic cartesian coordinates in a non-uniformly rotating system are obtained from equations (29) or (30) and equation (5). When $\mu=0$, we have

$$\left.\begin{aligned}
x_1 &= \xi_1^2-\xi_2^2\ ;\ x_2=2\xi_1\xi_2\ ;\ x_3=\xi_3\\
p_1 &= \frac{1}{2z}(\pi_1\xi_1-\pi_2\xi_2)\\
p_2 &= \frac{1}{2z}(\xi_1\pi_2+\xi_2\pi_1)\\
p_3 &= \pi_3
\end{aligned}\right\} \tag{31}$$

Here $$z=a\,(1-e\cos l).$$

In the uniformly rotating system, the coordinates $(\bar{\xi}, \bar{\eta}, \bar{\phi})$ are given by

$$\left.\begin{aligned}
\bar{\xi} &= rx_1\ ;\ \bar{\eta}=rx_2\ ;\ \bar{\phi}=rx_3\\
\dot{\bar{\xi}} &= \dot{r}x_1+r\dot{x}_1\\
\dot{\bar{\eta}} &= \dot{r}x_2+r\dot{x}_2\\
\dot{\bar{\phi}} &= \dot{r}x_3+r\dot{x}_3,
\end{aligned}\right\} \tag{32}$$

where $\dot{r}$ is worked out from

$$\frac{a'(1-e'^2)}{r}=1+e'\cos f.$$

The sidreal cartesian coordinate are obtained by considering the transformation

$$\left.\begin{aligned}
X_1 &= \bar{\xi}\cos f-\bar{\eta}\sin f\\
X_2 &= \bar{\xi}\sin f+\bar{\eta}\cos f\\
X_3 &= \bar{\phi}\\
\dot{X}_1 &= (\dot{\bar{\xi}}-\bar{\eta})\cos f-(\bar{\xi}+\dot{\bar{\eta}})\sin f\\
\dot{X}_2 &= (\dot{\bar{\xi}}-\bar{\eta})\sin f+(\bar{\xi}+\dot{\bar{\eta}})\cos f\\
\dot{X}_3 &= \dot{\bar{\phi}}
\end{aligned}\right\} \tag{33}$$

All the above differentiations are with respect to f, where f is given by

$$df = r_1\, d\tau$$

or
$$f - f_0 = \frac{a}{[\overline{H}^2 - 2(G + C_0)]^{1/2}} [l - e \sin l], \tag{34}$$

and f_0 is a constant.

In terms of the canonical variables introduced, the complete Hamiltonian may be written as

$$\begin{aligned} K &= K_0 + \mu K_1 \\ &= L\,[\overline{H}^2 - 2(G + C_0)]^{1/2} - 1 + \mu\Big[\frac{1}{2}\, r_1^3 - \frac{r_1 r_2^2}{2} + 1 - \frac{r_1}{r_2} \\ &\qquad + e_1 \cos f\,(1 + \tfrac{1}{2} r_1^3) + r_1\, \frac{C_0 - C}{\mu} \\ &\quad + \int_0^{r_1}\int_0^{f} \frac{e_1 \sin f}{(1 + \mu e_1 \cos f)^2} \times \Big\{ \tfrac{1}{2}(\overline{1-\mu}\, r_1^2 + \mu r_2^2) \\ &\qquad + \frac{1-\mu}{r_1} + \frac{\mu}{r_2} - \frac{1}{2}(1 + \mu e_1 \cos f)\xi_3^2 \Big\} dr_1\, df \Big] \end{aligned} \tag{35}$$

where ξ_1, ξ_2 and ξ_3 are given by (28). If we denote the coefficient of μ by R after K has been transformed completely in terms of the canonical variabe, we can write the complete Hamiltonian as

$$K = L[\overline{H}^2 - 2\,(G + C_0)]^{1/2} - 1 + \mu\, R + o(\mu^2)$$

The equations of motion for the complete Hamiltonian are

$$\left.\begin{aligned} \frac{dl}{d\tau} &= \frac{\partial K}{\partial L} = [\overline{H}^2 - 2\,(G + C_0)]^{1/2} + \mu\, \frac{\partial R}{\partial L} \\ \frac{dg}{d\tau} &= \frac{\partial K}{\partial G} = \frac{-L}{[\overline{H}^2 - 2\,(G + C_0)]^{1/2}} + \mu\, \frac{\partial R}{\partial G} \\ \frac{dh}{d\tau} &= \frac{\partial K}{\partial \overline{H}} = \frac{L\overline{H}}{[\overline{H}^2 - 2(G + C_0)]^{1/2}} + \mu\, \frac{\partial R}{\partial \overline{H}} \\ \frac{dL}{d\tau} &= -\frac{\partial K}{\partial l} = -\mu \frac{\partial R}{\partial l} \\ \frac{dG}{d\tau} &= -\frac{\partial K}{\partial g} = -\mu \frac{\partial R}{\partial g} \\ \frac{d\overline{H}}{d\tau} &= -\frac{\partial K}{\partial h} = -\mu \frac{\partial R}{\partial h} \end{aligned}\right\} \quad \text{....(36)}$$

These equations form the basis of a general perturbation theory for the problem in question.

4. Existence of periodic orbits when $\mu \neq 0$. We shall follow the procedure as in (Choudhry, 1966) for proving the existence of periodic orbits when $\mu \neq 0$.

When $\mu=0$, the equations (36) become

$$\left.\begin{aligned}
\frac{dl}{d\tau}&=\frac{\partial K_0}{\partial L}=[\bar{H}^2-2(G+C_0)]^{1/2}=\text{const.}\\
\frac{dg}{d\tau}&=\frac{\partial K_0}{\partial G}=\frac{-L}{[\bar{H}^2-2(G+C_0)]^{1/2}}=\text{const.}\\
\frac{dh}{d\tau}&=\frac{\partial K_0}{\partial \bar{H}}=\frac{L\bar{H}}{[\bar{H}^2-2(G+C_0)]^{1/2}}=\text{const.}\\
\frac{dL}{d\tau}&=-\frac{\partial K_0}{\partial l}=0\\
\frac{dG}{d\tau}&=-\frac{\partial K_0}{\partial g}=0\\
\frac{d\bar{H}}{d\tau}&=-\frac{\partial K_0}{\partial h}=0
\end{aligned}\right\} \tag{37}$$

Let $x_1=L,\ x_2=G,\ x_3=\bar{H}\ ;\ y_1=l,\ y_2=g,\ y_3=h.$

The equations (37) may be written as

$$\left.\begin{aligned}
&\frac{dx_i}{d\tau}=0\ ;\ \frac{dy_i}{d\tau}=n_i^{(0)}\ \text{(say)}\\
\text{i e.}\quad &x_i=a_i\ ;\ y_i=n_i^{(0)}+\tau+\omega_i\quad (i=1,\ 2,\ 3)
\end{aligned}\right\} \tag{38}$$

These are the generating solution of the problem of two bodies in a three dimensional coordinate system. Here a_i, ω_i $(i=1,\ 2,\ 3)$ are constants.

Further
$$\left.\begin{aligned}
n_1^{(0)}&=\left(-\frac{\partial K_0}{\partial x_1}\right)_{x_1=a_1}\\
n_2^{(0)}&=\left(-\frac{\partial K_0}{\partial x_2}\right)_{x_2=a_2}\\
n_3^{(0)}&=\left(-\frac{\partial K_0}{\partial x_3}\right)_{x_3=a_3}
\end{aligned}\right\} \tag{39}$$

The generating solution will be periodic with the period τ_0 if

$$\left.\begin{aligned}
&x_i\ (\tau_0)-x_i\ (0)=0\\
&y_i\ (\tau_0)-y_i\ (0)=n_i^{(0)}\ \ \tau_0=2K_i\pi
\end{aligned}\right\} \tag{40}$$

Here K_i $(i=1,\ 2,\ 3)$ are integers so that $n_i^{(0)}$ are commensurable.

Let the general solution in the neighbourhood of the generating solution be periodic with the period $\tau_0\ (1+\alpha)$ where α is a negligible quantity of the $o(\mu)$. Let us introduce a new independent variable ϕ by the equation

$$\phi=\frac{\tau}{1+\alpha}.$$

The period of the general solution will then be τ_0. This period coincides with that of the generating solution.

The equations of motion can be written as

$$\left.\begin{aligned} \frac{dx_i}{d\phi} &= -(1+\alpha)\frac{\partial K}{\partial y_i} \\ \frac{dy_i}{d\phi} &= -(1+\alpha)\frac{\partial K}{\partial x_i} \quad (i=1,2,3) \end{aligned}\right\} \tag{41}$$

Let us take the general solution in the neighbourhood of the generating solution as

$$x_i = a_i + \beta_i + \xi_i(\phi)$$
$$y_i = n_i^{(0)}\phi + \omega_i + \nu_i + \eta_i(\phi), \; (i=1,2,3)$$

Then ξ_i, η_i are given by the equations (Bhatnagar, 1969)

$$\frac{\xi_k(\tau_0, \beta_i, \nu_i, \mu)}{-\mu\,\tau_0} = \frac{\partial [K_1]}{\partial \omega_k} + \sum_{i=1}^{3} \frac{\partial^2 [K_1]}{\partial \omega_k\, \partial a_i} + \sum_{i=1}^{3} \frac{\partial^2 [K_1]}{\partial \omega_k\, \partial \omega_i} = 0 \tag{42}$$

where $$[K_1] = \frac{1}{\tau_0}\int_0^{\tau_0} K_1\{\phi, a_i, n_i^{(0)}\phi + \omega\}\, d\phi$$

$$\eta_1\left(\tau_0, \beta_i, \nu_i, \mu\right) = \alpha\,\tau_0 \frac{\partial K_0}{\partial a_1} + \beta_1 \tau_0 \frac{\partial^2 K_0}{\partial a_1^2} + \beta_2 \tau_0 \frac{\partial^2 K_0}{\partial a_1\, \partial a_2} + \beta_3 \tau_0 \frac{\partial^2 K_0}{\partial a_1\, \partial a_3} + o(\mu) = 0. \tag{43}$$

$$\eta_2\left(\tau_0, \beta_i, \nu_i, \mu\right) = \alpha\,\tau_0 \frac{\partial K_0}{\partial a_2} + \beta_1 \tau_0 \frac{\partial^2 K_0}{\partial a_1\, \partial a_2} + \beta_2 \tau_0 \frac{\partial^2 K_0}{\partial a_2^2} + \beta_3 \tau_0 \frac{\partial^2 K_0}{\partial a_2\, \partial a_3} + o(\mu) = 0. \tag{44}$$

$$\eta_3\left(\tau_0, \beta_i, \nu_i, \mu\right) = \alpha\tau_0 \frac{\partial K_0}{\partial a_3} + \beta_1 \tau_0 \frac{\partial^2 K_0}{\partial a_3\, \partial a_1} + \beta_2 \tau_0 \frac{\partial^2 K_0}{\partial a_3\, \partial a_2} + \beta_3 \tau_0 \frac{\partial^2 K_0}{\partial a_3^2} + o(\mu) = 0. \tag{45}$$

Periodic orbits will exist if (Duboslim, 1964)

$$\frac{\partial [K_1]}{\partial \omega_i} = 0 \quad (i=1,2,3) \tag{46}$$

$$\frac{\partial [K_1]}{\partial a_i} = 0 \quad (i=1,2,3) \tag{47}$$

$$\frac{\partial(\xi_2, \xi_3, \eta_1, \eta_2, \eta_3)}{\partial(\nu_2, \nu_3, \beta_1, \beta_2, \beta_3)} \neq 0, \text{ when } \mu = \beta_i = \nu_i = 0 \tag{48}$$

We may note that

$$\frac{\partial^2[K_1]}{\partial a_j \partial \omega_i}=0, \text{ for } \frac{\partial[K_1]}{\partial \omega_i}=0 \; [i \neq j,\, i, j=1, 2, 3]$$

$$K_0=a_1[a_2{}^2-2(a_2+C_0)]^{1/2}-1$$

and

$$\begin{vmatrix} \frac{\partial^2 K_0}{\partial a_1{}^2} & \frac{\partial^2 K_0}{\partial a_2\, \partial a_1} & \frac{\partial^2 K_0}{\partial a_3\, \partial a_1} \\ \frac{\partial^2 K_0}{\partial a_1 \partial a_2} & \frac{\partial^2 K_0}{\partial a_2{}^2} & \frac{\partial^2 K_0}{\partial a_3\, \partial a_2} \\ \frac{\partial^2 K_0}{\partial a_1 \partial a_3} & \frac{\partial^2 K_0}{\partial a_2\, \partial a_3} & \frac{\partial^2 K_0}{\partial a_3{}^2} \end{vmatrix} = \frac{-a_1}{[a_3{}^2-2(a_2+C_0)]^{3/2}} \neq 0 \quad (49)$$

It can be seen easily from condition (49) that the determinant (48) will not be equal to zero, if

$$\begin{vmatrix} \frac{\partial^2[K_1]}{\partial \omega_2{}^2} & \frac{\partial^2[K_1]}{\partial \omega_3\, \partial \omega_2} \\ \frac{\partial^2[K_1]}{\partial \omega_3\, \partial \omega_2} & \frac{\partial^2[K_1]}{\partial \omega_3{}^2} \end{vmatrix} \neq 0 \qquad (50)$$

For calculating the partial derivatives occurring in (50), we may note from (35), that

$$K_1=\frac{1}{2}r_1{}^3-\frac{r_1\, r_2{}^2}{2}+1-\frac{r_1}{r_2}+e_1 \cos f \left(1+\frac{1}{2}r_1{}^3\right)+r_1.\frac{C_0-C_1}{\mu}$$

$$+\int_0^{r_1}\int_0^{f} \frac{e_1 \sin f}{(1+\mu\, e_1 \cos f)^2} \times \left[\frac{1}{2}\left\{(1-\mu)\, r_1{}^2+\mu r_2{}^2\right\}+\frac{1-\mu}{r_1}\right.$$

$$\left.+\frac{\mu}{r_2}-\frac{1}{2}(1+\mu\; e_1 \cos f)\xi_3{}^2\right] dr_1\, df$$

Taking only zero-order term, we have

$$[K_1]=\frac{1}{2}r_1{}^3-\frac{r_1\, r_2{}^2}{2}+1-\frac{r_1}{r_2},$$

where

$$z=\xi^2=a\;;\; r_1{}^2=a^2+\xi_3{}^2$$

$$r_2{}^2=1+a^2+2a \cos \phi+\xi_3{}^2,$$

$$\xi_3=\omega_3+\text{terms containing } \omega_1 \text{ and } \omega_2.$$

$$2\phi=\omega_1+\omega_2+n_1{}^{(0)}+n_2{}^{(0)}$$

Now

$$\frac{\partial(K_1)}{\partial \omega_2}=a_1\; r_1 \sin 2\phi\left[1-\frac{1}{r_2{}^3}\right] \qquad (51)$$

$$=0, \text{ gives}$$

$$2\phi=0,\; \pi \text{ or } r_2=1$$

But

$$\frac{\partial^2\,[K_1]}{\partial \omega_2{}^2}=a_1\; r_1 \cos 2\phi\left[1-\frac{1}{r_2{}^3}\right]+a_1\; r_1 \sin\; 2\phi\left(-\frac{3}{r_1{}^4}\right)\left(-\frac{a \sin 2\phi}{r_2}\right)$$

Therefore when $2\phi=0$, π or $r_2=1$,

$$\frac{\partial^2 [K_1]}{\partial \omega_2^2} \neq 0 \tag{52}$$

Again $$\frac{\partial^2 (K_1)}{\partial \omega_2\, \partial \omega_3} = \frac{\partial}{\partial \omega_3}\left[a_1\ r_1 \sin 2\phi\left(1-\frac{1}{r_2^3}\right)\right]$$

2ϕ is independent of ω_3, therefore

$$\frac{\partial^2 (K_1)}{\partial \omega_2\, \partial \omega_3} = a_1 \sin 2\phi\ \frac{\partial}{\partial \omega_3}\left[r_1\left(1-\frac{1}{r_2^3}\right)\right]$$

$$=0, \text{ for } 2\phi=0 \text{ or } \pi. \tag{53}$$

Further $$\frac{\partial K_1}{\partial \omega_3} = \frac{3}{2} r_1^2 \frac{\partial r_1}{\partial \omega_3} - \frac{r_2^2}{2}\frac{\partial r_1}{\partial \omega_3} - \frac{r_1}{2}.\ 2r_2 \frac{\partial r_2}{\partial \omega_3} - \frac{1}{r_2}\frac{\partial r_1}{\partial \omega_3} + \frac{r_1}{r_2^2}\frac{\partial r_2}{\partial \omega_3}$$

But $$\frac{\partial r_1}{\partial \omega_3} = \frac{\partial r_1}{\partial \xi_3}.\frac{\partial \xi_3}{\partial \omega_3} = \frac{\xi_3}{r_1}$$

and $$\frac{\partial r_2}{\partial \omega_3} = \frac{\partial r_2}{\partial \xi_3}.\frac{\partial \xi_3}{\partial \omega_3} = \frac{\xi_3}{r_2}$$

Thus $$\frac{\partial K_1}{\partial \omega_3} = B\xi_3 \text{ (say)} \tag{54}$$

Therefore $\dfrac{\partial K_1}{\partial \omega_3}=0$, gives, $\xi_3=0$ or $B=0$.

Now $$\frac{\partial^2 [K_1]}{\partial \omega_3^2} = B\frac{\partial \xi_3}{\partial \omega_3} + \xi_3\ \frac{\partial B}{\partial \omega_3}$$

$$\neq 0, \text{ when either } \xi_3=0, \text{ or } B=0 \tag{55}$$

From (52), (53), (55), it follows that determinant (50) $\neq 0$.

Hence condition (48) is satisfied.

Now $$\frac{\partial (K_1)}{\partial \omega_1} = \frac{3}{2} r_1^2 \frac{\partial r_1}{\partial \omega_1} - \frac{r_2^2}{2}\frac{\partial r_1}{\partial \omega_1} - \frac{r_1}{2}.\ 2r_2 \frac{\partial r_1}{\partial \omega_1} - \frac{1}{r_2}\frac{\partial r_1}{\partial \omega_1} + \frac{1}{r_2^2}.\frac{\partial r_2}{\partial \omega_1}$$

But $$\frac{\partial r_1}{\partial \omega_1}=0\ ;\ \frac{\partial r_2}{\partial \omega_1} = \frac{\partial r_2}{\partial (2\phi)}.\frac{\partial (2\phi)}{\partial \omega_1} = \frac{-2a \sin 2\phi}{2r_2}$$

Thus $$\frac{\partial (K_1)}{\partial \omega_1} = 0 \text{ for } 2\phi=0 \text{ or } \pi \tag{56}$$

This satisfies condition (46).

Let us consider the condition (47). Since K_1 is independent of a_1, a_2, a_3, therefore

$$\frac{\partial [K_1]}{\partial a_i} = 0,\ (i=1,\ 2,\ 3)$$

Hence all the conditions, *viz.* (46), (47), (48) are satisfied for the existence of the periodic orbits when $\mu \neq 0$.

5. Periodic Orbit of Collision when $\mu \neq 0$. In this section we shall prove the existence of periodic orbits of collision when $\mu \neq 0$.

Proceeding as in (Bhatnagar, 1969), in our case, the condition of collision should be of the form

$$G+\mu F(l, L, g, G, h, \bar{H})=0 \tag{57}$$

We, again, consider the case when $e=1$. In that case the orbit starts as an ejection from the origin and returns to it after time $\tau/4$.

Levi-Civita's condition for collision is

$$\dot{\theta}+1=\rho f(\rho, 0) \tag{58}$$

where $\tan \theta=\dfrac{x_2}{x_1-\mu}$; $\rho=\sqrt{r_1}$

The condition (58) in our case, becomes

$$2\dot{\phi}+1=\sqrt{r_1}\, f(\sqrt{r_1}, \theta)$$

or

$$2\frac{d\phi}{d\tau}\frac{d\tau}{dt}+1=\sqrt{r_1}\, f(\sqrt{r_1}, \theta)$$

But

$$2\phi'=\frac{G}{\xi^2}-\xi^2$$

$$\therefore \quad G-\xi^4+r_1\xi^2=\xi^2 r_1^{3/2} f(r_1^{1/2}, 2\phi). \tag{59}$$

This corresponds to (57). Obviously this is satisfied since at $\tau=0$,

$$G=0,\ \xi=0,\ \xi_3=0\ (i.e.\ r_1=0)$$

Since condition (59) is satisfied along the entire orbit, the infinitesional body will approach the origin with characteristics of a collision orbit. The proof of the existence of such periodic orbits in the collision is, therefore, fully established.

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