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THERMAL STRESSES DUE TO PRESCRIBED FLUX OF HEAT WITHIN A CIRCULAR REGION IN AN INFINITE ELASTIC HALF SPACE

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Abstract. In this paper an exact solution has been obtained for the thermoelastic problem of an isotropic material occupying a half space with stress free edges subjected to two different temperature distributions, *i.e.* (i) a constant flux of heat within a circular region of exposure, the exterior of circular region being free from any flux of heat and (ii) paraboloid distribution of temperature within the circular region, the exterior being insulated. Numerical results have been given.

1. Introduction. The thermal stress problem in an elastic half-space at constant temperature $T=T_0$ inside a circle of radius a, the exterior of the circle being thermally insulated, was considered by Nowacki [2]. Recently Bhattacharya [1] has considered the problem of determining the thermal stresses due to prescribed flux of heat on the surface of thick plate. The object of this paper is to find the exact solution of the thermoelastic problem of an isotropic material with stress free edges subjected to two different temperature distributions. In the first case, we assume a constant flux of heat within a circular region of exposure, the exterior of the circular region being free from any flux of heat. Secondly, we assume a paraboloid distribution of flux within the circular region, the exterior being insulated. The numerical calculations for the variation of $(\widehat{\rho\rho}+\widehat{\theta\theta})$ on he free surface have also been obtained in both cases.

2. Solutions of the Equations of Thermoelasticity. We shall consider the temperature and displacement field in a perfectly elastic solid which is conducting heat. With regards to both its mechanical and thermal properties the solid will be assumed to be isotropic and homogeneous. It will be assumed that there is symmetry about z-axis and any point of the solid may be expressed in terms of cylindrical co-ordinates (ρ, θ, z). For symmetrical deformations of the solid the displacement and the only nonvanishing components of stress tensor will be $(u, 0, w)$ and $\widehat{\rho\rho}$, $\widehat{\theta\theta}$, $\widehat{zz}$, $\widehat{\rho z}$ respectively.

The temperature field is given by the Laplace's equation

$$\nabla^2 T \equiv \frac{\partial^2 T}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial T}{\partial \rho} + \frac{\partial^2 T}{\partial z^2} = 0, \tag{2.1}$$

in the steady state and in the absence of thermal sources. Stress components are obtained by using the potential of thermoelastic displacement ϕ given by the equations

$$u_T = \frac{\partial \phi}{\partial \rho} \text{ and } w_T = \frac{\partial \phi}{\partial z}. \tag{2.2}$$

From the equations of equilibrium and the stress strain relations in problem of thermal stresses, we have

$$\frac{\partial^2 \phi}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial \phi}{\partial \rho} + \frac{\partial^2 \phi}{\partial z^2} = \beta T\,;\; \beta = \frac{1+\eta}{1-\eta}\,\alpha, \tag{2.3}$$

where T is the deviation of the absolute temperature from the temperature of the solid in a state of zero stress and strain, α is the coefficient of linear expansion of the solid and η is its poisson ratio.

A particular integral of the equation (2.3) is given by

$$\phi = \frac{\beta}{2}\int_0^\infty \xi^{-3} A(\xi)(1+\xi z)e^{-\xi z} J_0(\xi\rho)\, d\xi, \tag{2.4}$$

where $A(\xi)$ is a function of ξ only.

From the relations (2.3) and (2.4), we obtain

$$T = \int_0^\infty \xi^{-1} A(\xi)\, e^{-\xi z} J_0(\xi\rho)\, d\xi \tag{2.5}$$

which satisfies equation (2.1).

The components of displacement and stress can now be written as

$$u_T = -\frac{\beta}{2}\int_0^\infty \xi^{-2} A(\xi)\,(1+\xi z)e^{-\xi z} J_1(\xi\rho)\, d\xi, \tag{2.6}$$

$$w_T = -\frac{\beta}{2}\, z\int_0^\infty \xi^{-1} A(\xi)\, e^{-\xi z} J_0(\xi\rho)\, d\xi, \tag{2.7}$$

$$\widehat{\rho z}_T = 2\mu\frac{\partial^2 \phi}{\partial \rho \partial z} = \mu\beta z\int_0^\infty A(\xi)\, e^{-\xi z} J_1(\xi\rho)\, d\xi. \tag{2.8}$$

$$\widehat{zz}_T=-2\mu\left(\frac{\partial^2\phi}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial\phi}{\partial\rho}\right) \tag{2.9}$$
$$=\mu\beta\int_0^\infty \xi^{-1} A(\xi)(1+\xi z)e^{-\xi z}J_0'(\xi\rho)\,d\xi,$$

$$\widehat{\rho\rho}_T=-2\mu\left(\frac{\partial^2\phi}{\partial z^2}+\frac{1}{\rho}\frac{\partial\phi}{\partial\rho}\right) \tag{2.10}$$
$$=\mu\beta\left\{\int_0^\infty \xi^{-2} A(\xi)(1+\xi z)\,e^{-\xi z}\,\frac{J_1(\xi\rho)}{\rho}\,d\xi\right.$$
$$\left.+\int_0^\infty \xi^{-1} A(\xi)(1-\xi z)\,e^{-\xi z}J_0(\xi\rho)\,d\xi\right\}$$

$$\widehat{\theta\theta}_T=-2\mu\left(\frac{\partial^2\phi}{\partial z^2}+\frac{\partial^2\phi}{\partial\rho^2}\right) \tag{2.11}$$
$$=\mu\beta\left\{2\int_0^\infty \xi^{-1} A(\xi)\,e^{-\xi z}J_0(\xi\rho)\,d\xi\right.$$
$$\left.-\frac{1}{\rho}\int_0^\infty \xi^{-2} A(\xi)(1+\xi z)e^{-\xi z}J_1(\xi\rho)\,d\xi\right\}.$$

The subscript T denotes that the stresses are due to the thermal, expansion, μ being the modulus of rigidity.

We observe that the shearing stress $\widehat{\rho z}_T$ vanishes for $z=0$, and the stress $\widehat{zz}_T$ does not vanish.

To satisfy the boundary conditions on the plane $z=0$, we superimpose an elementary stress system. The components of stresses and displacement are expressed by means of Love's function ψ by relations

$$u_c=\frac{1}{(1-2\eta)}\frac{\partial^2\psi}{\partial\rho\partial z}\ ;\ w_c=\frac{1}{(1-2\eta)}\left[2(1-\eta)\nabla^2\psi-\frac{\partial^2\psi}{\partial z^2}\right], \tag{2.12}$$

$$\widehat{\rho\rho}_c=\frac{2\mu}{(1-2\eta)}\frac{\partial}{\partial z}\left[\eta\nabla^2\psi-\frac{\partial^2\psi}{\partial\rho^2}\right],$$
$$\widehat{\theta\theta}_c=\frac{2\mu}{(1-2\eta)}\frac{\partial}{\partial z}\left[\eta\nabla^2\psi-\frac{1}{\rho}\frac{\partial\psi}{\partial\rho}\right], \tag{2.13}$$
$$\widehat{\rho z}_c=\frac{2\mu}{(1-2\eta)}\frac{\partial}{\partial\rho}\left[(1-\eta)\nabla^2\psi-\frac{\partial^2\psi}{\partial z^2}\right],$$
$$\widehat{zz}_c=\frac{2\mu}{(1-2\eta)}\frac{\partial}{\partial z}\left[(2-2\eta)\nabla^2\psi-\frac{\partial^2\psi}{\partial z^2}\right],$$

the function ψ satisfying the biharmonic equation

$$\nabla^2\nabla^2\psi=0. \tag{2.14}$$

The solution of the equation (2.14) can be assumed in the form

$$\psi=\int_0^\infty \xi^{-3}\,\beta(\xi)\,(2\eta+\xi z)\,e^{-\xi z}J_0(\xi\rho)\,d\xi, \tag{2.15}$$

where $B(\xi)$ is the function of ξ only.

The components of complementary stresses are given by

$$\widehat{\rho z}_c=\frac{2\mu z}{(1-2\eta)}\int_0^\infty \xi\, B(\xi)e^{-\xi z}J_1(\xi\rho)\,d\xi, \tag{2.16}$$

$$\widehat{zz}_c=\frac{2\mu}{(1-2\eta)}\int_0^\infty B(\xi)\,e^{-\xi z}(1+\xi z)\,J_0(\xi\rho)\,d\xi. \tag{2.17}$$

Now the boundary conditions to be satisfied are

$$[\widehat{\rho z}_c]_{z=0}=0\ ;\ [\widehat{zz}_T+\widehat{zz}_c]_{z=0}=0. \tag{2.18}$$

The first relation of (2.18) will be satisfied for $z=0$ and from equations (2.9) and (2.17), we have

$$B(\xi)=-\frac{(1-2\eta)}{2}\,\beta\,\xi^{-1}A(\xi) \tag{2.19}$$

Consequently the remaining stresses and the displacements are given by

$$\begin{aligned} u_c&=-\frac{\beta}{2}\int_0^\infty \xi^{-2}A(\xi)(2\eta+\xi z-1)e^{-\xi z}J_1(\xi\rho)\,d\xi,\\ w_c&=\frac{\beta}{2}\int_0^\infty \xi^{-2}A(\xi)(2-2\eta+\xi z)\,e^{-\xi z}J_0(\xi\rho)\,d\xi, \end{aligned} \tag{2.20}$$

$$\begin{aligned} \widehat{\rho\rho}_c=-\mu\beta\Big[&\int_0^\infty \xi^{-1}A(\xi)\,(1-\xi z)\,e^{-\xi z}J_0(\xi\rho)\,d\xi\\ &+\int_0^\infty \xi^{-2}A(\xi)\,(2\eta+\xi z-1)\,e^{-\xi z}J_1(\xi\rho)\,d\xi\Big], \end{aligned} \tag{2.21}$$

$$\begin{aligned} \widehat{\theta\theta}_c=-\mu\beta\Big[&2\eta\int_0^\infty \xi^{-1}A(\xi)\,e^{-\xi z}J_0(\xi\rho)\,d\xi\\ &-\frac{1}{\rho}\int_0^\infty \xi^{-2}A(\xi)\,(2\eta+\xi z-1)\,e^{-\xi z}J_1(\xi\rho)\,d\xi\Big]. \end{aligned} \tag{2.22}$$

Adding (2.6), (2.7) and (2.20), we find

$$\begin{aligned} u&=-\beta\int_0^\infty \xi^{-2}A(\xi)(\eta+\xi z)\,e^{-\xi z}J_1(\xi\rho)\,d\xi,\\ w&=\beta\,(1-\eta)\int_0^\infty \xi^{-2}A(\xi)\,e^{-\xi z}J_0(\xi\rho)\,d\xi. \end{aligned} \tag{2.23}$$

Also adding (2.10), (2.21) and (2.11), (2.22), we have finally

$$\widehat{\rho\rho}=2\mu\beta\,(1-\eta)\int_0^\infty \xi^{-2}A(\xi)\,e^{-\xi z}\frac{J_1(\xi\rho)}{\rho}\,d\xi,$$

(2.24) $$\widehat{\theta\theta}=2\mu\beta\,(1-\eta)\left[\int_0^\infty \xi^{-1}\,A(\xi)\,e^{-\xi z}J_0(\xi\rho)\,d\xi - \frac{1}{\rho}\int_0^\infty \xi^{-2}\,A(\xi)\,e^{-\xi z}J_1\,(\xi\rho)\,d\xi\right].$$

Hence we have

$$\widehat{\rho\rho}+\widehat{\theta\theta}=2\mu\beta(1-\eta)\int_0^\infty \xi^{-1}\,A(\xi)e^{-\xi z}J_0(\xi\rho)\,d\xi$$

(2.25)

$$\widehat{\rho\rho}-\widehat{\theta\theta}=2\mu\beta\,(1-\eta)\int_0^\infty \xi^{-1}\,A(\xi)e^{-\xi z}\left\{J_0\,(\xi\rho)-\frac{2J_1(\xi\rho)}{\xi\rho}\right\}d\xi.$$

3. Temperature Distribution. In this section we shall suppose that there is prescribed flux of heat within a circular region $0\leqslant\rho<1$, on the free surface $z=0$, the rest of the surface being free from any flux of heat. So the boundary conditions are, on the plane $z=0$

(3.1) $$\frac{\partial T}{\partial z}=-f\,(\rho)\;;\;0\leqslant\rho<1,$$
$$=0\;;\;\rho>1.$$

Now from the equation (2.5), we have

$$T=\int_0^\infty \xi^{-1}\,A(\xi)\,e^{-\xi z}J_0(\xi\rho)\,d\xi.$$

Therefore

$$\frac{\partial T}{\partial z}=-\int_0^\infty A(\xi)\,e^{-\xi z}J_0\,(\xi\rho)\,d\xi.$$

So that on the boundary surface $z=0$, we have

$$\int_0^\infty \xi\,\frac{A(\xi)}{\xi}J_0\,(\xi\rho)d\xi=\begin{cases}f\,(\rho), & 0<\rho<1,\\ 0, & \rho>1.\end{cases}$$

Hence by Hankel's inversion theorem

(3.2) $$\frac{A(\xi)}{\xi}=\int_0^1 \rho\,f(\rho)\,J_0\,(\xi\rho)\,d\rho.$$

Under the same set of transformations, we have on $z=0$

$$u=-\beta\eta\int_0^\infty \xi^{-2}\,A(\xi)\,J_1\,(\xi\rho)\,d\xi,$$

(3.3)

$$w=\beta(1-\eta)\int_0^\infty \xi^{-2}\,A(\xi)\,J_0(\xi\rho)\,d\xi,$$

(3.4)

$$\widehat{\rho\rho}+\widehat{\theta\theta}=2\mu\beta\,(1-\eta)\int_0^\infty \xi^{-1}\,A(\xi)\,J_0(\xi\rho)\,d\xi,$$

$$\widehat{\rho\rho}-\widehat{\theta\theta}=-2\mu\beta(1-\eta)\int_0^\infty \xi^{-1}\,A(\xi)\left\{J_0(\xi\rho)-\frac{2J_1(\xi\rho)}{\xi\rho}\right\}d\xi.$$

Case (i). Let us assume that the flux function is constant so that $f(\rho)=K$. Then from (3.2), we have

$$\frac{A(\xi)}{\xi}=K\int_0^1 \rho J_0\,(\xi\rho)\,d\rho,$$
$$=K\frac{J_1\,(\xi)}{\xi}.$$

Hence

(3.5) $$A(\xi)=KJ_1\,(\xi).$$

This value of $A(\xi)$ substituted in the relations of displacement and stress components gives complete solutions.

We find the values of $(\widehat{\rho\rho}+\widehat{\theta\theta})_{z=0}$ and $(\widehat{\rho\rho}-\widehat{\theta\theta})_{z=0}$ with this expression for $A(\xi)$ as

$$(\widehat{\rho\rho}+\widehat{\theta\theta})_{z=0}=2\,\iota\,(1+\eta\;\alpha K\int_0^\infty \frac{J_1(\xi)J_0(\xi\rho)d\xi}{\xi},$$

and

$$(\widehat{\rho\rho}-\widehat{\theta\theta})_{z=0}=-2\mu(1+\eta)\alpha K\int_0^\infty \left\{\frac{J_0(\xi\rho)J_1(\xi)}{\xi}-\frac{2J_1(\xi)J_1(\xi\rho)}{\xi^2\,\rho}\right\}d\xi.$$

If we assume that $2\mu(1+\eta)\alpha K=\delta$, then

$$(\widehat{\rho\rho}+\widehat{\theta\theta})_{z=0}=\delta\begin{cases}\dfrac{1}{2\rho}\,F\left(\dfrac{1}{2},\,\dfrac{1}{2},2\,;\dfrac{1}{\rho^2}\right) & \text{for } \rho>1,\\ 2/\pi & \text{for } \rho=1,\\ F\left(\dfrac{1}{2},\,\dfrac{1}{2},\,1\,;\rho^2\right) & \text{for } \rho<1.\end{cases}$$

$$(\widehat{\rho\rho}-\widehat{\theta\theta})_{z=0}=-\delta\begin{cases}\dfrac{1}{2\rho}\,F\left(\dfrac{1}{2},\,\dfrac{1}{2},\,2\,;\dfrac{1}{\rho^2}\right)-F\left(\dfrac{1}{2},-\dfrac{1}{2},\,2\,;\dfrac{1}{\rho^2}\right) & \text{for } \rho>1,\\ -2/3\pi & \text{for } \rho=1,\\ F\left(\dfrac{1}{2},\,\dfrac{1}{2},\,1\,;\,\rho^2\right)-\rho F\left(\dfrac{1}{2},-\dfrac{1}{2},2\,;\,\rho^2\right) & \text{for } \rho<1.\end{cases}$$

Case (ii). Next we assume that if there is parabolic flux given by $f(\rho)=K(1-\rho^2)$, then we have from (3.2)

(3.6)

$$\frac{A(\xi)}{\xi}=K\int_0^1 \rho(1-\rho^2)\,J_0(\xi\rho)\,d\rho$$

$$=K\left[\frac{4J_1(\xi)}{\xi^3}-\frac{2}{\xi^2}\,J_0(\xi)\right].$$

Therefore from (3.4),

$$(\widehat{\rho\rho}-\widehat{\theta\theta})_{z=0}=\delta\int_0^\infty\left[\frac{4J_1(\xi)J_0(\xi\rho)}{\xi^3}-\frac{2J_0(\xi)J_0(\xi\rho)}{\xi^2}\right]d\xi,$$

and

$$(\widehat{\rho\rho}-\widehat{\theta\theta})_{z=0}=-\delta\left\{\int_0^\infty\left[\frac{4J_1(\xi)J_0(\xi\rho)}{\xi^3}-\frac{2J_0(\xi)\,J_0(\xi\rho)}{\xi^2}\right]d\xi\right.$$
$$\left.-\frac{2}{\rho}\int_0^\infty\left[\frac{4J_1(\xi)J_1(\xi\rho)}{\xi^4}-\frac{2J_0(\xi)J_1(\xi\rho)}{\xi^3}\right]d\xi\right\}.$$

Hence in this case

$$(\widehat{\rho\rho}+\widehat{\theta\theta})_{z=0}=\delta\; I_1,$$
$$(\widehat{\rho\rho}-\widehat{\theta\theta})_{z=0}=-\delta\left[I_1-\frac{2}{\rho}\;I_2\right].$$

where

$$I_1=4\int_0^\infty\frac{J_1(\xi)J_0(\xi\rho)}{\xi^3}\,d\xi\;-2\int_0^\infty\frac{J_0(\xi\rho)J_0(\xi)}{\xi^2}\,d\xi$$
$$=\begin{cases}2\rho\left\{F\left(-\frac{1}{2},\frac{-1}{2},1;\frac{1}{\rho^2}\right)-F\left(\frac{-1}{2},\frac{-1}{2},2;\frac{1}{\rho^2}\right)\right\} & \text{for } \rho>1,\\ 8/9\pi & \text{for } \rho=1,\\ \frac{2}{3}\left\{3F\left(\frac{-1}{2},\frac{-1}{2},1;\rho\right\}-2F\left(\frac{-1}{2},\frac{-3}{2},1;\rho^2\right)\right\} & \text{for } \rho<1.\end{cases}$$

$$I_2=\left[4\int_0^\infty\frac{J_1(\xi)J_1(\xi\rho)}{\xi^4}-2\int_0^\infty\frac{J_0(\xi)J_1(\xi\rho)}{\xi^3}\right]d\xi$$
$$=\begin{cases}\frac{2\rho^2}{3}\left\{F\left(\frac{-1}{2},\frac{-3}{2},1;\frac{1}{\rho^2}\right)-F\left(\frac{-1}{2},\frac{-3}{2},2;\frac{1}{\rho^2}\right)\right\} & \text{for } \rho>1,\\ 32/45\pi & \text{for } \rho=1,\\ \frac{\rho}{3}\left\{3F\left(\frac{-1}{2},\frac{-1}{2},2;\rho^2\right)-2F\left(\frac{-1}{2},\frac{-3}{2},2;\rho^2\right)\right\} & \text{for } \rho<1.\end{cases}$$

4. **Numerical Results.** The variation of $\dfrac{[\widehat{\rho\rho}+\widehat{\theta\theta}]_{z=0}}{2\mu(1+\eta)\alpha K}$ for different values of ρ within the circle $\rho\leqslant 1$ in both the cases when $f(\rho)$ is (i) constant, and (ii) parabolic flux is given by the following table :

ρ		0·0	0·2	0·4	0·6	0·8	1
$\dfrac{[\widehat{\rho\rho}+\widehat{\theta\theta}]_{z=0}}{2\mu(1+\eta)\alpha K}$	(i)	1·0000	0·9899	0·9588	0·9039	0·8208	·6373
	(ii	0·6666	0·6468	0·5896	0·4854	0·3856	·2828

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