Jñānabha Sect. A, Vol. 5, July 1975.

Asymptotic Solutions of Certain Periodic Differential Equations and Lauricella's Function $F_D^{(n)}$

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(Received : October 7, 1974)

1. **Introduction**. The leading terms of asymptotic expansions of solutions of the differential equations with periodic coefficients (1·1) and (1·2), below, are obtained by means of what is essentially the *WKB* method. We restrict ourselves to the case where no turning points occur on the real line.

$$(1\cdot1)\qquad y''=\left\{\sum_{r=0}^{n} a_r \cos 2rx\right\} y,$$

$$(1\cdot2)\qquad y''=\left\{\sum_{r=0}^{n} a_r\, sn^{2r}(x, m)\right\} y.$$

For the purpose of this present discussion, it is assumed that the parameters and variables of (1·1) and (1·2) are real. It is of interest to note that the Lauricella function $F_D^{(n)}$ figures prominently in the analysis. This function was first defined and studied by G. Lauricella [3], and has the following series representation :

$$(1\cdot3)\qquad F_D^{(n)}(a, b_1,\ldots, b_n ; c ; x_1,\ldots, x_n)$$

$$=\sum_{m_1,\ldots,m_n=0}^{\infty} \frac{(a)_{m_1+\ldots+m_n}(b_1)_{m_1}\ldots(b_n)_{m_n}\, x_1^{m_1}\ldots x_n^{m_n}}{(c)_{m_1+\ldots+m_n}\, m_1!\ldots m_n!}$$

which converges for $|x_i|<1$ $(1\leqslant i\leqslant n)$ for any values of its parameters, and also if $Re\,(c-a-\Sigma b_i)>0$, $x_1=1$; a result which follows readily from the elementary properties of the function ${}_2F_1$.

We now write (1.1) and (1.2) in the form (assuming a_k to be positive)

$$(1{\cdot}4)\qquad y''=\left\{a_k\left[\sum_{r=0}^{k}\alpha_r \cos 2rx\right]+\sum_{r=k+1}^{n} a_r \cos 2rx\right\} y$$

and

$$(1{\cdot}5)\qquad y''=\left\{a_k\left[\sum_{r=0}^{k}\alpha_r\, sn^{2r}(x, m)\right]+\sum_{r=k+1}^{n} a_r\, sn^{2r}(x, m)\right\} y$$

where $\alpha_r=a_1/a_k$, $0\leqslant r\leqslant k-1$, $\alpha_k=1$.

If it assumed that $\alpha_1, \alpha_2,\ldots \alpha_{k-1}$ and $a_{k+1}, a_{k+2},\ldots, a_n$ are small compared with a_k, we may develop solutions of (1·1) and (1·2) valid for large a_k.

Set $a_k=u^2$, so that, if coefficients of $a_k y$ in (1·4) and (1·5) are factorised, we have, respectively,

$$(1{\cdot}6)\qquad y''=\left\{u^2\prod_{j=1}^{k}(\cos 2x-\beta_j)+\sum_{r=k+1}^{n} a_r \cos 2rx\right\} y$$

and

$$(1{\cdot}7)\qquad y''=\left\{u^2\prod_{j=1}^{k}(sn^2(x, m)-B_j)+\sum_{r=k+1}^{n} a_r\, sn^{2r}(x, m)\right\} y$$

If β_j is real and modulus greater than unity, $\cos 2x-\beta_j$ and $sn^2(x, m)-\beta_j$ are non-zero for all real x, with the result that, with this restriction, (1·4) and (1·5) possess no real turning points.

2. **Reduction to standard form.** Liouville's transformation is now supplied to (1·6), by which, if new variables

$$(2{\cdot}1)\qquad Y=\dot{x}^{-\frac{1}{2}}\, y \text{ and}$$

$$(2\cdot2)\qquad X=\int_0^x \prod_{j=k_1}^{k} (\cos 2s-\beta_j)^{\frac{1}{2}}\, ds$$

are taken, where dots denote differentiations with respect to X, (1·6) takes the form

$$(2\cdot3)\qquad d^2Y/dX^2=(u^2+f(X))\, Y \text{ and}$$

$$f(X)=\dot{x}^2 \sum_{r=k+1}^{n} a_r \cos 2rx+(3\ddot{x}^2-2\dot{x}\dddot{x})/4\dot{x}^2.$$

A suitable comparison equation for (2·3) is

$$(2\cdot4)\qquad d^2Y/dX^2=u^2Y$$

for large u. Hence, we have the asymptotic solutions of (2·3),

$$(2\cdot5)\qquad Y_2\sim\exp(uX(x))+0(1/u)$$

$$(2\cdot6)\qquad Y_2\sim\exp(-uX(x))+0(1/u).$$

3. **Discussion of the function X(x).** If, in the integral (2.2), we let

$$(3.1)\qquad \cos 2s=w/(\cos 2x-1)+1,$$

we find that

$$X(x)=2^{\frac{1}{2}} \prod_{j=1}^{k} (1-\beta_j)^{\frac{1}{2}} (1-\cos 2x)^{\frac{1}{2}}/4$$

$$(3\cdot2)\qquad X\int_0^1 w^{-\frac{1}{2}}(1-(1-\cos 2x)\, w/2)^{-\frac{1}{2}}$$

$$\prod_{j=1}^{k} (1-(1-\cos 2x)\, w/(1-\beta_j))^{\frac{1}{2}}\, dw.$$

Now,

$$(3\cdot3)\qquad \int_0^1 w^{\rho-1}(1-w)^{\sigma-\rho-1} \prod_{j=1}^{n} (1-wx_j)^{-\nu_j}\, dw$$

$$=\Gamma(\rho)\,\Gamma(\sigma-\rho)/\Gamma(\sigma)\, F_D^{(n)}(\rho, \nu_1, \ldots, \nu_n;\sigma;x_1,\ldots,x_n),$$

provided that Re (ρ) and Re $(\sigma-\rho)>0$.

Since various analytical continuation formulae for $F_D^{(n)}$ outside the unit hypercube are known, [1], [2], [4], this function may be computed fairly easily for moderate value of n provided that its parameters are not too large.

By comparing (3·2) and (3·3), it is clear that

$$X(x)=2^{\frac{1}{2}} \prod_{j=1}^{k} (1-\beta_j)^{\frac{1}{2}} (1-\cos 2x)^{\frac{1}{2}}/2 \tag{3·4}$$

$$\times F_{D}^{(k+1)}(\tfrac{1}{2}, \tfrac{1}{2}, -\tfrac{1}{2}, \ldots, -\tfrac{1}{2}; \tfrac{3}{2}; (1-\cos 2x)/2, (1-\cos 2x)/(1-\beta_1), \ldots, (1-\cos 2x)/(1-\beta_k))$$

provided that all the β's are distinct. If this is not the case, $X(x)$ still assumes a similar form in that the F_D function has fewer variables.

4. **Conclusion.** The leading terms of the asymptotic solutions of (1·1) under consideration are thus

$$y_1 \sim \prod_{j=1}^{k} (\cos 2x-\beta_j)^{-\frac{1}{4}} \exp\left(a_k^{\frac{1}{2}} X(x)\right) \tag{4·1}$$

and

$$y^2 \sim \prod_{j=1}^{k} (\cos 2x-\beta_j)^{-\frac{1}{4}} \exp\left(-a_k^{\frac{1}{2}} X(x)\right). \tag{4·2}$$

Simiiarly, in case of (1·2), the corresponding leading terms are

$$y_1 \sim \prod_{j=1}^{k} (sn^2(x, m)-\beta_j)^{-\frac{1}{4}} \exp\left(a_k^{\frac{1}{2}} Z(x)\right) \tag{4·3}$$

and

$$y_2 \sim \prod_{j=1}^{k} (sn^2(x, m)-\beta_j)^{-\frac{1}{4}} \exp\left(-a_k^{\frac{1}{2}} Z(x)\right), \tag{4·4}$$

where

$$Z(x)=\prod_{j=1}^{k} (-\beta_j)^{\frac{1}{2}} sn(x, m)$$

$$X\ F_D^{(k+2)}(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, -\tfrac{1}{2}, \ldots, -\tfrac{1}{2}; \tfrac{3}{2}, \tag{4·5}$$

$$sn^2(x, m), m^2\, sn^2(x, m), sn^2(x, m)/\beta_1, \ldots, sn^2(x, m)/\beta_k)$$

obtained by letting the variable of integration in the integral corresponding to (2.2) be

$$w=sn^2(s, m)/sn^2(x, m). \tag{4·6}$$

Special cases of (1·1) are Mathieu's equation and the paraboloidal wave equation, and of (1·2) are Lame's equation and the ellipsoidal wave equation to which the above analysis may readily be applied. It is hoped to study these cases in greater detail subsequently.

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