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A Transformation Formula for Certain Functions of Several Variables

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ABSTRACT

The object of this paper is to establish a transformation formula for a function of several variables defined recently by H. M. Srivastava. Several particular cases involving a certain class of the generalized Lauricella functions due to H. M. Srivastava and M. C. Daoust are considered briefly.

1. In the present paper we propose to derive and study a transformation formula for the function $f(z_1,\ldots,z_r)$ of r variables defined recently by Srivastava [4, p. 1079 (1)] by means of the following equality :

$$(1\cdot1)\qquad f(z_1,\ldots,z_r)=\sum_{k_1,\ldots,k_r=0}^{\infty} C(k_1,\ldots,k_r)\prod_{i=1}^{r}\frac{(z_i)^{k_i}}{k_i!}$$

where the coefficients $C(k_1,\ldots,k_r)$, $k_i\geqslant 0$, $1\leqslant i\leqslant r$, are arbitrary constants subject to the appropriate conditions of (absolute) convergence so that the two sides are meaningful in every case.

We establish the formula :

$$(1\cdot2)\quad (1-z_1)^{-A_1}\ldots(1-z_r)^{-A_r}\, f\left[x_1\left(\frac{-z_1}{1-z_1}\right)^{\delta},\ldots,\; x_r\left(\frac{-z_r}{1-z_r}\right)^{\delta}\right]$$

$$=\sum_{m_1,\ldots,m_r=0}^{\infty} \prod_{i=1}^{r} \left[\frac{(A_i,m_i)z_i^{m_i}}{m_i\,!}\right]$$

$$\times \sum_{k_1=0}^{\left[\frac{m_1}{\delta}\right]} \cdots \sum_{k_r=0}^{\left[\frac{m_r}{\delta}\right]} C\,(k_1,\ldots\ldots,k_r) \prod_{i=1}^{r} \left[\frac{(-m_i,\delta k_i)x_i^{k_i}}{k_i!\,(A_i,\delta k_i)}\right],$$

where $|z_i| < 1$ and $|x_i|$, $\left|\frac{z_i}{1-z_i}\right|$, $i=1,\ldots,r$ are restricted suitably. The function f on the left-hand side of (1·2) is given by (1·1) subject to such conditions that the series involved is either (absolutely) convergent or terminating. For convenience, the abbreviation $\Delta\,(\delta,\alpha)$ has been used to denote the set of parameters

$$\frac{\alpha}{\delta},\ \frac{\alpha+1}{\delta},\ldots\ldots,\frac{\alpha+\delta-1}{\delta}\ \delta\geqslant 1, \text{ and}$$

$$(1{\cdot}3) \qquad (\alpha,\,n) = \frac{\Gamma(\alpha+n)}{\Gamma\,(\alpha)} = \begin{cases} 1, & \text{if } n=0, \\ \alpha(\alpha+1)\ldots(\alpha+n-1), & \text{if } n=1,2,\ldots \end{cases}$$

Proof. To prove (1·2) we start from its left-hand side, L say, substitute from (1·1) to obtain :

$$0=\sum_{k_1,\ldots,k_r=0}^{\infty} C\,(k_1,\ldots,k_r) \prod_{i=1}^{r} \left[\frac{(-z_i)^{\delta k_i}}{k_i!} (1-z_i)^{-(A_i+\delta k_i)}\, x_i^{k_i}\right]$$

$$\mathrm{L}=\sum_{k_1,\ldots,k_r=0}^{\infty} C\,(k_1,\ldots,k_r) \prod_{i=1}^{r} \left[\frac{(-1)^{\delta k_i}\ x_i^{k_i}}{k_i\,!}\right]$$

$$\times \sum_{s_1,\ldots,s_r=0}^{\infty} \prod_{i=1}^{r} \left[\frac{(A_i+\delta k_j,\ s_i)\ z_i^{\delta k_i+s_i}}{s_i\,!}\right].$$

Inverting the order of the summations, which is justified due to absolute convergence of the series involved, substituting $\delta k_i+s_i=m_i$, and making use of the elementary relationships [3, pp. 22, 32] :

$$(1{\cdot}4) \quad (\alpha,\,\delta k) = \delta^{\delta k} \prod_{i=1}^{\delta} \left\{\left(\frac{\alpha+i-1}{\delta},\ k\right)\right\},\ (-n,\,k)=\frac{n!(-1)^k}{(n-k)!},$$

formula (1·2) is established after a little simplification.

Formula (1.2) contains a large number of particular cases scattered throughout the literature, while with $x_1 = \ldots = x_r = 1$, it provides us a transformation formula whenever we can sum the righthand side by choosing $C(k_1, \ldots, k_r)$ suitably.

To facilitate the discussion, we substitute :

$$(1{\cdot}5) \qquad C(k_1,\ldots,k_r) = \frac{\prod_{i=1}^{A} (a_i, k_1 + \ldots + k_r) \prod_{i=1}^{B_1} \left(b_{i,}^{(1)} k_1 \right) \ldots \prod_{i=1}^{B_r} \left(b_{i}^{(r)}, k_r \right)}{\prod_{i=1}^{C} (c_i, k_1 + \ldots + k_r) \prod_{i=1}^{D_1} \left(d_{i,}^{(1)} k_1 \right) \ldots \prod_{i=1}^{D_r} \left(d_{i,}^{(r)} k_r \right)},$$

so that the formula (1·2) assumes the form :

$$(1{\cdot}6) \quad (1-z_1)^{-A_1} \ldots (1-z_r)^{-A_r} F\left[\begin{matrix} (a_A) : \left(b^{(1)}_{B_1} \right) ;\ldots; \left(b^{(r)}_{B_r} \right) ; \\ (c_C) : \left(d^{(1)}_{D_1} \right) ;\ldots; \left(d^{(r)}_{D_r} \right) ; \end{matrix} \; x_1 \left(\frac{-z_1}{1-z_1} \right)^{\delta}, \ldots, x_r \left(\frac{-z_r}{1-z_r} \right)^{\delta} \right]$$

$$= \sum_{m_1,\ldots,m_r=0}^{\infty} \prod_{i=1}^{r} \left[\frac{(A_i m_i)\, z_i^{\,m_i}}{m_i!} \right]$$

$$F\left[\begin{matrix} (a_A) : \Delta(\delta, -m_1), \left(b^{(1)}_{B_1} \right) ;\ldots; \Delta(\delta, -m_r), \left(b^{(r)}_{B_r} \right) ; \\ (c_C) : \Delta(\delta, A_1), \left(d^{(1)}_{D_1} \right) ;\ldots; \Delta(\delta, A_r), \left(d^{(r)}_{D_r} \right) ; \end{matrix} \; x_1, \ldots, x_r \right],$$

which is valid for $|z_i| < 1$, $A + B_i \geqslant C + D_i$, but if $A + B_i = C + D_i + 1$, then $|x_i|$, $|z_i|$ are constrained appropriately. Here (a_A) is taken to abbreviate the sequence of parameters $a_1, \ldots, a_A$; $\left(b^{(k)}_{B_k} \right)$ stands for the sequence of B_k parameters $b^{(1)}_{1,}, \ldots, b^{(k)}_{B_k}$; $i, k = 1, \ldots, r$; r is a positive integer, with similar interpretations for (C_C) and $\left(d^{(k)}_{D_k} \right)$. Colon (:) and Semicolon (;) separate the forms of $(\alpha, m_1 + \ldots + m_r)$ and $(B_1, m_1), \ldots, (B_r, m_r)$. An empty product is to be treated as unity, and this interpretation will be retained throughout the paper.

The hypergeometric series involved in (1·6) is a particular case of the multiple hypergeometric series due to Srivastava and Daoust [5, p. 454 (4·1)].

2. **Particular cases.** In (1·6) taking $\delta=r=2$, we obtain:

$$(2\cdot1)\quad (1-z_1)^{-A_1}(1-z_2)^{-A_2}\, F\left[\begin{matrix}(a_A):(b_B):(b'_{B'});\\(c_C):(d_D);(d'_{D'});\end{matrix}\ \frac{xz_1^2}{(1-z_1)^2},\frac{yz_2^2}{(1-z_2)^2}\right]$$

$$=\sum_{m_1,m_2=0}^{\infty}\frac{(A_1,m_1)(A_2,m_2)}{m_1!\,m_2!}z_1^{m_1}z_2^{m_2}\times$$

$$F\left[\begin{matrix}(a_A):\frac{-m_1}{2},\frac{-m_1+1}{2},(b_B),\frac{-m_2}{2},\frac{-m_2+1}{2}(b'_{B'});\\(c_C):\frac{A_1}{2},\frac{A_1+1}{2},(d_D);\frac{A_2}{2},\frac{A_2+1}{2},(d'_{D'});\end{matrix}\ x,y\right],$$

where $|z_1|$, $|z_2|<1$, $A+B\leqslant C+D$ and $A+B'\leqslant C+D'$, but is $A+B=C+D+1$, $A+B'=C+D'+1$, then $|x|$, $|y|$, $|z_1|$ and $|z_2|$ are constrained appropriately.

In (2.1) let $z_1=t$, $z_2\to0$, replace A by $A+2$, set $B=D=0$, and $a_1=A_1=a$, $a_2=\frac{a+1}{2}$, so that we obtain the following result due to Brafman [1, p. 947 (24)] :

$$(2.2)\quad (1-t)^{-a}{}_{A+2}F_C\left[\begin{matrix}\frac{a}{2},\frac{a+1}{2},(a_A);\\(c_C);\end{matrix}\ \frac{xt^2}{(1-t)^2}\right]$$

$$=\sum_{n=0}^{\infty}\frac{(a,n)}{n!}\,{}_{A+2}F_C\left[\begin{matrix}\frac{-n}{2},\frac{-n+1}{2}(a_A);\\(c_C);\end{matrix}\ x\right]t^n.$$

Next we substitute $\delta=1$, $r=2$ in (1.6) to obtain :

$$(2.3) \quad (1-z_1)^{-A_1}(1-z_2)^{-A_2}\, F\left[\begin{matrix}(a_A) : (b_B) \ ; \ (b'_{B'}) \ ; \\ (c_C) : (d_D) \ ; \ (d'_{D'}) \ ;\end{matrix}\ \frac{z_1 x}{z_1-1} \ ; \ \frac{z_2 y}{z_2-1}\right]$$

$$=\sum_{m_1,m_2=0}^{\infty} \frac{(A_1,m_1)\,(A_2,m_2)}{m_1!\, m_2!}\, z_1^{m_1}\, z_2^{m_2}$$

$$F\left[\begin{matrix}(a_A) : -m_1, (b_B) \ ; \ -m_2, (b'_{B'}) \ ; \\ (c_C) : A_2, (d_D) \ ; \ A_1, (d'_{D'}) \ ;\end{matrix}\ x, y\right],$$

where the conditions of the covergence are the same as in (2.1) ; this provides the following result due to Chaundy [2, p. 62 (25)] on letting $z_1=t$, $z_2\to 0$, $B=1$, $D=0$, $b_1=A_1=a$;

$$(2.4) \quad (1-t)^{-a}\ {}_{A+1}F_C\left[\begin{matrix} a, (a_A) \ ; \\ (c_C) \ ;\end{matrix}\ \frac{-xt}{1-t}\right]$$

$$=\sum_{n=0}^{\infty} \frac{(a,n)}{n!}\ {}_{A+1}F_C\left[\begin{matrix} -n, (a_A) \ ; \\ (c_C) \ ;\end{matrix}\ x\right] t^n.$$

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