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# Circular Hole in a Homogeneous, Isotropic, Compressible Medium Stressed at Infinity In Second Order Elasticity

*By*

**D. Pande and M. Lal**

*Department of Mathematics, Chirst Church College, Kanpur, India*

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1. **Abstract.** The present paper seeks to solve the problem of of a circular hole in an infinite, isotropic homogenous, compressible medium acted upon by a tensile force at infinity. The system is considered in the state of generalized plane stress. Work done in second order elasticity has established that the results so obtained significantly differ from those obtained in the classical linear elasticity. This is due to the fact that the results in the second order case are influenced by the elastic properties of the material to a greater extent. The same fact has been confirmed in this paper. The problem chosen is of immense practical importance and effectively covers a wider range of materials than permissible under the linear elasticity case.

2. **Introduction and Preliminaries.** The well known complex variable technique in infinitesimal elasticity was extended and made available for use in the second order elasticity case by Green, Adkins and his associates [1, 2]. This technique was further developed and modified by Bhargava, Pande and Dube [3, 5] so that a larger and more complex set of problems became tractable. This technique mainly consists of finding the two sets of potential functions $\{\Omega(z), \omega(z)\}$ and $\{\triangle(z), \delta(z)\}$, which determine the Airy's stress function $\phi$ and the displacement vector $D$ and eventually the entire elastic field, by means of an integro-differential equation. A brief resume of the basic preliminaries from the view point of second order elasticity will be found relevent. The same is given below.

Let

$$\left.\begin{aligned} \zeta &= x_1 + ix_2, \quad \bar{\zeta} = x_1 - ix_2 \\ z &= y_1 + iy_2, \quad \bar{z} = y_1 - iy_2 \end{aligned}\right\} \qquad \ldots(1)$$

respectively be the complex coordinates of a point of the body in its undeformed and deformed states. Let the displacement vector be

$$\underline{D} = \underline{u}_1 + i\,\underline{u}_2 \qquad \ldots(2)$$

$$z = \zeta + D \text{ and } \bar{z} = \bar{\zeta} + \bar{D}. \qquad \ldots(2')$$

In the absence of body forces the equilibrium equations are satisfied if

$$T^{11} = \bar{T}^{22} = -4\frac{\partial^2 \phi}{\partial z^2}, \quad T^{12} = 4\frac{\partial^2 \phi}{\partial z \partial \bar{z}} \qquad \ldots(3)$$

where $\phi$ is the Airy's stress function, $T^{\alpha\beta}$ are the stress components given by

$$\left.\begin{aligned} T^{11} &= \bar{T}^{22} = p_{y_1 y_1} - p_{y_2 y_2} + 2ip_{y_1 y_2}, \\ \text{and} \qquad T^{12} &= p_{y_1 y_2} + p_{y_2 y_2} \end{aligned}\right\} \qquad \ldots(4)$$

where $p_{y_1 y_1}, p_{y_2 y_2}$ are the normal and $p_{y_1 y_2}$ the shearing stress components of the system.

When the resulant forces along the boundary is zero.

$$\frac{\partial \phi}{\partial z} = \frac{\partial \phi}{\partial \bar{z}} = 0. \qquad \ldots(5)$$

The Airy's stress function $\phi$, the displacement vector $\underline{D}$ may be expanded in terms of a real small parameter $\varepsilon$, dependent upon the physical conditions of the problem, in the following absolutely and uniformly convergent series

$$\left.\begin{aligned} \phi &= \mu\varepsilon\,\{\phi_0\,(z, \bar{z}) + \varepsilon\phi_1\,(z, \bar{z}) + \ldots\}, \\ \text{and} \qquad D &= \varepsilon\{D_0\,(z, \bar{z}) + \varepsilon\, D_1\,(z, \bar{z}) + \ldots\} \end{aligned}\right\} \qquad \ldots(6)$$

$\mu$ being the modulus of rigidity, $\phi_0$, $D_0$ cover the case of infinitesimal elasticity and jointly $(\phi_0, D_0)$ and $(\phi_1, D_1)$ account for the second order case. It is well known [4]

$$\left.\begin{aligned} \phi_{\circ}(z, \bar{z}) &= \bar{z}\,\Omega(z)+z\,\overline{\Omega}(\bar{z})+\omega(z)+\bar{\omega}(\bar{z}), \\ \text{and} \quad D_{\circ}(z, \bar{z}) &= \kappa\,\Omega(z)-z\,\overline{\Omega}'(\bar{z})-\bar{\omega}'(\bar{z}) \end{aligned}\right\}$$

where $\kappa=(5\lambda+6\mu)/(3\lambda+2\mu)$ for the plane stress case. $\lambda$ and $\mu$ being the known Lame's constants.

The set $(\Omega, \omega)$ of potential function is determined by (5) taking $\phi=\phi_{\circ}$ and $(\triangle, \delta)$ is determined by

$$\frac{\partial\phi_1}{\partial\bar{z}}=\triangle(z)+z\overline{\triangle}^1(\bar{z})+\bar{\delta}'(\bar{z})+\gamma\,\Gamma(z, \bar{z})$$

$$+B_3\,\Omega(z)\,\overline{\Omega}^1(\bar{z})-B_1\,z\,\{\overline{\Omega}'(\bar{z})\}^2 \quad \ldots(8)$$

$$D_1(z, z)=\kappa\,\triangle(z)-z\,\overline{\triangle}'(\bar{z})-\bar{\delta}'(\bar{z})-\gamma\,\Lambda(z, \bar{z})$$

$$+B_1''z\,\{\overline{\Omega}'(\bar{z})\}^2-B_3\,\Omega(z)\,\overline{\Omega}^1(\bar{z})$$

$$+B'_1\int^{\bar{z}}\overline{\Omega}'(\bar{z})\bar{\omega}''(\bar{z})\,d\bar{z}-B_4\int^{z}\{\Omega'(z)\}^2\,dz \quad \ldots(9)$$

where $\gamma$, $B_1$, $B_j$ and $B''_1$ are elastic constants given in the appendix. It is found convenient to express $D$ in terms of a minimum number of elastic constants. In the problem $D$ has been expressed in term of $B_1$, $B_3$ and $\kappa$.

Also

$$\Gamma(z,\bar{z})=-\left\{D_0\frac{\partial}{\partial z}+\bar{D}_0\frac{\partial}{\partial\bar{z}}\right\}\frac{\partial\phi_0}{\partial\bar{z}}, \quad \ldots(10)$$

$$\Lambda(z,z)=\left\{D_0\frac{\partial}{\partial z}+\bar{D}_0\frac{\partial}{\partial\bar{z}}\right\}D_0, \quad \ldots(11)$$

For the uniqueness and single valuedness of $\phi_0$ and $D_0$ the following conditions should be satisfied.

$$[\Omega'(z)]_c=0,\ [\omega''(z)]_c=0,\ [k\Omega(z)-\omega'(z)]_c=0, \quad \ldots(12)$$

$$\left.\begin{aligned} &[\triangle'(z)]_c=0,\ [\delta''(z)+B_3\overline{\Omega}(\bar{z})\Omega''(z)]_c=0, \\ &[\kappa\triangle(z)-\bar{\delta}'(\bar{z})]_c=[B_4\int^{z}\{\Omega'(z)\}^2dz \\ &-B_1'\int^{\bar{z}}\overline{\Omega}'(\bar{z})\bar{\omega}'(\bar{z})d\bar{z} \\ &+B_3\Omega(z)\overline{\Omega}'(\bar{z})]_c \end{aligned}\right\} \quad \ldots(13)$$

where $[\ ]_c$ denotes the change in value of the function inside the brackets during a complete circuit of a contour C lying entirely within the deformed body.

3. **The Problem.** Let a homogeneous, isotropic, compressible, infinite elastic medium have a hole in it. Let a tensile force $T$ act at the outer boundary of the medium such that the whole becomes circular in the final state and the system is in the state of plane stress. Let $T$ be inclined at $\alpha$ to $y_1$-axis and the equation of the circular boundary be

$$z\bar{z}=a^2 \qquad \text{...(14)}$$

$a$ being its radius.

The transformation

$$z=a|\zeta \qquad \text{...(14)}$$

maps the region $z\bar{z}>a^2$ on to the region $\zeta<1$, and the region $z\bar{z}<a^2$ on to $\zeta\geqslant 1$, $|\zeta|=1$ the boundary of the circle corresponds to the boundary of the circular hole in the medium. Also if $z$ moves in an anticlockwise sense along the circular boundary, $\sigma$ describes the unit circle in the clockwise sense. Thus if $z=ae^{i\theta}$, $\sigma=e^{-i\theta}$. The potentiai functions $\Omega(z)$, $(\omega)'(z)$, $\Delta(z)$ $\delta'(z)$ will be transformed as follows

$$\left.\begin{aligned} \Omega(z) &= \Omega(a/\zeta) = \Omega_1(\zeta) \\ \omega'(z) &= \omega'(a/\zeta) = \omega_1'(\zeta) \\ \Delta(z) &= \Delta(a/\zeta) = \Delta_1(\zeta) \\ \delta'(z) &= \delta'(a|\zeta) = \delta'_1(\zeta) \end{aligned}\right\} \qquad \text{...(15)}$$

The functions $\Omega_1(\zeta)$, $\omega'_1(\zeta)$ may be evaluated by the known method [6] of linear elasticity. They are

$$\left.\begin{aligned} \Omega_1(\zeta) &= \frac{Ta}{4}\left[2\zeta e^{2i\alpha}+\frac{1}{\zeta}\right], \\ \text{and} \quad \omega_1'(\zeta) &= \frac{Ta}{2}\left[e^{2i\alpha}\zeta^3-\zeta-\frac{e^{-2i\alpha}}{\zeta}\right] \end{aligned}\right\} \qquad \text{...(16)}$$

Taking

$$\varepsilon=T|4\mu. \qquad \text{...(17)}$$

The boundary conditions for the hole are

$$\left.\begin{aligned} \frac{\partial\phi_0}{\partial\bar{z}} &= \frac{\partial\phi_0}{\partial z} = 0 \\ \frac{\partial\phi_1}{\partial\bar{z}} &= \frac{\partial\phi_1}{\partial z} = 0. \end{aligned}\right\} \qquad \text{...(18)}$$

Also at the outer boundary where

$$\left.\begin{aligned} e^{-2i\alpha}\frac{\partial^2\phi_0}{\partial\bar{z}^2} &= e^{2i\alpha}\frac{\partial^2\phi_0}{\partial z^2} = -\frac{\partial^2\phi_0}{\partial z\partial\bar{z}} \\ &= -1+0\left(\frac{1}{|z|^2}\right), \\ e^{-2i\alpha}\frac{\partial^2\phi_1}{\partial\bar{z}^2} &= e^{2i\alpha}\frac{\partial^2\phi_1}{\partial z^2} = -\frac{\partial^2\phi_1}{\partial z\partial\bar{z}} \\ &= 0\left(\frac{1}{|z|^2}\right) \end{aligned}\right\} \quad \text{...(19)}$$

with the above boundary conditions, (8) yields

$$\Delta_1(\sigma)-\frac{1}{\sigma^3}\bar{\Delta}'(\bar{\sigma})+\bar{\delta}_1(\bar{\sigma})+\gamma\Gamma(\sigma,\bar{\sigma})$$

$$+\ B_3\Omega_1(\sigma)\,\bar{\Omega}'_1(\bar{\sigma})-\frac{a}{\sigma}B_1\{\bar{\Omega}'_1(\bar{\sigma})\}^2 \quad \text{...(20)}$$

or
$$\Delta_1(\sigma)-\frac{1}{\sigma^3}\bar{\Delta}'_1(\bar{\sigma})+\bar{\delta}_1(\bar{\sigma})=F(\sigma), \quad \text{...(20A)}$$

where

$$F(\sigma)=-\{\gamma\Gamma(\sigma,\bar{\sigma})+B_3\Omega_1(\sigma)\bar{\Omega}'_1(\bar{\sigma})$$

$$-\frac{a}{\sigma}B_1\{\bar{\Omega}'(\bar{\sigma})\}^2$$

$$=\frac{B_3T^2\gamma}{16\sigma^3}\left(3\sigma^2-2e^{2i\alpha}\sigma^4+2e^{-2i\alpha}\right)$$

$$-\frac{B_1T^2a}{16\sigma^5}\left(4e^{4i\alpha}\sigma^8+3\sigma^4+4e^{-2i\alpha}\sigma^2\right) \quad \text{...(21)}$$

multiplying both sides of equation (20A) by $\zeta/\sigma(\sigma-\zeta)$ and integrating round the unit circle $\gamma(\sigma=1)$, we get

$$\Delta_1(\zeta)=-\frac{T^2a}{8}\left(2B_1\,e^{2i\alpha}\zeta^3+B_3\zeta e^{2i\alpha}\right) \quad \text{...(22)}$$

Taking the conjugate of (21), multiplying both sides by $1/(\sigma-\zeta)$ and integrating round $\gamma$, we get

$$\delta'_1(\zeta)=-\frac{T^2a}{8}\left[B_1\left(12e^{4i\alpha}\zeta^5+4e^{2i\alpha}\zeta^3+3\zeta\right)-3B_3\zeta\right]. \quad \text{...(23)}$$

The sets of potential functions $\{\Omega(z), \omega'(z)\}$ and $\{\Delta(z), \delta'(z)\}$ being evaluated the entire elastic field is obtained by making appropriate substitutions in (3) and (9).

$$D=\frac{T^2a\,(1+\kappa)}{16\mu\sigma}\,[1+2e^{2i\alpha}\,\sigma^2]-\frac{T^3a}{64\mu^2\,\sigma^5}\,[B_1\,\{\tfrac{2}{3}\,\kappa e^{4i\alpha}\,\sigma^8$$

$$+2\,(1-\kappa)\,e^{2i\alpha}\sigma^6+\tfrac{1}{2}\,(13+\kappa)\,\sigma^4+\tfrac{2}{5}\,e^{-4i\alpha}\}+$$

$$B_3\,(1+\kappa)\left\{-\frac{2}{3}\,e^{4i\alpha}\,\sigma^8+\frac{\sigma^4}{2}+4e^{2i\alpha}\,\sigma^6\right\}$$

$$-\frac{(\kappa+1)}{2}\left\{\frac{4\kappa}{3}\,e^{4i\alpha}\,\sigma^8+4\,(1-\kappa)e^{2i\alpha}\,\sigma^6\right.$$

$$\left.+(11-\kappa)\,\sigma^4+\frac{4}{5}\,e^{-4i\alpha}\right\}\Bigg]. \qquad \ldots(24)$$

The hoop stress $p_{\eta\eta}$ is given by

$$p_{\eta\eta}=-\frac{T}{2\sigma^2}\,[e^{2i\alpha}\,\sigma^4-\sigma^2+e^{-2i\alpha}]+\frac{T^2}{64\mu\sigma^4}\,[\gamma\,\{8\,e^{4i\alpha}\,\sigma^8$$

$$+2\,(\kappa-7)\,e^{2i\alpha}\,\sigma^6-(7\kappa-17)\,\sigma^4+2\,(\kappa-7)\,e^{-2i\alpha}\,\sigma^2$$

$$+8e^{-4i\alpha}\}+5B_3\,\sigma^4] \qquad \ldots(25)$$

4. **Conclusion**. As indicated by (24) it may be observed that the displacements are such that the initial configuration of hole differs significantly from the final. Also, unlike the case of infinitesimal elasticity, the hoop stress is influenced by the elastic properties of the material in the present case as is evident from (25).

APPENDIX

$$\left.\begin{aligned} C_1&=-\frac{2}{\mu}\left[\frac{\partial^2 W'}{\partial J_1^2}\right]_0, & C_2&=-\frac{2}{\mu}\left[\frac{\partial^2 W'}{\partial J_3^2}\right]_0 \\ C_3&=-\frac{2}{\mu}\left[\frac{\partial W'}{\partial J_3}\right]_0, & C_4&=-\frac{2}{\mu}\left[\frac{\partial^3 W'}{\partial J_1^3}\right]_0 \end{aligned}\right\} \qquad \ldots(26)$$

where $[\ \ ]_0$ indicates the value of the function when $J_i=0$.

$$\left.\begin{aligned} B_1&=\frac{13C_1-4C_1C_3-4C_2-7}{3C_1-1} \\ B_2&=\frac{(3C_1-1)(13C_1^2-10C_1+1)+12C_2(3C_1-1)-12(C_1-1)^2C_1C_3-8C_4}{(3C_1-1)^3} \end{aligned}\right\}(27).$$

$$\left.\begin{aligned} &B_1'=B_1-(\kappa+1)\ ;\ B_1''=\tfrac{1}{2}\,B_1'+B_1\ ;\ B_2'=B_2-\tfrac{1}{2}\,(\kappa+1)^2 \\ &B_3=B_1-\frac{2B_2}{(\kappa+1)}\ ;\ B_4=\tfrac{1}{2}\,B_1'-B_2'\ ;\ \gamma=\frac{B_1}{(\kappa+1)} \end{aligned}\right\}(15)$$

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