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On Uniform Summability-Λ of Orthonormal Expansions of Functions of Class L^p $(1<p<\infty)$

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Let $f(x)$ be an integrable function and $\{\phi_\nu(x)\}$, $\nu=0, 1, 2, \ldots$, be an orthonormal system in $[0, 1]$.

Let

$$f(x) \sim \sum_{\nu=0}^{\infty} c_\nu \phi_\nu(x)$$

where

$$c_\nu = \int_0^1 f(t)\phi_\nu(t)dt \qquad (\nu=0, 1, 2, \ldots)$$

be an orthonormal expansion of $f(x)$ in the system $\{\phi_\nu(x)\}$.

Let $\{\lambda_\nu\}_{\nu=0}^{\infty}$ be an infinite sequence of real numbers.

If the series

$$\sum_{\nu=0}^{\infty} \lambda_\nu c_\nu \phi_\nu(x)$$

converges uniformly in x, then $\{\lambda_\nu\}_0^\infty$ is called a sequence of uniform convergence factors of orthonormal expansions of $f(x)$.

We note

$$S_n(f,x)=\sum_{\nu=0}^{n}\lambda_\nu c_\nu \phi_\nu(x)=\int_0^1 f(t)\left(\sum_{\nu=0}^{n}\lambda_\nu \phi_\nu(x)\phi_\nu(t)\right)dt$$

$$=\int_0^1 f(t)\,K_n(x,t)\,dt$$

where

$$K_n(x,t)=\sum_{\nu=0}^{n}\lambda_\nu \phi_\nu(x)\phi_\nu(t).$$

In this paper, we consider the orthonormal system $\{\phi_\nu(x)\}$ closed in C, the class of continuous functions and $|\phi_\nu(x)|<A_\nu$, independent of x.

In ([3]) we have proved.

Theorem 1. A necessary and sufficient condition that $\{\lambda_\nu\}$ be a sequence of uniform convergence factors of orthonormal expansions of $f(x)\in L^p$ $(1<p<\infty)$, is that there exists an N such that

$$\int_0^1 |K_n(x,t)|^q\,dt<M$$

for all x and $n>N$.

Here p and q are conjugate exponents, *i.e.*,

$$\frac{1}{p}+\frac{1}{q}=1.$$

Let $\|\lambda_{\nu\mu}\|$ $(\nu,\mu=0,1,2,\ldots)$ be an infinite matrix of real elements.

If

$$\Lambda_{\mu,x}(f)=\sum_{\nu=0}^{\infty}\lambda_{\nu\mu}c_\nu \phi_\nu(x)$$

is uniformly convergent for each $\mu=0,1,2,\ldots$ and

$$\lim_{\mu\to\infty}\Lambda_{\mu,x}(f)=\Lambda_x(f)$$

exists uniformly in x, then orthonormal expansions of $f(x)$ is said to be uniformly Summable-Λ.

The Λ-Summation method is not necessarily regular, *i.e.*, every convergent series is not summed by it to its sum in the sense of convergence.

We note if

$$f(x) \sim \sum_{\nu=0}^{\infty} c_\nu \phi_\nu (x)$$

then

$$\begin{aligned} \Lambda_{n, \mu, x} (f) &= \sum_{\nu=0}^{n} \lambda_{\nu\mu} c_\nu \phi_\nu (x) \\ &= \int_0^1 f(t) \left(\sum_{\nu=0}^{n} \lambda_{\nu\mu} \phi_\nu (x) \phi_\nu (t)\, dt \right) \\ &= \int_0^1 f(t)\, K_{n, \mu} (x, t)\, dt \end{aligned}$$

where

$$K_{n, \mu} (x, t) = \sum_{\nu=0}^{n} \lambda_{\nu\mu} \phi_\nu (x) \phi_\nu (t).$$

We may write

$$\begin{aligned} \Lambda_{\mu, x} (f) &= \sum_{\nu=0}^{\infty} \lambda_{\nu\mu} c_\nu \phi_\nu (x) \\ &= \lim_{n \to \infty} \Lambda_{n, \mu, x} (f) \\ &= \lim_{n \to \infty} \int_0^1 f(t)\, K_{n, \mu} (x, t)\, dt. \end{aligned}$$

Formally we write

$$\begin{aligned} \Lambda_{\mu, x} (f) &= \int_0^1 f(t) \left(\sum_{\nu=1}^{\infty} \lambda_{\nu\mu} \phi_\nu (x) \phi_\nu (t) \right) dt \\ &= \int_0^1 f(t)\, K_\mu (x, t)\, dt. \end{aligned}$$

We do not mean

$$K_{\mu}(x,t)=\sum_{\nu=0}^{\infty}\lambda_{\nu\mu}\,\phi_{\nu}(x)\,\phi_{\nu}(t)$$

in the sense of ordinary congruence, but rather a formal relationship which can be made precise by a suitable definition for the sum of the series.

If $f(t)\in L^p$ $(1<p<\infty)$ and

$$\lim_{n\to\infty}\int_0^1 f(t)\,K_{n,\mu}(x,t)\,dt$$

exists for every $f(t)\in L^p$, then $K_{n,\mu}(x,t)$ converges weakly in L^q. But since L^q is weakly complete ([2], p. 240), there exists a $K_{\mu}(x,t)\in L^q$ such that

$$\wedge_{\mu,x}(f)=\lim_{n\to\infty}\int_0^1 f(t)\,K_{n,\mu}(x,t)\,dt=\int_0^1 f(t)\,K_{\mu}(x,t)\,dt.$$

Now we get our main theorem.

Theorem 2. The necessary and sufficient condition for the uniform Summability-$\wedge$ of orthonormal expansions of $f(x)\in L^p$ $(1<p<\infty)$, are

(1) $\lim\limits_{\mu\to\infty}\lambda_{\nu\mu}$ exists $(\nu=0,1,2,\ldots)$

(2) $\int_0^1 |K_{n,\mu}(x,t)|^q\,dt<M_{\mu}$ $(\mu=0,1,2,\ldots)$ independent of n and x.

(3) $\int_0^1 |K_{\mu}(x,t)|^q\,dt<M$

independent of μ and x.

Here $1<p<\infty$ and $\dfrac{1}{p}+\dfrac{1}{q}=1$.

To prove the theorem we need lemma ([3]) in a different form.

Lemma. Let $\{A_{n,x}(f)\}$ be a sequence of linear functionals defined on the Banach space X, and Y be everywhere dense subset of X.

In order that $\lim_{n\to\infty} A_{n,x}(f)$ exists uniformly in x (x belonging to a set E say) for every $f \in X$, it is necessary and sufficient that

(A) there exists an N and an M such that

$$\|A_{n,x} - A_{m,x}\| < M$$

for all x, and $n, m > N$.

(B) $\lim_{n\to\infty} A_{n,x}(g)$ exists uniformly in x, for every $g \in Y$.

Proof of the theorem.

Clearly

$$\Lambda_{\mu,x}(f) = \lim_{n\to\infty} \Lambda_{n,\mu,x}(f)$$

is a continuous linear functional on L^p $(1<p<\infty)$.

Now for each $\mu = 0, 1, 2, \ldots$, the sequence $\{\lambda_{\nu\mu}\}$ be a sequence of uniform convergence factors of orthonormal expansions of $f(x) \in L^p$ $(1<p<\infty)$ the condition (2) is necessary and sufficient by Theorem 1. We show that condition (B) of lemma is satisfied in the theorem.

Since $\{\phi_\nu(x)\}$ is closed in C, the finite linear combinations

$$g(x) = \sum_{\rho=0}^{k} \gamma_\rho \phi_\rho(x)$$

are dense in C and hence dense in L^p $(1<p<\infty)$.

From orthonormality of $\{\phi_\nu(x)\}$ we have

$$\Lambda_{\mu,x}(g) = \lim_{n=\infty} \Lambda_{n,\mu,x}(g)$$

$$= \lim_{n\to\infty} \int_0^1 \left(\sum_{\rho=0}^{k} \gamma_\rho \phi_\rho(t)\right)\left(\sum_{\nu=0}^{n} \lambda_{\nu\mu} \phi_\nu(x)\phi_\nu(t)\right)dt$$

$$= \sum_{\rho=0}^{k} \gamma_\rho \lambda_{\rho\mu} \phi_\rho(x).$$

Hence if

$$\lim_{\mu\to\infty} \Lambda_{\mu, x}(g)$$

exists uniformly in x, then

$$\lim_{\mu\to\infty} \lambda_{\nu\mu}$$

exists for each $\nu=0, 1, 2, \ldots$

So condition (B) of lemma implies condition (1) of theorem.

Now consider

$$|\Lambda_{\mu', x}(g)-\Lambda_{\mu'', x}(g)|$$

$$=\left|\lim_{n\to\infty}\int_0^1 g(t)\,(K_{n, \mu'}(x, t)-K_{n, \mu''}(x, t))dt\right|$$

$$=\left|\sum_{\rho=0}^{k}\gamma_\rho\,(\lambda_{\rho\mu'}-\lambda_{\rho\mu''})\phi_\rho(x)\right|$$

$$\leq \sum_{\rho=0}^{k}|\gamma_\rho|\,|\lambda_{\rho\mu'}-\lambda_{\rho\mu''}|\,|\phi_\rho(x)|$$

$$\leq \sum_{\rho=0}^{k}|\gamma_\rho|\,|\lambda_{\rho\mu'}-\lambda_{\rho\mu''}|\,\Lambda_\rho\,.$$

Hence if

$$\lim_{\mu\to\infty} \lambda_{\nu\mu}$$

exists for each $\nu=0, 1, 2, \ldots$, then

$$\lim_{\mu\to\infty} \Lambda_{\mu, x}(g)$$

exists uniformly in x, for every g belonging to dense subset of L^p $(1<p<\infty)$. Thus condition (1) of theorem is equivalent to condition (B) of lemma.

We note

$$||\Lambda_{\mu, x}||=\left(\int_0^1 |K_\mu(x, t)|^q dt\right)^{1/q}.$$

Now we show that condition (A) of lemma is equivalent to uniform boundedness of the norm, *i.e.*,

$$\|\wedge_{\mu, x}\| < M$$

for all μ and x, hence equivalent to condition (3) of theorem.

Assume

$$\|\wedge_{\mu, x}\| < M$$

for all x, and $\mu > N$.

We get

$$\|\wedge_{\mu', x} - \wedge_{\mu'', x}\| \leq \|\wedge_{\mu', x}\| + \|\wedge_{\mu'', x}\| < 2M$$

for all x, and $\mu', \mu'' > N$.

So condition (1) of theorem implies condition (A) of lemma.

Now since $|\phi_\nu(x) < A_\nu$, independent of x

$$|K_\mu(x, t)| \lessdot c_\mu$$

independent of x and t.

Consider

$$|\wedge_{\mu, x}(f)| = \left|\int_0^1 f(t)\, K_\mu(x, t)\, dt\right| \leqslant \|f\|_p$$

$$\text{ess sup } |K_\mu(x, t) \leqslant \|f\|_p\, \mu.$$

Therefore

$$\| < _{\mu, x} \| \leqslant C_\mu.$$

Now condition (A) of lemma gives

$$\|\wedge_{\mu', x} - \wedge_{\mu'', x}\| < M$$

for all x, and $\mu', \mu'' > N$.

Therefore

$$\|\wedge_{\mu', x}\| \leqslant \|\wedge_{\mu'', x}\| + M \leqslant C_{\mu''} + M = M'$$

for all x, and $\mu' > N - N'$.

Hence condition (A) of lemma is equivalent to condition (3) of theorem.

Thus we have showed that conditions (1) and (3) of theorem are necessary and sufficient for the uniform convergence of $\Lambda_{\mu, x}(f)$. The proof of theorem is complete.

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