Jñānabha Sect. A, Vol. 5, July 1975.

A Note on a Recurrence Formula for Appell's Function F_4

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(Received : January 15, 1975)

In this note, the application of a recurrence formula for the hypergeometric function ${}_0F_1$ to a double Laplace integral representation of Appell's function F_4 gives a recurrence relation for this latter function Reduction formulae involving products of pairs of Gauss functions. are then deduced which may find practical application in, for example, the evaluation of certain coefficients in two-centre expansions employed in certain aspects of molecular physics, and first obtained by Sack [1].

The expression, [3, p. 101]

$$(1) \qquad (\lambda)_m = \{1/\Gamma(\lambda)\} \int_0^\infty e^{-t}\, t^{\lambda+m-1}\, dt, \ \mathrm{Re}(\lambda)>0,\ m=0,\ 1,\ 2,\ldots,$$

readily gives the result

$$(2) \qquad F_4(a, b\ ;\ c, c'\ ;\ x, y)$$

$$=\{1/\Gamma(a)\,\Gamma(b)\} \int_0^\infty \int_0^\infty e^{-(s+t)}\, s^{a-1}\, t^{b-1}\ {}_0F_1(-;\ c;\ xst)$$

$${}_0F_1(-;\ c';\ yst)\ ds\ dt,\ \mathrm{Re}\ (a),\ (b) > 0,$$

where

$$(3) \quad F_4(a, b\ ;\ c, c'\ ;\ x,y) = \sum_{m,\ n=0}^{\infty} \frac{(a)_{m+n}\,(b)_{m+n}\, x^m y^n}{(c)_m\,(c')_n\, m!\ m!},\ \sqrt{|x|}+\sqrt{|y|}<1.$$

If the well-known recurrence relation for the Bessel function

$$(4) \qquad J_{\nu-1}(x)+J_{\nu+1}(x)=(2\nu/x)\,J_{\nu}(x)$$

is expressed in terms of the function ${}_0F_1$ by means of the formula

$$(5) \qquad J_{\nu-\frac{1}{2}}(x)=\{(\tfrac{1}{2}x)^{\nu-\frac{1}{2}}/\Gamma(\nu+\tfrac{1}{2})\}\,{}_0F_1(-;\,\nu+\tfrac{1}{2};\,-\tfrac{1}{4}x^2),$$

we obtain

$$(6) \qquad {}_0F_1(-;\,c\,;\,x)={}_0F_1(-;\,c-1\,;\,x)\,-\{x/c(c-1)\}\,{}_0F_1(-;c+1;\,x).$$

This last expression is now applied to one of the hypergeometric functions of the integrand of (2), when the main result of this note follows :

$$(7) \quad F_4(a,\,b;\,c,\,c':\,x,\,y)=F_4(a,\,b;\,c-1,\,c';\,x,\,y)+\{abx/c(1-c)\}\,F_4(a+1,\,b+1;\,c+1,\,c';\,x,\,y).$$

Application. Slater [2, *p*. 222] cites a special F_4 series as a product of two ${}_2F_1$ series :

$$(8) \qquad F_4(a,\,b;\,c,\,1+a+b-c\,;\,x/[(x-1)\,(1-y)],\,y/[(x-1)(1-y)])$$
$$={}_2F_1(a,\,b;\,c;\,x/(x-1))\,{}_2F_1(a,\,b;\,1+a+b-c;\,y/(y-1)).$$

If, in (7), c' is taken to be equal to $2+a+b-c$, then both the F_4 series on the right-hand side may be reduced by means of (8) giving

$$(9) \qquad F_4(a,\,b;\,c,\,2+a+b-c;\,x/[(x-1)(1-y)],\,y/[(x-1)\,(1-y)])$$
$$={}_2F_1(a,\,b;\,c-1\,;\,x/(x-1))\,{}_2F_1(a,\,b;\,2+a+b-c\,;\,y/(y-1))$$
$$+\frac{abx}{c(1-c)(x-1)(1-y)}\,{}_2F_1(a+1,\,b+1;\,c+1;\,x/(x-1))$$
$${}_2F_1(a+1,\,b+1;\,2+a+b-c;\,y/(y-1)).$$

If this process is repeated several times, then an expression for $F_4(a,\,b;\,c,\,n+1+a+b-c\,;\,x/[(x-1)(1-y)],\,y/[(x-1)(1-y)])$ will follow, where n is a positive integer. In practice, this result would have somewhat limited value because the number of terms involving products of pairs of Gauss functions would increase rapidly with n. For lower values of n, however, the somewhat intractable nature of the function F_4 may justify the use of the above type of reduction.

REFERENCE

1. R.A. Sack, *Two-center expansions for the powers of the distance between two points*, J. Math. Phys. **5** (1964), 260-268.

2. L.J. Slater, *Generalised Hypergeometric Functions*, Cambridge University Press, 1966.

3. H. M. Srivastava, *Some integrals representing triple hypergeometric functions*, Rend. Circ. Mat. Palermo Ser. II 16 (1967), 99-115.