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PARAMETRIC SET-VALUED OPTIMIZATION PROBLEMS UNDER GENERALIZED CONE CONVEXITY

By

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Abstract

In this paper, we establish sufficient Karush-Kuhn-Tucker (*KKT*) conditions for a parametric set-valued optimization problem under contingent epiderivative and generalized cone convexity assumptions. We also study duality results of Mond-Weir, Wolfe, and mixed types for weak solutions of a pair of set-valued optimization problems.

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1 Introduction

Parametric optimization problem is a special class of optimization problems. It is mainly studied in various fields of mathematics, operational research, and economics. Optimization problems with parameters for single-valued case were studied by various authors like Ioffe [16], Khanh [18, 19, 20], Khanh and Nuong [23, 24], and Nuong [25]. This class of optimization problems has applications in deriving the Pontryagin maximum principle for control problems with state constraints. Khanh and Luu [21, 22] studied this type of problems in set-valued case. They established the necessary optimality conditions of Fritz John and Kuhn-Tucker types under relaxed differentiability assumptions on the state variable and convexlikeness assumptions on the parameter. Parametric set-valued optimization problem arises in many situations where optimization problems involve set-valued maps and the equality constraint represents equations, like differential equations, and initial conditions. The case where the differential inclusions replace the differential equations to describe the system under consideration can also be considered as parametric set-valued optimization problem.

In this paper, a parametric set-valued optimization problem (PP) is considered, where the objective function and functions attached to constraints are set-valued maps. The sufficient *KKT* conditions are established for the problem (PP) via contingent epiderivative and generalized cone convexity assumptions. Finally, different types of duality are formulated and the relationships between the primal problem (PP) and the corresponding dual problems are studied.

This paper is organized as follows. ***Section* 2** deals with some definitions and preliminary concepts of set-valued maps. In ***Section* 3**, a parametric set-valued optimization problem (PP) is considered and the sufficient *KKT* conditions are established for the problem (PP). Various types of duality theorems are studied under contingent epiderivative and generalized cone convexity assumptions.

2 Definition and preliminaries

Let Y be a real normed space and K be a nonempty subset of Y. Then K is said to be a cone if $\lambda y \in K$, for all $y \in K$ and $\lambda \geq 0$. Further, K is called pointed if $K \cap (-K) = \{\theta_Y\}$, solid if $\mathrm{int}(K) \neq \emptyset$, closed if $\overline{K} = K$ and convex if $\lambda K + (1-\lambda)K \subseteq K$, for all $\lambda \in [0,1]$, where $\mathrm{int}(K)$ and $\overline{K}$ denote the interior and closure of K, respectively and θ_Y is the zero element of Y.

Let us define the non-negative orthant $\mathbb{R}^m_+$ of the m-dimensional Euclidean space $\mathbb{R}^m$ by

$$\mathbb{R}^m_+ = \{y = (y_1, ..., y_m) \in \mathbb{R}^m : y_i \geq 0, \forall i = 1, 2, ..., m\}.$$

Then $\mathbb{R}^m_+$ is a solid pointed closed convex cone and $\mathrm{int}(\mathbb{R}^m_+) \cup \{\mathbf{0}_{\mathbb{R}^m}\}$ is a solid pointed convex cone in $\mathbb{R}^m$, where $\mathbf{0}_{\mathbb{R}^m}$ is the zero element of $\mathbb{R}^m$.

Let Y^* be the space of all continuous linear functionals on Y and K be a solid pointed convex cone in Y. Then the dual cone K^+ to K and quasi-interior K^{+i} of K^+ are defined as

$$K^+ = \{y^* \in Y^* : \langle y^*, y\rangle \geq 0, \forall y \in K\}$$

and

$$K^{+i} = \{y^* \in Y^* : \langle y^*, y\rangle > 0, \forall y \in K \setminus \{\theta_Y\}\},$$

where $\langle .,. \rangle$ is the canonical bilinear form with respect to the duality between Y^* and Y.

Let B be a nonempty subset of a real normed space Y and $y^* \in Y^*$. Define a subset $\langle y^*, B\rangle$ of $\mathbb{R}$ by

$$\langle y^*, B\rangle = \cup_{y\in B}\{\langle y^*, y\rangle\}.$$

For any two nonempty subsets B, B' of Y and $y^* \in Y^*$, we also use the following notations

$$\langle y^*, B\rangle \geq 0(\text{or, } 0 \leq \langle y^*, B\rangle) \Leftrightarrow \langle y^*, b\rangle \geq 0, \quad \forall b \in B$$

and

$$\langle y^*, B\rangle \geq \langle y^*, B'\rangle(\text{or, } \langle y^*, B'\rangle \leq \langle y^*, B\rangle) \Leftrightarrow \langle y^*, b\rangle \geq \langle y^*, b'\rangle, \forall b \in B \text{ and } \forall b' \in B'.$$

Let K be a solid pointed convex cone in Y. There are two types of cone-orderings in Y with respect to K. For any two elements $y_1, y_2 \in Y$, we have

$$y_1 \leq y_2 \Leftrightarrow y_2 - y_1 \in K$$

and

$$y_1 < y_2 \Leftrightarrow y_2 - y_1 \in \text{int}(K).$$

We say $y_2 \geq y_1$, if $y_1 \leq y_2$ and $y_2 > y_1$, if $y_1 < y_2$. For any two nonempty subsets B, B' of Y, we use the following notations:

$$B \leq \theta_Y \Leftrightarrow y \leq \theta_Y, \forall y \in B,$$

$$B < \theta_Y \Leftrightarrow y < \theta_Y, \forall y \in B,$$

$$B \leq B' \Leftrightarrow y \leq y', \forall y \in B \text{ and } \forall y' \in B',$$

and

$$B < B' \Leftrightarrow y < y', \forall y \in B \text{ and } \forall y' \in B'.$$

The following notions of minimality are mainly used in a real normed space Y with respect to a solid pointed convex cone K of Y.

***Definition* 2.1** *Let B be a nonempty subset of Y. Then minimal and weakly minimal points of B are defined as*

(i) *$y' \in B$ is a minimal point of B if there is no $y \in B \setminus \{y'\}$ such that $y \leq y'$.*
(ii) *$y' \in B$ is a weakly minimal point of B if there is no $y \in B$ such that $y < y'$.*

The sets of minimal points and weakly minimal points of B are denoted by min(B) and w-min(B), respectively and characterized as

$$\min(B) = \{y' \in B : (y' - K) \cap B = \{y'\}\}$$

and

$$\text{w-min}(B) = \{y' \in B : (y' - \text{int}(K) \cap B = \emptyset\}.$$

Similarly, the sets of maximal points and weak maximal points of B can be defined.

Let X and Y be real normed spaces, 2^Y be the set of all subsets of Y and K be a solid pointed convex cone in Y. Let $F : X \to 2^Y$ be a set-valued map from X to Y i.e., $F(x) \subseteq Y$, for all $x \in X$. The effective domain, graph and epigraph of F are defined by

$$\text{dom}(F) = \{x \in X : F(x) \neq \emptyset\},$$

$$F(A) = \cup_{x\in A}F(x), \text{ for any } A(\neq \emptyset) \subseteq X,$$

$$\text{gr}(F) = \{(x, y) \in X \times Y : y \in F(x)\},$$

and

$$\text{epi}(F) = \{(x, y) \in X \times Y : y \in F(x) + K\}.$$

We recall the notion of contingent cone in a real normed space.

***Definition* 2.2** [1, 2] *Let Y be a real normed space, $\emptyset \neq B \subseteq Y$, and $y' \in \overline{B}$. The contingent cone to B at y' is denoted by $T(B, y')$ and is defined as follows*

An element $y \in T(B, y')$ if there exist sequences $\{\lambda_n\}$ in $\mathbb{R}$, with $\lambda_n \to 0^+$ and $\{y_n\}$ in Y, with $y_n \to y$, such that

$$y' + \lambda_n y_n \in B, \forall n \in \mathbb{N},$$

or, there exist sequences $\{t_n\}$ in $\mathbb{R}$, with $t_n > 0$ and $\{y'_n\}$ in B, with $y'_n \to y'$, such that

$$t_n(y'_n - y') \to y.$$

Let $F : X \to 2^Y$ be a set-valued map, with $\mathrm{dom}(F) = X$, $x' \in X$, and $y' \in F(x')$. Jahn and Rauh [17] introduced the notion of contingent epiderivative of set-valued maps which plays a vital role in various aspects of set-valued optimization problems.

***Definition* 2.3** [17] *A single-valued map $D_\uparrow F(x', y') : X \to Y$ whose epigraph coincides with the contingent cone to the epigraph of F at (x', y'), i.e.*

$$\mathrm{epi}(D_\uparrow F(x', y')) = T(\mathrm{epi}(F), (x', y')),$$

is said to be the contingent epiderivative of F at (x', y').

***Proposition* 2.1** *When $f : X \to \mathbb{R}$ is a real-valued map, being continuous at $x_0 \in X$ and convex,*

$$D_\uparrow f(x_0, f(x_0))(u) = f'(x_0)(u), \forall u \in X,$$

where $f'(x_0)(u)$ is the directional derivative of f at x_0 in the direction u.

Borwein [3] introduced the notion of cone convexity of set-valued maps.

***Definition* 2.4** [3] *Let A be a nonempty convex subset of X. A set-valued map $F : X \to 2^Y$, with $A \subseteq \mathrm{dom}(F)$, is called K-convex on A if $\forall x_1, x_2 \in A$ and $\lambda \in [0, 1]$,*

$$\lambda F(x_1) + (1 - \lambda)F(x_2) \subseteq F(\lambda x_1 + (1 - \lambda)x_2) + K.$$

It is clear that if the set-valued map $F : X \to 2^Y$ is K-convex on A, then $\mathrm{epi}(F)$ is a convex subset of $X \times Y$.

A cone convex set-valued map cab be characterized in terms of contingent epiderivative of set-valued maps.

***Lemma* 2.1** [17] *If $F : X \to 2^Y$ is K-convex on a nonempty convex subset A of X, then $\forall x, x' \in A$ and $y' \in F(x')$,*

$$F(x) - y' \subseteq D_\uparrow F(x', y')(x - x') + K.$$

3 Main results

Das and Nahak [4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15] introduced the notion of ρ-cone convex set-valued maps. They establish the sufficient *KKT* conditions and study the duality results for various types of set-valued optimization problems under contingent epiderivative and ρ-cone convexity assumptions. For $\rho = 0$, we have the usual notion of cone convexity of set-valued maps introduced by Borwein [3].

***Definition* 3.1** [4, 7] *Let X, Y be real normed spaces, A be a nonempty convex subset of X, K be a solid pointed convex cone in Y, $e \in \mathrm{int}(K)$, and $F : X \to 2^Y$ be a set-valued map, with $A \subseteq \mathrm{dom}(F)$. Then F is said to be ρ-K-convex with respect to e on A if there exists $\rho \in \mathbb{R}$ such that*

$$\lambda F(x_1) + (1 - \lambda)F(x_2) \subseteq F(\lambda x_1 + (1 - \lambda)x_2) + \rho\lambda(1 - \lambda)\|x_1 - x_2\|^2 e + K, \forall x_1, x_2 \in A \text{ and } \forall \lambda \in [0, 1].$$

Das and Nahak [7] constructed an example of ρ-cone convex set-valued map, which is not cone convex. They also characterized ρ-cone convex set-valued maps in terms of contingent epiderivative of set-valued maps.

***Theorem* 3.1** [7] *Let A be a nonempty convex subset of X, $e \in \mathrm{int}(K)$, and $F : X \to 2^Y$ be ρ-K-convex with respect to e on A. Let $x' \in A$ and $y' \in F(x')$. Then,*

$$F(x) - y' \subseteq D_\uparrow F(x', y')(x - x') + \rho\|x - x'\|^2 e + K, \forall x \in A.$$

***Remark* 3.1** *If $\rho > 0$, then F is said to be strongly ρ-K-convex, if $\rho = 0$, we have the usual notion of K-convexity, and if $\rho < 0$, then F is said to be weakly ρ-K-convex.*
Obviously, strongly ρ-K-convexity $\Rightarrow$ K-convexity $\Rightarrow$ weakly ρ-K-convexity.

***Remark* 3.2** *For the case of single-valued map,* **Definition 3.1** *coincides with the existing one. Let X, Y be real normed spaces, A be a nonempty convex subset of X, K be a solid pointed convex cone in Y, $x' \in X$, and $e \in \mathrm{int}(K)$. Let $f : X \to Y$ be continuously differentiable function and convex. By considering $F(x) = \{f(x)\}$, from* **Definition 3.1** *and* **Proposition 2.1**, *we can conclude that f is called ρ-K-convex with respect to e on A if there exists $\rho \in \mathbb{R}$ such that*

$$f(x) - f(x') \in f'(x')(x - x') + \rho\|x - x'\|^2 e + K, \forall x \in X.$$

The followings are some special cases.

When $Y = \mathbb{R}^m$, $K = \mathbb{R}^m_+$, $f = (f_1, f_2, ..., f_m)$, and $e = (1, 1, ..., 1)$, we have

$$f_i(x) - f_i(x') \geq f_i'(x')(x - x') + \rho\|x - x'\|^2, \forall x \in X \text{ and } i = 1, 2, ..., m.$$

When $Y = \mathbb{R}$, $K = \mathbb{R}_+$, and $e = 1$, we have

$$f(x) - f(x') \geq f'(x')(x - x') + \rho\|x - x'\|^2, \forall x \in X.$$

When $X = \mathbb{R}^n$, $Y = \mathbb{R}$, $K = \mathbb{R}_+$, and $e = 1$, we have

$$f(x) - f(x') \geq (x - x')^T \nabla f(x') + \rho\|x - x')\|^2, \forall x \in X,$$

where $\nabla f(x')$ is the gradient of f at x'.

Let X, Y, Z, and W be real normed spaces and K, L, and M be solid pointed convex cones in Y, Z, and W, respectively. Let U be an arbitrary set and A be a nonempty subset of X. Suppose that $F : X\times U \to 2^Y$, $G : X\times U \to 2^Z$ are set-valued maps and $p : X \times U \to W$ is a single-valued map with

$$A \times U \subseteq \mathrm{dom}(F) \cap \mathrm{dom}(G).$$

We consider a parametric set-valued optimization problem (PP).

$$(PP)\quad \begin{array}{ll} \underset{(x,u)\,\in\, A\times U}{\text{minimize}} & F(x,u) \\ \text{subject to} & G(x,u) \cap (-L) \neq \emptyset \\ & p(x,u) = 0, \end{array}$$

where x is the state variable and u is the parameter.

The feasible set S of the problem (PP) is defined by

$$S = \{(x,u) \in A \times U : G(x,u) \cap (-L) \neq \emptyset \text{ and } p(x,u) = 0\}.$$

The minimizer and weak minimizer of the problem (PP) are defined in the following ways.

***Definition* 3.2** *A point $(x', u', y') \in X \times U \times Y$, with $(x', u') \in S$ and $y' \in F(x', u')$, is called a minimizer of the problem (PP) if there exists no point $(x, u, y) \in X \times U \times Y$, with $(x, u) \in S$ and $y \in F(x, u)$, such that*

$$y - y' \in -K \setminus \{\theta_Y\}.$$

***Definition* 3.3** *A point $(x', u', y') \in X \times U \times Y$, with $(x', u') \in S$ and $y' \in F(x', u')$, is called a weak minimizer of the problem (PP) if there exists no point $(x, u, y) \in X \times U \times Y$, with $(x, u) \in S$ and $y \in F(x, u)$, such that*

$$y - y' \in -\mathrm{int}(K).$$

3.1 Sufficient optimality conditions

Let $(x, u), (x', u), (x, u'), (x', u') \in X \times U$, $y' \in F(x', u')$, and $z' \in G(x', u')$. Throughout the paper, we use the following assumptions.

(3.1) $F(x, u) - F(x', u) \subseteq K$,

(3.2) $G(x, u) - G(x', u) \subseteq L$,

(3.3) $F(x, u') - y' \subseteq -K$,

(3.4) $G(x, u') - z' \subseteq -L$,

and

(3.5) $p(x, u') + p(x', u) \in -M$.

We now prove the following ***Lemma* 3.1** which is required in establishing the sufficient KKT conditions of the parametric set-valued optimization problem (PP).

***Lemma* 3.1** *Let A be a nonempty convex subset of X and $(x', u') \in X \times U$, with $y' \in F(x', u')$, $z' \in G(x', u')$, and $p(x', u') \geq 0$. Let $e \in \mathrm{int}(K)$, $e' \in \mathrm{int}(L)$, and $e'' \in \mathrm{int}(M)$. Suppose that $F(., u') : X \to 2^Y$ is ρ_1-K-convex with respect to e, $G(., u') : X \to 2^Z$ is ρ_2-L-convex with respect to e', and $p(., u') : X \to W$ is ρ_3-M-convex with respect to e'', on A. Assume that the contingent epiderivatives $D_\uparrow F(., u')(x', y')$ and $D_\uparrow G(., u')(x', z')$ exist and the Gâteaux derivative $p'(., u')(x')$ exists. If Eqs.* (3.1) - (3.5) *are satisfied, then we have*

$$\begin{aligned}(3.6)\quad & \langle y^*, F(x, u) - y'\rangle + \langle z^*, G(x, u) - z'\rangle \\ & \geq \langle y^*, D_\uparrow F(., u')(x', y')(x - x') + F(x', u) - y'\rangle \\ & +\langle z^*, D_\uparrow G(., u')(x', z')(x - x') + G(x', u) - z'\rangle \\ & +\langle w^*, p'(., u')(x')(x - x') + p(x', u)\rangle \\ & +\|x - x'\|^2(\rho_1\langle y^*, e\rangle + \rho_2\langle z^*, e'\rangle + \rho_3\langle w^*, e''\rangle), \forall (x, u) \in A \times U.\end{aligned}$$

Proof. Let $(x, u) \in A \times U$. As $F(., u') : X \to 2^Y$ is ρ_1-K-convex with respect to e on A and $y' \in F(x', u')$, we have

(3.7) $F(x, u') - y' \subseteq D_{\uparrow}F(., u')(x', y')(x - x') + \rho_1\|x - x'\|^2 e + K.$

As $G(., u') : X \to 2^Z$ is ρ_2-L-convex with respect to e' on A and $z' \in G(x', u')$, we have

(3.8) $G(x, u') - z' \subseteq D_{\uparrow}G(., u')(x', z')(x - x') + \rho_2\|x - x'\|^2 e' + L.$

Again, as $p(., u') : X \to W$ is ρ_3-M-convex with respect to e'' on A, we have

(3.9) $p(x, u') - p(x', u') \in p'(., u')(x')(x - x') + \rho_3\|x - x'\|^2 e'' + M.$

Hence, from Eqs. (3.7) - (3.9), we have

$$\begin{aligned}
&\langle y^*, F(x, u') - y' + F(x', u) - y'\rangle + \langle z^*, G(x, u') - z' + G(x', u) - z'\rangle \\
&\quad + \langle w^*, p(x, u') - p(x', u') + p(x', u)\rangle \\
&\geq \\
&\langle y^*, D_{\uparrow}F(., u')(x', y')(x - x') + F(x', u) - y'\rangle \\
&\quad + \langle z^*, D_{\uparrow}G(., u')(x', z')(x - x') + G(x', u) - z'\rangle \\
&\quad + \langle w^*, p'(., u')(x')(x - x') + p(x', u)\rangle \\
&\quad + \|x - x'\|^2(\rho_1\langle y^*, e\rangle + \rho_2\langle z^*, e'\rangle + \rho_3\langle w^*, e''\rangle).
\end{aligned}$$

By Eqs. (3.1) - (3.5), we have

$$\langle y^*, F(x, u) - y'\rangle \geq \langle y^*, F(x', u) - y'\rangle,$$

$$\langle z^*, G(x, u) - z'\rangle \geq \langle z^*, G(x', u) - z'\rangle,$$

$$\langle y^*, F(x, u') - y'\rangle \leq 0,$$

$$\langle z^*, G(x, u') - z'\rangle \leq 0,$$

and

$$\langle w^*, p(x, u') + p(x', u)\rangle \leq 0.$$

By assumption, we have

$$p(x', u') \geq 0.$$

Therefore,

$$\begin{aligned}
&\langle y^*, F(x, u) - y'\rangle + \langle z^*, G(x, u) - z'\rangle \\
&\geq \\
&\langle y^*, F(x, u') - y' + F(x', u) - y'\rangle + \langle z^*, G(x, u') - z' + G(x', u) - z'\rangle \\
&\quad + \langle w^*, p(x, u') - p(x', u') + p(x', u)\rangle.
\end{aligned}$$

Consequently,

$$\begin{aligned}
&\langle y^*, F(x, u) - y'\rangle + \langle z^*, G(x, u) - z'\rangle \\
&\geq \\
&\langle y^*, D_{\uparrow}F(., u')(x', y')(x - x') + F(x', u) - y'\rangle \\
&\quad + \langle z^*, D_{\uparrow}G(., u')(x', z')(x - x') + G(x', u) - z'\rangle \\
&\quad + \langle w^*, p'(., u')(x')(x - x') + p(x', u)\rangle \\
&\quad + \|x - x'\|^2(\rho_1\langle y^*, e\rangle + \rho_2\langle z^*, e'\rangle + \rho_3\langle w^*, e''\rangle).
\end{aligned}$$

It completes the proof of the ***Lemma* 3.1**.

The sufficient *KKT* conditions of the parametric set-valued optimization problem (PP) are established under contingent epiderivative and generalized cone convexity assumptions.

***Theorem* 3.2** *(Sufficient optimality conditions) Let A be a nonempty convex subset of X and $(x', u') \in X \times U$, with $(x', u') \in S$, $y' \in F(x', u')$, $z' \in G(x', u') \cap (-L)$, and $p(x', u') \geq 0$. Let $e \in \mathrm{int}(K)$, $e' \in \mathrm{int}(L)$, and $e'' \in \mathrm{int}(M)$. Suppose that $F(., u') : X \to 2^Y$ is ρ_1-K-convex with respect to e, $G(., u') : X \to 2^Z$ is ρ_2-L-convex with respect to e', and $p(., u') : X \to W$ is ρ_3-M-convex with respect to e'', on A. Assume that the contingent epiderivatives*

$D_{\uparrow}F(.,u')(x',y')$ and $D_{\uparrow}G(.,u')(x',z')$ exist and the Gâteaux derivative $p'(.,u')(x')$ exists. Suppose that the conditions of ***Lemma* 3.1** *hold at $(x',u',y',z',y^*,z^*,w^*)$ for some $(y^*,z^*,w^*) \in K^+ \times L^+ \times M^+$, with $y^* \neq \theta_{Y^*}$ and*

(3.10) $\rho_1\langle y^*, e\rangle + \rho_2\langle z^*, e'\rangle + \rho_3\langle w^*, e''\rangle \geq 0,$

such that

(3.11)
$$\langle y^*, D_{\uparrow}F(.,u')(x',y')(x-x') + F(x',u) - y'\rangle$$
$$+\langle z^*, D_{\uparrow}G(.,u')(x',z')(x-x') + G(x',u) - z'\rangle$$
$$+\langle w^*, p'(.,u')(x')(x-x') + p(x',u)\rangle$$
$$\geq 0, \forall (x,u) \in A \times U$$

and

(3.12) $\langle z^*, z'\rangle = 0,$

then (x',u',y') is a weak minimizer of the problem (PP).

Proof. Suppose that (x',u',y') is not a weak minimizer of the problem (PP).

Then there exist $(x,u) \in S$ and $y \in F(x,u)$ such that

$$y < y'.$$

As $y^* \in K^+ \setminus \{\theta_{Y^*}\}$,

$$\langle y^*, y - y'\rangle < 0.$$

As $(x,u) \in S$, there exists

$$z \in G(x,u) \cap (-L).$$

So,

$$\langle z^*, z\rangle \leq 0, \text{ as } z^* \in L^+.$$

Since $\langle z^*, z'\rangle = 0$, we have

$$\langle z^*, z - z'\rangle = \langle z^*, z\rangle \leq 0.$$

Therefore,

(3.13) $\langle y^*, y - y'\rangle + \langle z^*, z - z'\rangle < 0.$

As the conditions of ***Lemma* 3.1** hold at $(x',u',y',z',y^*,z^*,w^*)$, from Eqs. (3.6), (3.10), and (3.11), we have

$$\langle y^*, F(x,u) - y'\rangle + \langle z^*, G(x,u) - z'\rangle \geq 0.$$

Hence,

$$\langle y^*, y - y'\rangle + \langle z^*, z - z'\rangle \geq 0,$$

which contradicts (3.13).

Consequently, (x',u',y') is a weak minimizer of the problem (PP).

3.2 Mond-Weir type dual

We consider a Mond-Weir type dual (MWD), where $F(.,u')$ and $G(.,u')$ are contingent epiderivable set-valued maps and $p(.,u')$ is a Gâteaux derivable single-valued map, where $u' \in U$.

maximize y',

subject to,

(MWD)
$$\begin{aligned}
&\langle y^*, D_{\uparrow}F(.,u')(x',y')(x-x') + F(x',u) - y'\rangle \\
&+ \langle z^*, D_{\uparrow}G(.,u')(x',z')(x-x') + G(x',u) - z'\rangle \\
&+ \langle w^*, p'(.,u')(x')(x-x') + p(x',u)\rangle \geq 0, \forall (x,u) \in A \times U, \\
&\langle z^*, z'\rangle \geq 0, \\
&x' \in A, u' \in U, y' \in F(x',u'), z' \in G(x',u'), p(x',u') \geq 0, \\
&(y^*,z^*,w^*) \in K^+ \times L^+ \times M^+, \text{ and } \langle y^*, e\rangle = 1.
\end{aligned}$$

***Definition* 3.4** *A point $(x', u', y', z', y^*, z^*, w^*)$ satisfying all the constraints of the problem (MWD) is called a feasible point of (MWD).*

***Definition* 3.5** *A feasible point $(x', u', y', z', y^*, z^*, w^*)$ of the problem (MWD) is called a weak maximizer of the problem (MWD) if there exists no feasible point $(x, u, y, z, y_1^*, z_1^*, w_1^*)$ of (MWD) such that*

$$y - y' \in \text{int}(K).$$

***Theorem* 3.3** *(Weak duality) Let A be a nonempty convex subset of X, $(x_0, u_0) \in S$, $(x', u', y', z', y^*, z^*, w^*)$ be a feasible point of the problem (MWD), and $p(x', u') \geq 0$. Let $e \in \text{int}(K)$, $e' \in \text{int}(L)$, and $e'' \in \text{int}(M)$. Suppose that $F(., u') : X \to 2^Y$ is ρ_1-K-convex with respect to e, $G(., u') : X \to 2^Z$ is ρ_2-L-convex with respect to e', and $p(., u') : X \to W$ is ρ_3-M-convex with respect to e'', on A. Assume that the contingent epiderivatives $D_\uparrow F(., u')(x', y')$ and $D_\uparrow G(., u')(x', z')$ exist and the Gâteaux derivative $p'(., u')(x')$ exists. Suppose that the conditions of **Lemma 3.1** hold at $(x', u', y', z', y^*, z^*, w^*)$. Assume that*

(3.14) $\rho_1 + \rho_2\langle z^*, e'\rangle + \rho_3\langle w^*, e''\rangle \geq 0.$

Then,

$$F(x_0, u_0) - y' \subseteq Y \setminus -\text{int}(K).$$

Proof. We prove the theorem by the method of contradiction.
Suppose that for some $y_0 \in F(x_0, u_0)$,

$$y_0 - y' \in -\text{int}(K).$$

Therefore,

$$\langle y^*, y_0 - y'\rangle < 0, \text{ as } \theta_{Y^*} \neq y^* \in K^+.$$

Again, since $(x_0, u_0) \in S$, we have

$$G(x_0, u_0) \cap (-L) \neq \emptyset \text{ and } p(x_0, u_0) = 0.$$

We choose $z_0 \in G(x_0, u_0) \cap (-L)$.
So,

$$\langle z^*, z_0\rangle \leq 0, \text{ as } z^* \in L^+.$$

Again, from the constraints of (MWD), we have

$$\langle z^*, z'\rangle \geq 0.$$

Therefore,

$$\langle z^*, z_0 - z'\rangle = \langle z^*, z_0\rangle - \langle z^*, z'\rangle \leq 0.$$

Hence,

(3.15) $\langle y^*, y_0 - y'\rangle + \langle z^*, z_0 - z'\rangle < 0.$

As the conditions of ***Lemma* 3.1** hold at $(x', u', y', z', y^*, z^*, w^*)$, from Eqs. (3.6), (3.14), and the constraints of (MWD), we have

$$\langle y^*, F(x_0, u) - y'\rangle + \langle z^*, G(x_0, u) - z'\rangle \geq 0.$$

Hence,

$$\langle y^*, y_0 - y'\rangle + \langle z^*, z_0 - z'\rangle \geq 0,$$

which contradicts (3.15).
Therefore,

$$F(x_0, u_0) - y' \subseteq Y \setminus -\text{int}(K).$$

It completes the proof of the ***Theorem* 3.3**.

***Theorem* 3.4** *(Strong duality) Let (x', u', y') be a weak minimizer of the problem (PP) and $z' \in G(x', u') \cap (-L)$. Assume that for some $(y^*, z^*, w^*) \in K^+ \times L^+ \times M^+$, with $\langle y^*, e\rangle = 1$, Eqs. (3.11) and (3.12) are satisfied at $(x', u', y', z', y^*, z^*, w^*)$. Then $(x', u', y', z', y^*, z^*, w^*)$ is a feasible solution for (MWD). If the weak duality **Theorem 3.3** between (PP) and (MWD) holds, then $(x', u', y', z', y^*, z^*, w^*)$ is a weak maximizer of (MWD).*

Proof. As the Eqs. (3.11) and (3.12) are satisfied at $(x', u', y', z', y^*, z^*, w^*)$,

$$\langle y^*, D_\uparrow F(., u')(x', y')(x - x') + F(x', u) - y'\rangle$$
$$+ \langle z^*, D_\uparrow G(., u')(x', z')(x - x') + G(x', u) - z'\rangle$$
$$+ \langle w^*, p'(., u')(x')(x - x') + p(x', u)\rangle \geq 0, \forall (x, u) \in A \times U$$

and

$$\langle z^*, z'\rangle = 0.$$

As $(x', u') \in S$,

$$p(x', u') = 0.$$

Hence, (x', y', z', y^*, z^*) is a feasible solution for (MWD).

Suppose that the weak duality ***Theorem* 3.3** between the problems (PP) and (MWD) holds and $(x', u', y', z', y^*, z^*, w^*)$ is not a weak maximizer of the problem (MWD).

Let $(x, u, y, z, y_1^*, z_1^*, w_1^*)$ be a feasible point for (MWD) such that

$$y' - y \in -\text{int}(K).$$

It contradicts the weak duality ***Theorem* 3.3** between (PP) and (MWD).
Consequently, $(x', u', y', z', y^*, z^*, w^*)$ is a weak maximizer for (MWD).

***Theorem* 3.5** *(Converse duality) Let A be a nonempty convex subset of X, $p(x', u') \geq 0$, and $(x', u', y', z', y^*, z^*, w^*)$ be a feasible point of the problem (MWD). Let $e \in \text{int}(K)$, $e' \in \text{int}(L)$, and $e'' \in \text{int}(M)$. Suppose that $F(., u') : X \to 2^Y$ is ρ_1-K-convex with respect to e, $G(., u') : X \to 2^Z$ is ρ_2-L-convex with respect to e', and $p(., u') : X \to W$ is ρ_3-M-convex with respect to e'', on A. Assume that the contingent epiderivatives $D_\uparrow F(., u')(x', y')$ and $D_\uparrow G(., u')(x', z')$ exist and the Gâteaux derivative $p'(., u')(x')$ exists. Suppose that the conditions of **Lemma 3.1** hold at $(x', u', y', z', y^*, z^*, w^*)$ and* (3.14) *is satisfied. If $(x', u') \in S$, then (x', u', y') is a weak minimizer of (PP).*

Proof. Suppose that (x', u', y') is not a weak minimizer of the problem (PP). Then there exist $(x, u) \in S$ and $y \in F(x, u)$ such that

$$y < y'.$$

As $y^* \in K^+ \setminus \{\theta_{Y^*}\}$,

$$\langle y^*, y - y'\rangle < 0.$$

As $(x, u) \in S$, there exists

$$z \in G(x, u) \cap (-L).$$

So,

$$\langle z^*, z\rangle \leq 0, \text{ as } z^* \in L^+.$$

By the constraints of (MWD), we have

$$\langle z^*, z'\rangle \geq 0.$$

Therefore,

$$\langle z^*, z - z'\rangle = \langle z^*, z\rangle - \langle z^*, z'\rangle \leq 0.$$

Therefore,

$$\langle y^*, y - y'\rangle + \langle z^*, z - z'\rangle < 0. \quad (3.16)$$

As the conditions of ***Lemma* 3.1** hold at $(x', u', y', z', y^*, z^*, w^*)$, from Eqs. (3.6), (3.14), and the constraints of (MWD), we have

$$\langle y^*, F(x, u) - y'\rangle + \langle z^*, G(x, u) - z'\rangle \geq 0.$$

Hence,

$$\langle y^*, y - y'\rangle + \langle z^*, z - z'\rangle \geq 0,$$

which contradicts (3.16).

Consequently, (x', u', y') is a weak minimizer of the problem (PP).

3.3 Wolfe type dual

We consider a Wolfe type dual (WD), where $F(.,u')$ and $G(.,u')$ are contingent epiderivable set-valued maps and $p(.,u')$ is a Gâteaux derivable single-valued map, where $u' \in U$.

maximize $y' + \langle z^*, z' \rangle e$,

subject to,

$$
\begin{aligned}
(WD)\quad & \langle y^*, D_\uparrow F(.,u')(x',y')(x-x') + F(x',u) - y' \rangle \\
& + \langle z^*, D_\uparrow G(.,u')(x',z')(x-x') + G(x',u) - z' \rangle \\
& + \langle w^*, p'(.,u')(x')(x-x') + p(x',u) \rangle \geq 0, \forall (x,u) \in A \times U, \\
& x' \in A, u' \in U, y' \in F(x',u'), z' \in G(x',u'), p(x',u') \geq 0, \\
& (y^*, z^*, w^*) \in K^+ \times L^+ \times M^+, \text{ and } \langle y^*, e \rangle = 1.
\end{aligned}
$$

***Definition* 3.6** *A point $(x', u', y', z', y^*, z^*, w^*)$ satisfying all the constraints of (WD) is called a feasible point of the problem (WD).*

***Definition* 3.7** *A feasible point $(x', u', y', z', y^*, z^*, w^*)$ of the problem (WD) is called a weak maximizer of (WD) if there exists no feasible point $(x, u, y, z, y_1^*, z_1^*, w_1^*)$ of (WD) such that*

$$(y + \langle z_1^*, z \rangle e) - (y' + \langle z^*, z' \rangle e) \in \text{int}(K).$$

We prove the duality results of Wolfe type of the problem (PP). The proofs are very similar to ***Theorems* 3.3 - 3.5**, and hence omitted.

***Theorem* 3.6** *(Weak duality) Let A be a nonempty convex subset of X, $(x_0, u_0) \in S$, $(x', u', y', z', y^*, z^*, w^*)$ be a feasible point of the problem (WD), and $p(x', u') \geq 0$. Let $e \in \text{int}(K)$, $e' \in \text{int}(L)$, and $e'' \in \text{int}(M)$. Suppose that $F(.,u') : X \to 2^Y$ is ρ_1-K-convex with respect to e, $G(.,u') : X \to 2^Z$ is ρ_2-L-convex with respect to e', and $p(.,u') : X \to W$ is ρ_3-M-convex with respect to e'', on A. Assume that the contingent epiderivatives $D_\uparrow F(.,u')(x',y')$ and $D_\uparrow G(.,u')(x',z')$ exist and the Gâteaux derivative $p'(.,u')(x')$ exists. Suppose that the conditions of **Lemma 3.1** hold at $(x', u', y', z', y^*, z^*, w^*)$ and (3.14) is satisfied.*

Then,

$$F(x_0, u_0) - (y' + \langle z^*, z' \rangle e) \subseteq Y \setminus -\text{int}(K).$$

***Theorem* 3.7** *(Strong duality) Let (x', u', y') be a weak minimizer of the problem (PP) and $z' \in G(x', u') \cap (-L)$. Assume that for some $(y^*, z^*, w^*) \in K^+ \times L^+ \times M^+$, with $\langle y^*, e \rangle = 1$, Eqs. (3.11) and (3.12) are satisfied at $(x', u', y', z', y^*, z^*, w^*)$. Then $(x', u', y', z', y^*, z^*, w^*)$ is a feasible solution for (WD). If the weak duality **Theorem 3.6** between (PP) and (WD) holds, then $(x', u', y', z', y^*, z^*, w^*)$ is a weak maximizer of the problem (WD).*

***Theorem* 3.8** *(Converse duality) Let A be a nonempty convex subset of the space X and $(x', u', y', z', y^*, z^*, w^*)$ be a feasible point of the problem (WD) with $\langle z^*, z' \rangle \geq 0$ and $p(x', u') \geq 0$. Let $e \in \text{int}(K)$, $e' \in \text{int}(L)$, and $e'' \in \text{int}(M)$. Suppose that $F(.,u') : X \to 2^Y$ is ρ_1-K-convex with respect to e, $G(.,u') : X \to 2^Z$ is ρ_2-L-convex with respect to e', and $p(.,u') : X \to W$ is ρ_3-M-convex with respect to e'', on A. Assume that the contingent epiderivatives $D_\uparrow F(.,u')(x',y')$ and $D_\uparrow G(.,u')(x',z')$ exist and the Gâteaux derivative $p'(.,u')(x')$ exists. Suppose that the conditions of **Lemma 3.1** hold at $(x', u', y', z', y^*, z^*, w^*)$ and (3.14) is satisfied. If $(x', u') \in S$, then (x', u', y') is a weak minimizer of (PP).*

3.4 Mixed type dual

We consider a mixed type dual (MD), where $F(.,u')$ and $G(.,u')$ are contingent epiderivable set-valued maps and $p(.,u')$ is a Gâteaux derivable single-valued map, where $u' \in U$.

maximize $y' + \langle z^*, z' \rangle e$,

subject to,

$$
\begin{aligned}
(MD)\quad & \langle y^*, D_\uparrow F(.,u')(x',y')(x-x') + F(x',u) - y' \rangle \\
& + \langle z^*, D_\uparrow G(.,u')(x',z')(x-x') + G(x',u) - z' \rangle \\
& + \langle w^*, p'(.,u')(x')(x-x') + p(x',u) \rangle \geq 0, \forall (x,u) \in A \times U, \\
& \langle z^*, z' \rangle \geq 0, \\
& x' \in A, u' \in U, y' \in F(x',u'), z' \in G(x',u'), p(x',u') \geq 0, \\
& (y^*, z^*, w^*) \in K^+ \times L^+ \times M^+, \text{ and } \langle y^*, e \rangle = 1.
\end{aligned}
$$

***Definition* 3.8** *A point $(x', u', y', z', y^*, z^*, w^*)$ satisfying all the constraints of (MD) is called a feasible point of the problem (MD).*

***Definition* 3.9** *A feasible point $(x', u', y', z', y^*, z^*, w^*)$ of the problem (MD) is called a weak maximizer of (MD) if there exists no feasible point $(x, u, y, z, y_1^*, z_1^*, w_1^*)$ of (MD) such that*

$$(y + \langle z_1^*, z\rangle e) - (y' + \langle z^*, z'\rangle e) \in \text{int}(K).$$

We prove the duality results of mixed type of the problem (PP). The proofs are very similar to ***Theorems* 3.3 - 3.5**, and hence omitted.

***Theorem* 3.9** *(Weak duality) Let A be a nonempty convex subset of X with respect to $\eta : A \times A \to X$, $(x_0, u_0) \in S$, $(x', u', y', z', y^*, z^*, w^*)$ be a feasible point of the problem (MD), and $p(x', u') \geq 0$. Let $e \in \text{int}(K)$, $e' \in \text{int}(L)$, and $e'' \in \text{int}(M)$. Suppose that $F(., u') : X \to 2^Y$ is ρ_1-K-convex with respect to e, $G(., u') : X \to 2^Z$ is ρ_2-L-convex with respect to e', and $p(., u') : X \to W$ is ρ_3-M-convex with respect to e'', on A. Assume that the contingent epiderivatives $D_{\uparrow}F(., u')(x', y')$ and $D_{\uparrow}G(., u')(x', z')$ exist and the Gâteaux derivative $p'(., u')(x')$ exists. Suppose that the conditions of **Lemma 3.1** hold at $(x', u', y', z', y^*, z^*, w^*)$ and (3.14) is satisfied. Then,*

$$F(x_0, u_0) - (y' + \langle z^*, z'\rangle e) \subseteq Y \setminus -\text{int}(K).$$

***Theorem* 3.10** *(Strong duality) Let (x', u', y') be a weak minimizer of the problem (PP) and $z' \in G(x', u') \cap (-L)$. Assume that for some $(y^*, z^*, w^*) \in K^+ \times L^+ \times M^+$, with $\langle y^*, e\rangle = 1$, Eqs. (3.11) and (3.12) are satisfied at $(x', u', y', z', y^*, z^*, w^*)$. Then $(x', u', y', z', y^*, z^*, w^*)$ is a feasible solution for (MD). If the weak duality **Theorem 3.9** between (PP) and (MD) holds, then $(x', u', y', z', y^*, z^*, w^*)$ is a weak maximizer of (MD).*

***Theorem* 3.11** *(Converse duality) Let A be a nonempty convex subset of X, $p(x', u') \geq 0$, and $(x', u', y', z', y^*, z^*, w^*)$ be a feasible point of the problem (MD). Let $e \in \text{int}(K)$, $e' \in \text{int}(L)$, and $e'' \in \text{int}(M)$. Suppose that $F(., u') : X \to 2^Y$ is ρ_1-K-convex with respect to e, $G(., u') : X \to 2^Z$ is ρ_2-L-convex with respect to e', and $p(., u') : X \to W$ is ρ_3-M-convex with respect to e'', on A. Assume that the contingent epiderivatives $D_{\uparrow}F(., u')(x', y')$ and $D_{\uparrow}G(., u')(x', z')$ exist and the Gâteaux derivative $p'(., u')(x')$ exists. Suppose that the conditions of **Lemma 3.1** hold at $(x', u', y', z', y^*, z^*, w^*)$ and (3.14) is satisfied. If $(x', u') \in S$, then (x', u', y') is a weak minimizer of (PP).*

4 Conclusions

In this paper, we establish the sufficient Karush-Kuhn-Tucker (*KKT*) conditions for the parametric set-valued optimization problem (*PP*) under generalized cone convexity and contingent epiderivative assumptions. We also formulate the duals of Mond-Weir (*MWD*), Wolfe (*WD*), and mixed (*MD*) types and prove the duality results for weak minimizers between the primal problem (*PP*) and corresponding dual problems.

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