***Jñānābha*, Vol. 50(2) (2020), 63-74**
(Dedicated to Honor Dr. R. C. Singh Chandel on His 75^{th} Birth Anniversary Celebrations)

SOME APPLICATIONS OF MULTINOMINAL THEOREM IN SOLVING ALGEBRAIC EXPRESSIONS UNDER LINEAR MODELS

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(*Received : May* 07, 2020*; Revised: June* 03, 2020*; Final Form : December* 11, 2020)

Abstract

In this paper, an attempt has been made to provide the step-wise detailed algebraic expression for evaluation of expected values of different orders of error term (u) by employing multinominal theorem. The expectation of even order moments of error term (u) has been provided in case error term is not necessarily following normal distribution.

2010 Mathematics Subject Classifications: 15A30, 60E10, 62J05
Keywords and phrases: Characteristic function, Expectations, Linear Models, Matrices, Multinominal Theorem.

1 Introduction

The literature review of linear regression has witnessed the extensive utilization of algebraic expression for providing proof of various problems. The several research articles have employed algebraic methods to provide important results but avoided the step-wise derivation for the sake of continuity. However, it take sufficient time for the young researchers to break down these expression by applying knowledge of Algebra. Therefore, the detailed derivations of expressions pertaining to expected value of disturbance term u for different distributional conditions have been obtained by employing well-known results of Multinominal theorem. Also, the detailed derivations of results pertaining to inversion of characteristic function have been obtained by using special methods such as integration by parts.

2 Model and measures of goodness of fit

Let us postulate the linear regression model as under

(2.1) $y = \alpha e + X\beta + u,$

where, y is a vector of $n \times 1$ observations on the variable to be explained, α is a scalar representing the intercept term, e is a $n \times 1$ vector with all elements of unity, X is a $n \times p$ full column rank matrix of n-observations on p-explanatory variables, β is a $p \times 1$ vector of the coefficients associated with them and u is a $n \times 1$ vector of disturbances. Further, it is assumed that the elements of disturbance vector (u) are distributed independently and identically.

The multinomial theorem has been employed to solve the expressions in this paper and statement of multinomial theorem is stated as here-under

***Theorem* 2.1**: *It is stated that for real numbers a_1, a_2, a_3, ..., a_m and non-negative integers n, s_1, s_2, s_3,...,s_m, the following proposition holds*

(2.2) $(a_1 + a_2 + a_3 + ... + a_m)^n = \sum \frac{n!}{s_1!s_2!s_3!...s_m!} a_1^{s_1} a_2^{s_2} a_3^{s_3} ... a_m^{s_m},$

where, $\sum$ denotes the sum of all possible combinations of (s_1, s_2, s_3,...,s_m), such that

$$s_1 + s_2 + s_3 + ... + s_m = n.$$

Proof. The standard proof of aforesaid theorem can be seen in any related book of mathematics.

3 Derivation of results

Lets us assume that expressions pertaining to real numbers (a_i's) are given by

(3.1) $[a_1 + a_2 + ... + a_n]$

and expressions for different orders of (3.1) can be obtained by employing multinomial theorem, as under

(3.2) $(a_1 + a_2 + a_3 + ... + a_n)^1 = \frac{1!}{1!} \sum_i a_i = \binom{n}{1} a_i = n a_i$

(3.3) $(a_1 + a_2 + a_3 + ... + a_n)^2 = \frac{2!}{2!} \sum_i a_i^2 + \frac{2!}{1!1!} \sum_{i \neq j} a_i a_j = \sum_i a_i^2 + 2 \sum_{i \neq j} a_i a_j$

$$= \binom{n}{1} a_i^2 + \binom{n}{2}\binom{2}{1} a_i a_j = n a_i^2 + n(n-1) a_i a_j$$

$$= n a_i^2 + (n^2 - n) a_i a_j = n a_i^2 + (n^2 - n) a_i a_j,$$

(3.4) $(a_1 + a_2 + a_3 + ... + a_n)^3 = \frac{3!}{3!} \sum_i a_i^3 + \frac{3!}{2!1!} \sum_{i \neq j} a_i^2 a_j + \frac{3!}{1!1!1!} \sum_{i \neq j \neq k} a_i a_j a_k$

$$= \sum_i a_i^3 + 3 \sum_{i \neq j} a_i^2 a_j + 6 \sum_{i \neq j \neq k} a_i a_j a_k$$

$$= \binom{n}{1} a_i^3 + \binom{n}{2}\binom{2}{1} a_i^2 a_j + \binom{n}{3}\binom{3}{0} a_i a_j a_k$$

$$= n a_i^3 + 3n(n-1) a_i^2 a_j + n(n-1)(n-2) a_i a_j a_k$$

$$= n a_i^3 + (3n^2 - 3n) a_i^2 a_j + (n^3 - 3n^2 + 2n) a_i a_j a_k,$$

(3.5) $(a_1 + a_2 + a_3 + ... + a_n)^4 = \frac{4!}{4!} \sum_i a_i^4 + \frac{4!}{3!1!} \sum_{i \neq j} a_i^3 a_j^2 + \frac{4!}{2!2!} \sum_{i \neq j} a_i^2 a_j^2$

$$+ \frac{4!}{2!} \sum_{i \neq j \neq k} a_i^2 a_j a_k + \frac{4!}{1!1!1!1!} \sum_{i \neq j \neq k \neq l} a_i a_j a_k a_l$$

$$= \sum_i a_i^4 + 4 \sum_{i \neq j} a_i^3 a_j^2 + 6 \sum_{i \neq j} a_i^2 a_j^2 + 12 \sum_{i \neq j \neq k} a_i^2 a_j a_k$$

$$+ 24 \sum_{i \neq j \neq k \neq l} a_i a_j a_k a_l$$

$$= \binom{n}{1}\binom{1}{1} a_i^4 + 4\binom{n}{2}\binom{2}{1} a_i^3 a_j^2 + 6\binom{n}{2}\binom{2}{2} a_i^2 a_j^2$$

$$+ 12\binom{n}{3}\binom{3}{1} a_i^2 a_j a_k + 24\binom{n}{4}\binom{4}{0} a_i a_j a_k a_l$$

$$= n a_i^4 + \frac{8n(n-1)}{2} a_i^3 a_j^2 + \frac{6n(n-1)}{2} a_i^2 a_j^2$$

$$+ \frac{36n(n^2 - 3n + 2)}{6} a_i^2 a_j a_k + \frac{24(n^3 - 3n^2 + 2n)(n-3)}{24} a_i a_j a_k a_l$$

$$= n a_i^4 + 4n(n-1) a_i^3 a_j^2 + 3n(n-1) a_i^2 a_j^2 + 6n(n-1)(n-2)$$

$$a_i^2 a_j a_k + n(n-1)(n-2)(n-3) a_i a_j a_k a_l$$

$$= n a_i^4 + (4n^2 - 4n) a_i^3 a_j^2 + (3n^2 - 3n) a_i^2 a_j^2 + (6n^3 - 18n^2 + 12n)$$

$$a_i^2 a_j a_k + (n^4 - 6n^3 + 11n^2 - 6n) a_i a_j a_k a_l$$

$$= n a_i^4 + (4n^2 - 4n) a_i^3 a_j^2 + (3n^2 - 3n) a_i^2 a_j^2 + (6n^3 - 18n^2 + 12n)$$

$$a_i^2 a_j a_k + (n^4 - 6n^3 + 11n^2 - 6n) a_i a_j a_k a_l,$$

(3.6) $(a_1 + a_2 + a_3 + ... + a_n)^5 = \frac{5!}{5!} \sum_i a_i^5 + \frac{5!}{4!1!} \sum_{i \neq j} a_i^4 a_j + \frac{5!}{3!2!} \sum_{i \neq j} a_i^3 a_j^2$

$$+ \frac{5!}{3!1!1!} \sum_{i \neq j \neq k} a_i^3 a_j a_k + \frac{5!}{2!2!1!} \sum_{i \neq j \neq k} a_i^2 a_j^2 a_k$$

$$+ \frac{5!}{2!1!1!1!} \sum_{i \neq j \neq k \neq l} a_i^2 a_j a_k a_l + \frac{5!}{1!1!1!1!1!} \sum_{i \neq j \neq k \neq l \neq m} a_i a_j a_k a_l a_m$$

$$= n a_i^5 + 5 \frac{n(n-1)}{2} 2 a_i^4 a_j + 10 \frac{n(n-1)}{2} 2 a_i^3 a_j^2$$

$$+ 20 \frac{n(n-1)(n-2)}{6} 3 a_i^3 a_j a_k + 30 \frac{n(n-1)(n-2)}{6} 3 a_i^2 a_j^2 a_k$$

$$+ 60 \frac{n(n-1)(n-2)(n-3)}{24} 4 a_i^2 a_j a_k a_l$$

$$+ 120 \frac{n(n-1)(n-2)(n-3)(n-4)}{120} a_i a_j a_k a_l a_m$$

$$
\begin{aligned}
&= na_i^5 + 5n(n-1)a_i^4 a_j + 10n(n-1)a_i^3 a_j^2 \\
&\quad + 10n(n-1)(n-2)a_i^3 a_j a_k + 15n(n-1)(n-2)a_i^2 a_j^2 a_k \\
&\quad + 10n(n-1)(n-2)(n-3)a_i^2 a_j a_k a_l \\
&\quad + n(n-1)(n-2)(n-3)(n-4)a_i a_j a_k a_l a_m \\
&= na_i^5 + 5n(n-1)a_i^4 a_j + 10n(n-1)a_i^3 a_j^2 \\
&\quad + 10n(n-1)(n-2)a_i^3 a_j a_k + 15n(n-1)(n-2)a_i^2 a_j^2 a_k \\
&\quad + 10n(n-1)(n-2)(n-3)a_i^2 a_j a_k a_l \\
&\quad + n(n-1)(n-2)(n-3)(n-4)a_i a_j a_k a_l a_m \\
&= na_i^5 + (5n^2 - 5n)a_i^4 a_j + (10n^2 - 10n)a_i^3 a_j^2 \\
&\quad + (10n^3 - 30n^2 + 20n)a_i^3 a_j a_k \\
&\quad + (15n^3 - 45n^2 + 30n)a_i^2 a_j^2 a_k \\
&\quad + (10n^4 - 60n^3 + 110n^2 - 60n)a_i^2 a_j a_k a_l \\
&\quad + (n^5 - 10n^4 + 35n^3 - 50n^2 + 24n)a_i a_j a_k a_l a_m,
\end{aligned}
$$

$$
\begin{aligned}
(3.7) \qquad (a_1 + a_2 + ... + a_n)^6 &= \tfrac{6!}{6!} \textstyle\sum_i a_i^6 + \tfrac{6!}{5!1!} \sum_{i \neq j} a_i^5 a_j + \tfrac{6!}{4!2!} \sum_{i \neq j} a_i^4 a_j^2 + \tfrac{6!}{4!1!1!} \sum_{i \neq j \neq k} a_i^4 a_j a_k \\
&\quad + \tfrac{6!}{3!3!} \textstyle\sum_{i \neq j} a_i^3 a_j^3 + \tfrac{6!}{3!2!1!} \sum_{i \neq j \neq k} a_i^3 a_j^2 a_k + \tfrac{6!}{3!1!1!1!} \sum_{i \neq j \neq k \neq l} a_i^3 a_j a_k a_l \\
&\quad + \tfrac{6!}{2!2!2!} \textstyle\sum_{i \neq j \neq k} a_i^2 a_j^2 a_k^2 + \tfrac{6!}{2!2!1!1!} \sum_{i \neq j \neq k \neq l} a_i^2 a_j^2 a_k a_l \\
&\quad + \tfrac{6!}{2!1!1!1!1!} \textstyle\sum_{i \neq j \neq k \neq l \neq m} a_i^2 a_j a_k a_l a_m \\
&\quad + \tfrac{6!}{1!1!1!1!1!1!} \textstyle\sum_{i \neq j \neq k \neq l \neq m \neq n} a_i a_j a_k a_l a_m a_n \\
&= \textstyle\sum_i a_i^6 + 6 \sum_{i \neq j} a_i^5 a_j + 15 \sum_{i \neq j} a_i^4 a_j^2 + 30 \sum_{i \neq j \neq k} a_i^4 a_j a_k + 20 \sum_{i \neq j} a_i^3 a_j^3 \\
&\quad + 60 \textstyle\sum_{i \neq j \neq k} a_i^3 a_j^2 a_k + 120 \sum_{i \neq j \neq k \neq l} a_i^3 a_j a_k a_l + 90 \sum_{i \neq j \neq k} a_i^2 a_j^2 a_k^2 \\
&\quad + 180 \textstyle\sum_{i \neq j \neq k \neq l} a_i^2 a_j^2 a_k a_l + 360 \sum_{i \neq j \neq k \neq l \neq m} a_i^2 a_j a_k a_l a_m \\
&\quad + 720 \textstyle\sum_{i \neq j \neq k \neq l \neq m \neq n} a_i a_j a_k a_l a_m a_n \\
&= \tbinom{n}{1}\tbinom{1}{1} a_i^6 + 6\tbinom{n}{2}\tbinom{2}{1} a_i^5 a_j + 15\tbinom{n}{2}\tbinom{2}{2} a_i^4 a_j^2 \\
&\quad + 30\tbinom{n}{3}\tbinom{3}{1} a_i^4 a_j a_k + 20\tbinom{n}{2}\tbinom{2}{2} a_i^3 a_j^3 \\
&\quad + 60\tbinom{n}{3}\tbinom{3}{1}\tbinom{2}{1} a_i^3 a_j^2 a_k + 120\tbinom{n}{4}\tbinom{4}{1} a_i^3 a_j a_k a_l \\
&\quad + 90\tbinom{n}{3}\tbinom{3}{3} a_i^2 a_j^2 a_k^2 + 180\tbinom{n}{4}\tbinom{4}{2} a_i^2 a_j^2 a_k a_l \\
&\quad + 360\tbinom{n}{5}\tbinom{5}{1} a_i^2 a_j a_k a_l a_m + 720\tbinom{n}{6}\tbinom{6}{0} a_i a_j a_k a_l a_m a_n \\
&= na_i^6 + 6\tfrac{n(n-1)}{2} 2a_i^5 a_j + 15\tfrac{n(n-1)}{2} 2a_i^4 a_j^2 \\
&\quad + 30\tfrac{n(n-1)(n-2)}{6} 3a_i^4 a_j a_k + 20\tfrac{n(n-1)}{2} a_i^3 a_j^3 \\
&\quad + 60\tfrac{n(n-1)(n-2)}{6} 6a_i^3 a_j^2 a_k + 120\tfrac{n(n-1)(n-2)(n-3)}{24} 4a_i^3 a_j a_k a_l \\
&\quad + 90\tfrac{n(n-1)(n-2)}{6} a_i^2 a_j^2 a_k^2 + 180\tfrac{n(n-1)(n-2)(n-3)}{24} 6a_i^2 a_j^2 a_k a_l \\
&\quad + 360\tfrac{n(n-1)(n-2)(n-3)(n-4)}{120} 5a_i^2 a_j a_k a_l a_m \\
&\quad + 720\tfrac{n(n-1)(n-2)(n-3)(n-4)(n-5)}{720} a_i a_j a_k a_l a_m a_n \\
&= na_i^6 + 6n(n-1)a_i^5 a_j + 15n(n-1)a_i^4 a_j^2 + 15n(n-1)(n-2)a_i^4 a_j a_k \\
&\quad + 10n(n-1)a_i^3 a_j^3 + 60n(n-1)(n-2)a_i^3 a_j^2 a_k \\
&\quad + 20n(n-1)(n-2)(n-3)a_i^3 a_j a_k a_l \\
&\quad + 15n(n-1)(n-2)a_i^2 a_j^2 a_k^2 + 45n(n-1)(n-2)(n-3)a_i^2 a_j^2 a_k a_l \\
&\quad + 15n(n-1)(n-2)(n-3)(n-4)a_i^2 a_j a_k a_l a_m \\
&\quad + n(n-1)(n-2)(n-3)(n-4)(n-5)a_i a_j a_k a_l a_m a_n
\end{aligned}
$$

$$
\begin{aligned}
&= na_i^6 + 6n(n-1)a_i^5a_j + 15n(n-1)a_i^4a_j^2 \\
&\quad + 15n(n-1)(n-2)a_i^4a_ja_k + 10n(n-1)a_i^3a_j^3 \\
&\quad + 60n(n-1)(n-2)a_i^3a_j^2a_k + 20n(n-1)(n-2)(n-3)a_i^3a_ja_ka_l \\
&\quad + 15n(n-1)(n-2)a_i^2a_j^2a_k^2 + 45n(n-1)(n-2)(n-3)a_i^2a_j^2a_ka_l \\
&\quad + 15n(n-1)(n-2)(n-3)(n-4)a_i^2a_ja_ka_la_m \\
&\quad + n(n-1)(n-2)(n-3)(n-4)(n-5)a_ia_ja_ka_la_ma_n \\
&= na_i^6 + (6n^2-6n)a_i^5a_j + (15n^2-15n)a_i^4a_j^2 \\
&\quad + (15n^3-45n^2+30n)a_i^4a_ja_k + (10n^2-10n)a_i^3a_j^3 \\
&\quad + (60n^3-180n^2+120n)a_i^3a_j^2a_k \\
&\quad + (20n^4-120n^3+220n^2-120n)a_i^3a_ja_ka_l \\
&\quad + (15n^3-45n^2+30n)a_i^2a_j^2a_k^2 \\
&\quad + (45n^4-270n^3+495n^2-270n)a_i^2a_j^2a_ka_l \\
&\quad + (15n^5-150n^4+525n^3-750n^2+360n)a_i^2a_ja_ka_la_m \\
&\quad + (n^6-15n^5+85n^4-155n^3+274n^2-120)a_ia_ja_ka_la_ma_n.
\end{aligned}
$$

From equations (3.2) to (3.7) we get respectively

(3.8)
$$
\begin{aligned}
(a_1+a_2+a_3+\dots+a_n)^1 &= n[a_i], \\
(a_1+a_2+a_3+\dots+a_n)^2 &= n\left[a_i^2\right] + (n^2-n)\left[a_ia_j\right], \\
(a_1+a_2+a_3+\dots+a_n)^3 &= n\left[a_i^3\right] + (3n^2-3n)\left[a_i^2a_j\right] \\
&\quad + (n^3-3n^2+2n)\left[a_ia_ja_k\right], \\
(a_1+a_2+a_3+\dots+a_n)^4 &= n\left[a_i^4\right] + (4n^2-4n)\left[a_i^3a_j^2\right] \\
&\quad + (3n^2-3n)\left[a_i^2a_j^2\right] \\
&\quad + (6n^3-18n^2+12n)\left[a_i^2a_ja_k\right] \\
&\quad + (n^4-6n^3+11n^2-6n)\left[a_ia_ja_ka_l\right], \\
(a_1+a_2+a_3+\dots+a_n)^5 &= n\left[a_i^5\right] + (5n^2-5n)\left[a_i^4a_j\right] \\
&\quad + (10n^2-10n)\left[a_i^3a_j^2\right] \\
&\quad + (10n^3-30n^2+20n)\left[a_i^3a_ja_k\right] \\
&\quad + (15n^3-45n^2+30n)\left[a_i^2a_j^2a_k\right] \\
&\quad + (10n^4-60n^3+110n^2-60n)\left[a_i^2a_ja_ka_l\right] \\
&\quad + (n^5-10n^4+35n^3-50n^2+24n)\left[a_ia_ja_ka_la_m\right], \\
(a_1+a_2+a_3+\dots+a_n)^6 &= n\left[a_i^6\right] + (6n^2-6n)\left[a_i^5a_j\right] + (15n^2-15n)\left[a_i^4a_j^2\right] \\
&\quad + (15n^3-45n^2+30n)\left[a_i^4a_ja_k\right] \\
&\quad + (10n^2-10n)\left[a_i^3a_j^3\right] \\
&\quad + (60n^3-180n^2+120n)\left[a_i^3a_j^2a_k\right] \\
&\quad + (20n^4-120n^3+220n^2-120n)\left[a_i^3a_ja_ka_l\right] \\
&\quad + (15n^3-45n^2+30n)\left[a_i^2a_j^2a_k^2\right] \\
&\quad + (45n^4-270n^3+495n^2-270n)\left[a_i^2a_j^2a_ka_l\right] \\
&\quad + (15n^5-150n^4+525n^3-750n^2+360n)\left[a_i^2a_ja_ka_la_m\right]
\end{aligned}
$$

$$+ (n^6 - 15n^5 + 85n^4 - 155n^3 + 274n^2 - 120)\left[a_i a_j a_k a_l a_m a_n\right].$$

In case, we required the expectations of $(u'u)$, $(u'u)^2$, $(u'u)^3$, $(u'u)^4$, $(u'u)^5$ and $(u'u)^6$, the same can be obtained by using (3.8) as stated in ***Theorem* 3.1**.

***Theorem* 3.1** *The expected value of different orders of disturbance term (u), when u follows normal distribution with mean zero and variance σ^2 is given by*

$$(3.9)\quad E(u'u) = n\sigma^2,$$
$$E(u'uu'u) = \sigma^4(n^2 - 2n),$$
$$E(u'uu'uu'u) = \sigma^6(n^3 + 6n^2 + 8n),$$
$$E(u'uu'uu'uu'u) = \sigma^8(n^4 + 12n^3 + 44n^2 + 48),$$
$$E(u'uu'uu'uu'uu'u) = \sigma^{10}(n^5 + 20n^4 + 140n^3 + 400n^2 + 384n),$$
$$E(u'uu'uu'uu'uu'uu'u) = \sigma^{12}(n^6 + 30n^5 + 340n^4 + 1870n^3 + 4384n^2 + 3840n).$$

Proof. As disturbances are following normal distribution, the generalized form of central moments (with mean zero and variance σ^2) is given by

$$(3.10)\quad m_{2n} = \sigma^{2n}(2n-1)!! \quad \text{for even moments,}$$
$$m_{2n+1} = 0 \quad \text{for odd moments.}$$

Using (3.9) one can see that all odd order moments of normal distribution are zero and even order central moments for normal distribution with mean zero and variance σ^2 as

$$(3.11)\quad m_2 = \sigma^2(2-1)!! = \sigma^2,$$
$$m_4 = \sigma^4(4-1)!! = \sigma^4(4-1)(4-3) = 3\sigma^4,$$
$$m_6 = \sigma^6(6-1)!! = \sigma^6(6-1)(6-3)(6-1) = 15\sigma^6,$$
$$m_8 = \sigma^8(8-1)!! = \sigma^8(8-1)(8-3)(8-5)(8-7) = 105\sigma^8,$$
$$m_{10} = \sigma^{10}(10-1)!! = \sigma^1 0(10-1)(10-3)(10-5)(10-7)(10-9) = 945\sigma^8,$$
$$m_{12} = \sigma^{12}(12-1)!! = \sigma^{12}(12-1)(12-3)(12-5)(12-7)(12-9)(12-11) = 10395\sigma^{12}.$$

By utilizing equations (3.1), (3.8) and (3.11) the expected value of disturbance term u following normal distribution with mean zero and variance σ^2 can be derived as

$$(3.12)\quad E(u_1^2 + u_2^2 + ... + u_n^2)^1 = E[a_1 + a_2 + a_3 + ... + a_n]^1 = nE[a_i] = nE[u_i^2] = n\sigma^2,$$

$$(3.13)\quad E(u_1^2 + u_2^2 + ... + u_n^2)^2 = E[a_1 + a_2 + a_3 + ... + a_n]^2 = nE\left[a_i^2\right] + (n^2 - n)E\left[a_i a_j\right]$$
$$= nE\left[u_i^4\right] + (n^2 - n)E\left[u_i^2 u_j^2\right] = nE\left[u_i^4\right] + (n^2 - n)E\left[u_i^2\right]\left[u_j^2\right]$$
$$= n[3\sigma^4] + (n^2 - n)[\sigma^2\sigma^2] = \sigma^4[n^2 - 2n],$$

$$(3.14)\quad E(u_1^2 + u_2^2 + ... + u_n^2)^3 = E[a_1 + a_2 + a_3 + ... + a_n]^3$$
$$= nE\left[a_i^3\right] + (3n^2 - 3n)E\left[a_i^2 a_j\right] + (n^3 - 3n^2 + 2n)E\left[a_i a_j a_k\right]$$
$$= nE\left[u_i^6\right] + (3n^2 - 3n)E\left[u_i^4 u_j^2\right] + (n^3 - 3n^2 + 2n)E\left[u_i^2 u_j^2 u_k^2\right]$$
$$= nE\left[u_i^6\right] + (3n^2 - 3n)E\left[u_i^4\right]E\left[u_j^2\right] + (n^3 - 3n^2 + 2n)E\left[u_i^2\right]E\left[u_j^2\right]E\left[u_k^2\right]$$
$$= n[15\sigma^6] + (3n^2 - 3n)[3\sigma^4\sigma^2] + (n^3 - 3n^2 + 2n)[\sigma^6]$$
$$= \sigma^6[15n + 9n^2 - 9n + n^3 - 3n^2 + 2n]$$
$$= \sigma^6[n^3 + 6n^2 + 8n],$$

$$(3.15)\quad E(u_1^2 + u_2^2 + ... + u_n^2)^4 = E[a_1 + a_2 + a_3 + ... + a_n]^4$$
$$= nE\left[a_i^4\right] + (4n^2 - 4n)E\left[a_i^3 a_j^2\right] + (3n^2 - 3n)E\left[a_i^2 a_j^2\right]$$

$$
\begin{aligned}
&+ (6n^3 - 18n^2 + 12n)E\left[a_i^2 a_j a_k\right] \\
&+ (n^4 - 6n^3 + 11n^2 - 6n)E\left[a_i a_j a_k a_l\right] \\
&= nE\left[u_i^8\right] + (4n^2 - 4n)E\left[u_i^6 u_j^4\right] + (3n^2 - 3n)E\left[u_i^4 u_j^4\right] \\
&+ (6n^3 - 18n^2 + 12n)E\left[u_i^4 u_j^2 u_k^2\right] \\
&+ (n^4 - 6n^3 + 11n^2 - 6n)E\left[u_i^2 u_j^2 u_k^2 u_l^2\right] \\
&= nE\left[u_i^8\right] + (4n^2 - 4n)E\left[u_i^6\right]E\left[u_j^4\right] \\
&+ (3n^2 - 3n)E\left[u_i^4\right]E\left[u_j^4\right] \\
&+ (6n^3 - 18n^2 + 12n)E\left[u_i^4\right]E\left[u_j^2\right]E\left[u_k^2\right] \\
&+ (n^4 - 6n^3 + 11n^2 - 6n)E\left[u_i^2\right]E\left[u_j^2\right]E\left[u_k^2\right]E\left[u_l^2\right] \\
&= n[105\sigma^8] + (4n^2 - 4n)[15\sigma^6 3\sigma^4] + (3n^2 - 3n)[3\sigma^4 3\sigma^4] \\
&+ (6n^3 - 18n^2 + 12n)[3\sigma^4 \sigma^2 \sigma^2] \\
&+ (n^4 - 6n^3 + 11n^2 - 6n)[\sigma^2 \sigma^2 \sigma^2 \sigma^2] \\
&= \sigma^8[105n + 45(4n^2 - 4n) + 9(3n^2 - 3n) \\
&+ 3(6n^3 - 18n^2 + 12n) \\
&+ (n^4 - 6n^3 + 11n^2 - 6n)] \\
&= \sigma^8[n^4 + 12n^3 + 44n^2 + 48],
\end{aligned}
$$

(3.16) $E(u_1^2 + u_2^2 + ... + u_n^2)^5 = E[a_1 + a_2 + a_3 + ... + a_n]^5$

$$
\begin{aligned}
&= nE\left[a_i^5\right] + (5n^2 - 5n)E\left[a_i^4 a_j\right] + (10n^2 - 10n)E\left[a_i^3 a_j^2\right] \\
&+ (10n^3 - 30n^2 + 20n)E\left[a_i^3 a_j a_k\right] \\
&+ (15n^3 - 45n^2 + 30n)E\left[a_i^2 a_j^2 a_k\right] \\
&+ (10n^4 - 60n^3 + 110n^2 - 60n)E\left[a_i^2 a_j a_k a_l\right] \\
&+ (n^5 - 10n^4 + 35n^3 - 50n^2 + 24n)E\left[a_i a_j a_k a_l a_m\right] \\
&= nE\left[u_i^{10}\right] + (5n^2 - 5n)E\left[u_i^8 u_j^2\right] + (10n^2 - 10n)E\left[u_i^6 u_j^4\right] \\
&+ (10n^3 - 30n^2 + 20n)E\left[u_i^6 u_j^2 u_k^2\right] \\
&+ (15n^3 - 45n^2 + 30n)E\left[u_i^4 u_j^4 u_k^2\right] \\
&+ (10n^4 - 60n^3 + 110n^2 - 60n)E\left[u_i^4 u_j^2 u_k^2 u_l^2\right] \\
&+ (n^5 - 10n^4 + 35n^3 - 50n^2 + 24n)E\left[u_i^2 u_j^2 u_k^2 u_l^2 u_m^2\right] \\
&= nE\left[u_i^{10}\right] + (5n^2 - 5n)E\left[u_i^8\right]E\left[u_j^2\right] + (10n^2 - 10n)E\left[u_i^6\right]E\left[u_j^4\right] \\
&+ (10n^3 - 30n^2 + 20n)E\left[u_i^6\right]E\left[u_j^2\right]E\left[u_k^2\right] \\
&+ (15n^3 - 45n^2 + 30n)E\left[u_i^4\right]E\left[u_j^4\right]E\left[u_k^2\right] \\
&+ (10n^4 - 60n^3 + 110n^2 - 60n)E\left[u_i^4\right]E\left[u_j^2\right]E\left[u_k^2\right]E\left[u_l^2\right] \\
&+ (n^5 - 10n^4 + 35n^3 - 50n^2 + 24n) \\
&\quad E\left[u_i^2\right]E\left[u_j^2\right]E\left[u_k^2\right]E\left[u_l^2\right]E\left[u_m^2\right] \\
&= n\left[945\sigma^{10}\right] + (5n^2 - 5n)E\left[105\sigma^8\sigma^2\right] \\
&+ (10n^2 - 10n)E\left[15\sigma^6 3\sigma^4\right] + (10n^3 - 30n^2 + 20n)
\end{aligned}
$$

$$E\left[15\sigma^6\sigma^2\sigma^2\right] + (15n^3 - 45n^2 + 30n)E\left[3\sigma^4 3\sigma^4\sigma^2\right]$$
$$+ (10n^4 - 60n^3 + 110n^2 - 60n)E\left[3\sigma^4\sigma^2\sigma^2\sigma^2\right]$$
$$+ (n^5 - 10n^4 + 35n^3 - 50n^2 + 24n)E\left[\sigma^2\sigma^2\sigma^2\sigma^2\sigma^2\right]$$
$$=\sigma^{10}\left[n^5 + 20n^4 + 140n^3 + 400n^2 + 384n\right],$$

(3.17) $$E(u_1^2 + u_2^2 + ... + u_n^2)^6 = E\left[a_1 + a_2 + a_3 + ... + a_n\right]^6$$
$$=nE\left[u_i^{12}\right] + (6n^2 - 6n)E\left[u_i^{10}u_j^2\right]$$
$$+ (15n^2 - 15n)E\left[u_i^8u_j^4\right] + (10n^2 - 10n)E\left[u_i^6u_j^6\right]$$
$$+ (15n^3 - 45n^2 + 30n)E\left[u_i^8u_j^2u_k^2\right]$$
$$+ (60n^3 - 180n^2 + 120n)E\left[u_i^6u_j^4u_k^2\right]$$
$$+ (20n^4 - 120n^3 + 220n^2 - 120n)E\left[u_i^6u_j^2u_k^2u_l^2\right]$$
$$+ (15n^3 - 45n^2 + 30n)E\left[u_i^4u_j^4u_k^4\right]$$
$$+ (45n^4 - 270n^3 + 495n^2 - 270n)E\left[u_i^4u_j^4u_k^2u_l^2\right]$$
$$+ (15n^5 - 150n^4 + 525n^3 - 750n^2 + 360n)E\left[u_i^4u_j^2u_k^2u_l^2u_m^2\right]$$
$$+ (n^6 - 15n^5 + 85n^4 - 155n^3 + 274n^2 - 120)$$
$$E\left[u_i^2u_j^2u_k^2u_l^2u_m^2u_n^2\right]$$
$$=nE\left[u_i^{12}\right] + (6n^2 - 6n)E\left[u_i^{10}\right]E\left[u_j^2\right]$$
$$+ (15n^2 - 15n)E\left[u_i^8\right]E\left[u_j^4\right] + (10n^2 - 10n)E\left[u_i^6\right]E\left[u_j^6\right]$$
$$+ (15n^3 - 45n^2 + 30n)E\left[u_i^8\right]E\left[u_j^2\right]\left[u_k^2\right]$$
$$+ (60n^3 - 180n^2 + 120n)E\left[u_i^6\right]E\left[u_j^4\right]E\left[u_k^2\right]$$
$$+ (20n^4 - 120n^3 + 220n^2 - 120n)$$
$$E\left[u_i^6\right]E\left[u_j^2\right]E\left[u_k^2\right]E\left[u_l^2\right]$$
$$+ (15n^3 - 45n^2 + 30n)E\left[u_i^4\right]E\left[u_j^4\right]E\left[u_k^4\right]$$
$$+ (45n^4 - 270n^3 + 495n^2 - 270n)$$
$$E\left[u_i^4\right]E\left[u_j^4\right]E\left[u_k^2\right]E\left[u_l^2\right]$$
$$+ (15n^5 - 150n^4 + 525n^3 - 750n^2 + 360n)$$
$$E\left[u_i^4\right]E\left[u_j^2\right]E\left[u_k^2\right]E\left[u_l^2\right]E\left[u_m^2\right]$$
$$+ (n^6 - 15n^5 + 85n^4 - 155n^3 + 274n^2 - 120)$$
$$E\left[u_i^2\right]E\left[u_j^2\right]E\left[u_k^2\right]E\left[u_l^2\right]E\left[u_m^2\right]E\left[u_n^2\right]$$
$$=nE\left[10395\sigma^{12}\right] + (6n^2 - 6n)E\left[945\sigma^{10}\right]E\left[\sigma^2\right]$$
$$+ (15n^2 - 15n)E\left[105\sigma^8\right]E\left[3\sigma^4\right]$$
$$+ (10n^2 - 10n)E\left[15\sigma^6\right]E\left[15\sigma^6\right]$$
$$+ (15n^3 - 45n^2 + 30n)E\left[105\sigma^8\right]E\left[\sigma^2\right]E\left[\sigma^2\right]$$
$$+ (60n^3 - 180n^2 + 120n)E\left[15\sigma^6\right]E\left[3\sigma^4\right]E\left[\sigma^2\right]$$
$$+ (15n^3 - 45n^2 + 30n)E\left[3\sigma^4\right]E\left[3\sigma^4\right]E\left[3\sigma^4\right]$$
$$+ (20n^4 - 120n^3 + 220n^2 - 120n)E\left[15\sigma^6\right]E\left[\sigma^2\right]$$

$$E\left[\sigma^2\right]E\left[\sigma^2\right] + (45n^4 - 270n^3 + 495n^2 - 270n)$$
$$E\left[3\sigma^4\right]E\left[3\sigma^4\right]E\left[\sigma^2\right]E\left[\sigma^2\right]$$
$$+ (15n^5 - 150n^4 + 525n^3 - 750n^2 + 360n)$$
$$E\left[3\sigma^4\right]\left[\sigma^2\right]\left[\sigma^2\right]\left[\sigma^2\right]\left[\sigma^2\right]$$
$$+ (n^6 - 15n^5 + 85n^4 - 155n^3 + 274n^2 - 120)$$
$$E\left[\sigma^2\right]\left[\sigma^2\right]\left[\sigma^2\right]\left[\sigma^2\right]\left[\sigma^2\right]\left[\sigma^2\right]$$
$$=\sigma^{12}\left[n^6 + 30n^5 + 340n^4 + 1870n^3 + 4384n^2 + 3840n\right].$$

Thus, using (3.12) to (3.17) we obtain the results (3.9) of the ***Theorem* 3.1**.

***Theorem* 3.2** *The even order moments of disturbance term (u), when u is distributed with non-normal distribution with finite moments as*

(3.18) $m_2 = \sigma^2,$

$$m_4 = \sigma^4(\gamma_2 + 3),$$
$$m_6 = \sigma^6(\gamma_4 + 15\gamma_2 + 10\gamma_1^2 + 15),$$
$$m_8 = \sigma^8(\gamma_6 + 28\gamma_4 + 56\gamma_3\gamma_1 + 35\gamma_2^2 + 210\gamma_2 + 280\gamma_1^2 + 105).$$

Further, the expected value of different orders of $u'u$ are given by

(3.19) $E(u'u) = \sigma^2\left[n\right],$

$$E(u'u)^2 = \sigma^4\left[\gamma_2 n + n^2 + 2n\right],$$
$$E(u'u)^3 = \sigma^6\left[\gamma_4 n + \gamma_2(3n^2 + 12n) + 10\gamma_1^2 n + 6n^2 + 8n\right],$$
$$E(u'u)^4 = \sigma^8\Big[n^4 + 12n^3 + 44n^2 + 48n + \gamma_2(6n^3 + 60n^2 + 144n)$$
$$+\gamma_4(4n^2 + 24n) + \gamma_6 n + \gamma_1^2(40n^2 + 240n) + \gamma_2^2(3n^2 + 32n) + 56\gamma_1\gamma_2 n\Big],$$

where, Pearson's measure of skewness and kurtosis are termed as γ_1 & γ_2 and γ_4, γ_6 may be treated as measures of deviation from the normality. Also, disturbance term is distributed with mean zero and variance σ^2 and elements of error term are i.i.d.

Proof. Using equations (3.1), (3.8) and (3.18) the expression for expectation of u under given conditions of non-normality as derived here-under

(3.20) $E(u'u) = nE(a_i) = nE(u_i^2) = \sigma^2[n],$

(3.21) $E(u'u)^2 = nE(a_i^2) + (n^2 - n)E(a_i a_j) = nE(a_i^2) + (n^2 - n)E(a_i)E(a_j)$

$$= n\sigma^4(\gamma_2 + 3) + (n^2 - n)\sigma^2\sigma^2 = \sigma^4\left[n(\gamma_2 + 3) + n^2 - n\right]$$
$$= \sigma^4\left[n\gamma_2 + n^2 + 2n\right],$$

(3.22) $E(u'u)^3 = nE(a_i^3) + (3n^2 - 3n)E(a_i^2 a_j) + (n^3 - 3n^2 + 2n)E(a_i a_j a_k)$

$$= nE(a_i^3) + (3n^2 - 3n)E(a_i^2)E(a_j^2) + (n^3 - 3n^2 + 2n)E(a_i)E(a_j)E(a_k)$$
$$= n\sigma^6(\gamma_4 + 15\gamma_2 + 10\gamma_1^2 + 15) + \sigma^6(3n^2 - 3n)(\gamma_2 + 3)$$
$$+\sigma^6(n^3 - 3n^2 + 2n)$$
$$= \sigma^6\left[\gamma_4 n + \gamma_2(3n^2 + 12n) + 10\gamma_1^2 n + 6n^2 + 8n\right],$$

(3.23) $E(u_1^2 + u_2^2 + ... + u_n^2)^4 = E[a_1 + a_2 + a_3 + ... + a_n]^4$

$$= nE\left[a_i^4\right] + (4n^2 - 4n)E\left[a_i^3 a_j^2\right] + (3n^2 - 3n)E\left[a_i^2 a_j^2\right]$$

$$+(6n^3-18n^2+12n)E\Big[a_i^2a_ja_k\Big]$$
$$+(n^4-6n^3+11n^2-6n)E\Big[a_ia_ja_ka_l\Big]$$
$$= nE\Big[u_i^8\Big]+(4n^2-4n)E\Big[u_i^6u_j^4\Big]+(3n^2-3n)E\Big[u_i^4u_j^4\Big]$$
$$+(6n^3-18n^2+12n)E\Big[u_i^4u_j^2u_k^2\Big]$$
$$+(n^4-6n^3+11n^2-6n)E\Big[u_i^2u_j^2u_k^2u_l^2\Big]$$
$$= nE\Big[u_i^8\Big]+(4n^2-4n)E\Big[u_i^6\Big]E\Big[u_j^4\Big]+(3n^2-3n)E\Big[u_i^4\Big]E\Big[u_j^4\Big]$$
$$+(6n^3-18n^2+12n)E\Big[u_i^4\Big]E\Big[u_j^2\Big]E\Big[u_k^2\Big]$$
$$+(n^4-6n^3+11n^2-6n)E\Big[u_i^2\Big]E\Big[u_j^2\Big]E\Big[u_k^2\Big]E\Big[u_l^2\Big]$$
$$= n\sigma^8\Big[n^4+12n^3+44n^2+48n+\gamma_2(6n^3+60n^2+144n)$$
$$+\gamma_4(4n^2+24n)+\gamma_6 n+\gamma_1^2(40n^2+240n)+\gamma_2^2(3n^2+32n)$$
$$+56\gamma_1\gamma_2 n\Big]+(4n^2-4n)[\sigma^6(\gamma_4+15\gamma_2+10\gamma_1^2+15)\sigma^2]$$
$$+(3n^2-3n)[\sigma^4(\gamma_2+3)]^2+(6n^3-18n^2+12n)[\sigma^8(\gamma_2+3)]$$
$$+(n^4-6n^3+11n^2-6n)[\sigma^2\sigma^2\sigma^2\sigma^2]$$
$$= \sigma^8[n^4-6n^3+11n^2-6n+105n+\gamma_2(6n^3-18n^2+12n)$$
$$+18n^3-54n^2+36n+\gamma_2^2(3n^2-3n)+27n^2-27n$$
$$+\gamma_2(18n^2-18n)+\gamma_4(4n^2-4n)+\gamma_2(60n^2-60n)$$
$$+\gamma_1^2(40n^2-40n)+60n^2-60n+\gamma_6 n+28\gamma_4 n+56\gamma_2\gamma_1 n$$
$$+35\gamma_2^2 n+210\gamma_2 n+280\gamma_1^2 n+105n$$
$$= \sigma^8\Big[n^4+12n^3+44n^2+48n+\gamma_2(6n^3+60n^2+144n)$$
$$+\gamma_4(4n^2+24n)+\gamma_6 n+\gamma_1^2(40n^2+240n)+\gamma_2^2(3n^2+32n)$$
$$+56\gamma_1\gamma_2 n\Big].$$

Thus, we obtain the results of the ***Theorem* 3.2**.

4 Derivation of expression pertaining to integrals

The derivation of integral utilized in solving expressions pertaining to inversion of characteristic function are provided in this section. Further, to carry out the detailed derivations of integrals expression by utilizing integration by parts method, let us state following theorem:

***Theorem* 4.1** *The integration of I_1, I_2 and I_3 as follows*

$$(4.1)\quad I_1=\frac{1}{2\Pi}\int_{-\infty}^{\infty} te^{-at^2/2}e^{-\iota tW_1}dt=-\Big[\tfrac{\iota W_1}{a}\Big]f(W_1),$$
$$I_2=\frac{1}{2\Pi}\int_{-\infty}^{\infty} t^2e^{-at^2/2}e^{-\iota tW_1}dt=\Big[\tfrac{\iota^2 W_1^2}{a^2}\Big]f(W_1),$$
$$I_3=\frac{1}{2\Pi}\int_{-\infty}^{\infty} t^3e^{-at^2/2}e^{-\iota tW_1}dt==-\iota\Big[\tfrac{3W_1}{a^2}-\tfrac{W_1^3}{a^3}\Big]f(W_1),$$

where,

$$(4.2)\quad \frac{1}{2\Pi}\int_{-\infty}^{\infty} e^{-at^2/2}e^{-\iota tW_1}dt=f(W_1),$$

given that W_1 follows normal distribution with mean zero and variance (σ^2) equals to a, such that $a>0$.

Proof. In order to proof results at (4.1), let us write $f(x_1)$, $f(x_2)$ and $f(x_3)$ as

$$(4.3)\quad f(x_1) = \int te^{-at^2/2},$$
$$f(x_2) = \int t^2 e^{-at^2/2},$$
$$f(x_3) = \int t^3 e^{-at^2/2},$$

where, integration of $f(x_1)$, $f(x_2)$ and $f(x_3)$ can be derived as given by

$$(4.4)\quad \int f(x_1) = \int te^{-at^2/2}$$
$$= \int e^{-au/2} \tfrac{du}{2} \qquad (Let \quad t^2 = u \implies 2tdt = du)$$
$$=\frac{1}{2}\Big[\frac{e^{-au/2}}{-a/2}\Big]$$
$$= -\frac{e^{-au/2}}{a}$$
$$= -\frac{1}{a}e^{-at^2/2} + C,$$

$$(4.5)\quad 3cmf(x_2) = \int t^2 e^{-at^2/2} dt$$
$$=t^2\Big[\frac{e^{-at^2/2}}{-at}\Big] - 2\int \tfrac{e^{-at^2/2}}{-a} dt$$
$$= -\frac{t}{a}e^{-at^2/2} - \frac{2}{a^2}\frac{e^{-at^2/2}}{t} + C,$$

$$(4.6)\qquad f(x_3) = \int t^3 e^{-at^2/2} dt$$
$$=t^3\Big[\frac{e^{-at^2/2}}{-at}\Big] - \int 3t^2 \tfrac{e^{-at^2/2}}{-at} dt$$
$$= -\frac{t^2}{a}e^{-at^2/2} - \frac{3}{a^2}e^{-at^2/2} + C.$$

Further, let $g(x) = \cos(tW_1) - \iota\sin(tW_1)$ and derivative of $g(x)$ is given by

$$(4.7)\quad g'(x) =\frac{d}{dx}[(\cos(tW_1) - \iota\sin(tW_1)]dt$$
$$= -W_1\sin(tW_1) - \iota W_1\cos(tW_1)$$
$$= -\iota W_1 g(x).$$

We may re-write I_1 as

$$(4.8)\quad I_1 = \frac{1}{2\Pi}\int_{-\infty}^{\infty} te^{-at^2/2}e^{-\iota tW_1}dt$$
$$= \frac{1}{2\Pi}\int_{-\infty}^{\infty} te^{-at^2/2}[\cos(tW_1) - \iota\sin(tW_1)]dt$$
$$= \frac{1}{2\Pi}\int_{-\infty}^{\infty} f(x_1)g(x),$$

where, $f(x_1) = \int te^{-at^2/2}$ and $g(x) = \cos(tW_1) - \iota\sin(tW_1)$ as defined in equations (4.3)) and (4.7). Further, employing integration by parts and utilizing results (4.4) and (4.7), we derive

$$(4.9)\quad I_1 = \frac{1}{2\Pi}\int_{-\infty}^{\infty} g(x)f(x_1)dt$$
$$= \frac{1}{2\Pi}\Big[g(x)\int_{-\infty}^{\infty} f(x_1)dt - \int_{-\infty}^{\infty} g'(x)\int_{-\infty}^{\infty} f(x_1)dtdt\Big]$$
$$= \frac{1}{2\Pi}\Big[-g(x)\frac{1}{a}e^{-at^2/2}\Big|_{-\infty}^{+\infty} - \frac{\iota W_1}{2\Pi}\frac{1}{a}\int_{-\infty}^{\infty} g(x)e^{-at^2/2}dt\Big]$$
$$= \frac{1}{2\Pi}\Big[-\frac{1}{a}g(x)e^{-at^2/2}\Big|_{-\infty}^{+\infty} - \frac{\iota W_1}{2\Pi}\frac{1}{a}\int_{-\infty}^{\infty} g(x)e^{-at^2/2}dt\Big]$$
$$= -\Big[\frac{\iota W_1}{a}\Big]f(W_1).$$

Similarly

(4.10)

$$
\begin{aligned}
I_2 =& \frac{1}{2\Pi}\int_{-\infty}^{\infty} t^2 e^{-at^2/2} e^{-\iota t W_1} dt \\
=& \frac{1}{2\Pi}\int_{-\infty}^{\infty} g(x) f(x_2) dt \\
=& \frac{1}{2\Pi}\Big[g(x)\int_{-\infty}^{\infty} f(x_2)dt + \iota W_1 \int_{-\infty}^{\infty} g(x)\Big(-\tfrac{t}{a}e^{-at^2/2} - \tfrac{2}{a^2}\tfrac{e^{-at^2/2}}{t}\Big)\Big]dt \\
=& \frac{1}{2\Pi}\Big[-\frac{\iota W_1}{a}\int_{-\infty}^{\infty} g(x) t e^{-at^2/2} dt - \tfrac{2\iota W_1}{a^2}\int_{-\infty}^{\infty} g(x)\tfrac{e^{-at^2/2}}{t}\Big] \\
=& \frac{1}{2\Pi}\Big[-\frac{-\iota W_1}{a}\Big(2\Pi\frac{-\iota W_1}{a} f(W_1)\Big)\Big] \\
=& \Big[\frac{\iota^2 W_1^2}{a^2}\Big] f(W_1).
\end{aligned}
$$

Using (4.7) and (4.8), we obtain

(4.11)

$$
\begin{aligned}
I_3 =& \frac{1}{2\Pi}\int_{-\infty}^{\infty} t^3 e^{-at^2/2} e^{-\iota t W_1} dt \\
=& \frac{1}{2\Pi}\int_{-\infty}^{\infty} g(x) f(x_3) dt \\
=& \frac{1}{2\Pi}\Big[g(x)\int_{-\infty}^{\infty} f(x_3)dt - \int_{-\infty}^{\infty} g'(x)\int_{-\infty}^{\infty} f(x_3)dtdt\Big] \\
=& +\iota W_1 \int_{-\infty}^{\infty} g(x)\Big[\tfrac{-t^2}{a}e^{-at^2/2} - \tfrac{3}{a^2}e^{-at^2/2}\Big]dt \\
=& -\Big[\frac{\iota W_1}{2\Pi a}\Big]\int_{-\infty}^{\infty} t^2 e^{-\iota t W_1} e^{-at^2/2} dt - \Big[\tfrac{3\iota W_1}{2\Pi a^2}\Big]\int_{-\infty}^{\infty} e^{-\iota t W_1} e^{-at^2/2} dt \\
=& -\frac{i W_1}{a}\Big[\frac{\iota^2 W_1^2}{a^2}\Big] f(W_1) - \Big[\frac{3\iota W_1}{a^2}\Big] f(W_1) \\
=& -\iota\Big[\frac{3W_1}{a^2} - \frac{W_1^3}{a^3}\Big] f(W_1).
\end{aligned}
$$

Thus, we obtain the results of ***Theorem* 4.1**.

5 Conclusion

The present work provide insights to the algebraic expression, utilized for the deducing the several important results available in literature, by providing step-wise derivation.

Acknowledgement. The authors are grateful to the Editor and Reviewer for their comments to bring the paper in its present form.

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