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(Dedicated to Honor Dr. R. C. Singh Chandel on His 75^{th} Birth Anniversary Celebrations)

A COMPUTATIONAL METHOD FOR SOLVING STOCHASTIC INTEGRAL EQUATIONS USING HAAR WAVELETS

By

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Abstract

In this article, we have developed a new technique for solving stochastic integral equations. A new Haar wavelets stochastic operational matrix of integration (*HWSOMI*) is developed in order to obtain efficient and accurate solution for stochastic integral equations. In the beginning we study the properties of stochastic integrals and Haar wavelets. Convergence and error analysis of Haar wavelet method is presented. Accuracy of the method investigated is justified through some examples.

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1 Introduction

Wavelet is a newly emerging area of mathematics. Wavelets have a number of applications in signal processing [11]. Integral equations are the most important tools describing knowledge models. Since many a times, the exact solution of integral equations does not exist, the numerical approximation of these equations become necessary. Different methods are used for approximating these equations and different basis functions are used.

Modeling various phenomena in science, engineering and physics requires stochastic integrals [4]. Numerical computations of stochastic integral equations have been studied by various authors. Some of which are Claeden and Platen [6], Oksendal [8], Maleknejad et al. [7], Cortes et al. [3], Douglas et al. [4], and Zhang [12].

Due to the large number of applications of Haar wavelets in solving differential, integral and integro differential equations, many authors have studied the computational methods for the solution of these equations using Haar wavelets. Some of which are found in [9], [10], [1] and [2]. In the present investigation with the help of Haar wavelets we are developing a novel stochastic operational matrix of Haar wavelets through which we can obtain an accurate solution for the stochastic integral equations. Here, we consider the following stochastic integral equation,

(1.1) $U(t) = g(t) + \int_0^t k_1(s,t)\,U(s)ds + \int_0^t k_2(s,t)\,U(s)dB(s), \quad t \in [0,T),$

where $U(t)$, $g(t)$, $k_1(s,t)$ and $k_2(s,t)$ for $s, t \in [0,T)$ are the stochastic processes on the same probability space $(\Omega,\ F,\ P)$ and $U(t)$ is unknown. Also $B(t)$ is a Brownian motion process and $\int_0^t k_2(s,t)\ U(s)\,dB(s)$ is the Itô integral [8].

The article is organized in the following way. Some definitions of stochastic calculus, properties of Haar wavelets and operational matrix of integration of Haar wavelets are studied. Also, *HWSOMI* is derived in ***Section* 2**. Method of solution is given in ***Section* 3**. In ***Section* 4**, convergence and error analysis of the proposed method is studied. ***Section* 5** presents some examples which shows the efficiency of the presented method. Lastly, ***Section* 6** gives the conclusion.

2 Stochastic Calculus and Wavelets

Here we examine some definitions existing in stochastic calculus. And we study the properties of Haar wavelets and operational matrix of integration of Haar wavelets (*HWOMI*). Stochastic operational matrix of integration of Haar wavelet is derived. Lastly, some results which will are used in further sections are mentioned.

2.1 Stochastic calculus

***Definition* 2.1** *A standard Brownian motion defined on the interval* $[0,\ T)$ *is a random variable* $B(t)$ *which depends on* $t \in [0,\ T)$ *and satisfies the following conditions:*

1. $B(0) = 0$ *with probability* 1.
2. *For* $0 \le s < t \le T$*, the random variable given by increment* $B(t) - B(s)$ *is distributed normally with mean zero and variance* $t - s$*, equivalently,* $B(t) - B(0) \sim \sqrt{t-s}N(0,1)$*, where* $N(0,1)$ *is a random variable distributed normally with mean zero and variance* 1.

3. *The increments* $B(t) - B(s)$ *and* $B(v) - B(u)$ *are independent for* $0 \le s < t < u < v \le T$.

Definition 2.2 [5] *The sequence* U_n *converge to* U *in* L^2 *if for each n,* $E\left(|U_n|^2\right) < \infty$. *Let us assume that* $0 \le s \le T$, *let* $v = v(s,T)$ *be the class of functions that* $g(t,w) : [0,\infty] \times \Omega \to R^n$, *satisfy,*

1. *the function* $(t,w) \to g(t,w)$ *is* $\beta \times G$ *measurable, where* β *is Borel algebra.*
2. *g is adapted to* G_t.
3. $E\left[\int_s^T g(t,w)^2\, dt\right] < \infty.$

Definition 2.3 *(The Itô-integral* [8]*) Let* $g \in v(s,T)$, *then the Itô-integral of g is defined by*

$$\int_s^T g(t,w)dB(t)(w) = \lim_{n\to\infty} \int_s^T \varphi_{t,w} dB(t)(w),$$

where, $\{\varphi\}$ *is the sequence of elementary functions such that*

$$E\left[\int_s^T (g-\varphi_n)^2\, dt\right] \to 0 \quad \text{a.s,} \quad n \to \infty.$$

2.2 Haar Wavelets

Haar wavelets $h_n(t)$ are defined as,

(2.1) $h_n(t) = \psi(2^j t - k),\ j \ge 0,\ \ 0 \le k < 2^j, n = 2^j + k, n, j, k \in \mathbb{Z},$

where

(2.2) $h_0(t) = 1, 0 \le t < 1, \ \ \psi(t) = \begin{cases} 1, & 0 \le t < \frac{1}{2}, \\ -1, & \frac{1}{2} \le t < 1. \end{cases}$

Every Haar wavelet $h_n(t)$ has the support $\left[\frac{k}{2^j}, \frac{k+1}{2^j}\right)$ and is elsewhere zero in the interval $[0,1)$.

Function Approximation: Any square integrable function $g(t)$ can be expressed with respect to Haar wavelets as

$$g(t) = g_0 h_0(t) + \sum_{i=1}^{\infty} g_i h_i(t),\ \ i = 2^j + k,\ \ j \ge 0,\ \ 0 \le k < 2^j,\ \ j,k \in \mathbb{N},$$

where g_i is given by

$$g_i = \int_0^1 g(t)h_i(t)dt,\ \ i = 0, 2^j + k,\ \ j \ge 0,\ \ 0 \le k < 2^j,\ \ j,k \in \mathbb{N}.$$

The above infinite series can be truncated after 2^J terms(J is the level of resolution) as

$$g(t) = \sum_{i=1}^{2^J-1} g(t)h_i(t),\ \ i = 2^j + k,\ \ 0 \le j \le J-1,\ \ 0 \le k < 2^j,\ \ j,k \in \mathbb{N}.$$

Rewriting this equation in the vector form as

$$g(t) \simeq G^T H(t) = GH^T(t),$$

where G and $H(t)$ are Haar wavelet coefficients given as

$$G = [g_0, g_1, ..., g_{2^J-1}],\ \ H(t) = [h_0(t), h_1(t), ..., h_{2^J-1}(t)].$$

Similarly, any two dimensional function $k(s,t) \in L^2([0,1) \times ([0,1))$ can be written in terms of Haar wavelets as

$$k_{ij} = \int_0^1 \int_0^1 k(s,t)h_i(t)h_j(s)dtds,\ \ i,j = 1,2,...N\ (N = 2^J).$$

For example, from equations (2.1) and (2.2), we can write

$$h_1(t) = \begin{cases} 1, & 0 \le t < \frac{1}{2}, \\ -1, & \frac{1}{2} \le t < 1, \end{cases}$$

$$h_2(t) = \begin{cases} 1, & 0 \le t < \frac{1}{4}, \\ -1, & \frac{1}{4} \le t < \frac{1}{2}, \end{cases}$$

$$h_3(t) = \begin{cases} 1, & \frac{1}{2} \le t < \frac{3}{4}, \\ -1, & \frac{3}{4} \le t < 1, \end{cases}$$

and so on.

2.3 Haar wavelets operational matrix of integration

HWOMI is computed as follows. Integrating equation (2.1), we get

(2.3) $\int_0^t H(s)ds \simeq PH(t)$,

where P is a matrix of order $N \times N$ and is called operational matrix of Haar wavelets.For example, for $N = 4$, we have

$$H(t) = [h_0(t), h_1(t), h_2(t), h_3(t)],$$

(2.4) $\int_0^t h_0(s)ds = t, \quad 0 \le t < 1,$

(2.5) $\int_0^t h_1(s)ds = \begin{cases} t, & 0 \le t < \frac{1}{2}, \\ 1 - t, & \frac{1}{2} \le t < 1, \end{cases}$

(2.6) $\int_0^t h_2(s)ds = \begin{cases} t, & 0 \le t < \frac{1}{4}, \\ (1/2) - t, & \frac{1}{4} \le t < \frac{1}{2}, \end{cases}$

(2.7) $\int_0^t h_3(s)ds = \begin{cases} t - \frac{1}{2}, & \frac{1}{2} \le t < \frac{3}{4}, \\ 1 - t, & \frac{3}{4} < t < 1. \end{cases}$

Thus, seeing equations (2.4), (2.5), (2.6) and (2.7), we can write P in general as,

(2.8) $P = \begin{cases} t - \frac{k}{m}, & \text{t} \in \left[\frac{k}{2^j}, \frac{k+0.5}{2^j}\right), \\ \frac{k+1}{m} - t, & \text{t} \in \left[\frac{k+0.5}{2^j}, \frac{k+1}{2^j}\right), \\ 0, & \text{elsewhere.} \end{cases}$

2.4 Haar wavelets stochastic operational matrix of integration

HWSOMI is written as follows,

(2.9) $\int_0^t H(s)dB(s) \simeq P_s H(t)$,

where P_s is a matrix of order $N \times N$ and is called stochastic operational matrix of Haar wavelets. For example, we obtain

(2.10) $\int_0^t h_0(s)dB(s) = B(t), \quad 0 \le t < 1,$

(2.11) $\int_0^t h_1(s)dB(s) = \begin{cases} B(t), & 0 \le t < \frac{1}{2}, \\ 2B(\frac{1}{2}) - B(t), & \frac{1}{2} \le t < 1, \end{cases}$

(2.12) $\int_0^t h_2(s)dB(s) = \begin{cases} B(t), & 0 \le t < \frac{1}{4}, \\ 2B(\frac{1}{4}) - B(t), & \frac{1}{4} \le t < \frac{1}{2}, \end{cases}$

(2.13) $\int_0^t h_3(s)dB(s) = \begin{cases} B(t) - B\left(\frac{1}{2}\right), & \frac{1}{2} \le t < \frac{3}{4}, \\ 2B\left(\frac{3}{4}\right) - B\left(\frac{1}{2}\right) - B(t), & \frac{3}{4} \le t < 1. \end{cases}$

Thus, seeing equations (2.10), (2.11), (2.12), and (2.13), we write the stochastic operational matrix of integration of Haar wavelets P_s in general as

(2.14) $P_s = \begin{cases} B(t) - B(\frac{k}{2^j}), & \text{t} \in \left[\frac{k}{2^j}, \frac{k+0.5}{2^j}\right), \\ B(\frac{k+0.5}{2^j}) - B(\frac{k}{2^j}) - B(t), & \text{t} \in \left[\frac{k+0.5}{2^j}, \frac{k+1}{2^j}\right), \\ 0, & \text{elsewhere.} \end{cases}$

***Remark* 2.1** *Using equation* (2.1) *for a N-vector G, we have*

(2.15) $H(t)H^T(t)G = \tilde{G}H(t)$,

where, $H(t)$ is the Haar wavelet coefficient matrix and $\tilde{G}$ is an $N \times N$ matrix given by

(2.16) $\tilde{G} = H\bar{G}H^{-1}$,

where $\bar{G} = diag(H^{-1}G)$. Also, for a $N \times N$ matrix X, we have

(2.17) $H^t XH(t) = \tilde{X^T} H(t)$,

where, $\tilde{X^T} = VH^{-1}$ and $V = diag(H^T XH)$ is a N-vector.

3 Method of solution

Consider the stochastic integral equation given in (1.1). Approximating the functions $U(t)$, $g(t)$, $k_1(x,t)$, and $k_2(x,t)$ using Haar wavelets, we get

(3.1) $U(t) \simeq U^T H(t) = UH^T(t)$,

(3.2) $g(t) \simeq G^T H(t) = GH^T(t)$,

(3.3) $k_1(s,t) \simeq H^T(s)K_1H(t) = H^T(t)K_1^T H(s)$,

(3.4) $k_2(s,t) \simeq H^T(s)K_2H(t) = H^T(t)K_2^T H(s)$,

where U and G are Haar wavelet coefficient vectors and K_1 and K_2 are Haar wavelet matrices. Substituting (3.1),(3.2),(3.3) and (3.4) in (1.1), we get

(3.5) $U^T H(t) \simeq G^T H(t) + H^T(t)K_1(\int_0^t H(s)H^T(s)Uds) + H^T(t)K_2(\int_0^t H(s)H^T(s)UdB(s))$.

By the use of *HWOMI*, *HWSOMI* and ***Remark* 2.1**, we have

(3.6) $U^T H(t) \simeq G^T H(t) + H^T(t)K_1\tilde{U}PH(t) + H^T(t)K_2\tilde{U}P_sH(t)$.

Using $\tilde{U_1} = K_1\tilde{U}P$ and $U_2 = K_2\tilde{U}P_s$ and using ***Remark* 2.1**, we get

(3.7) $U^T H(t) \simeq G^T H(t) + \tilde{U_1}H(t) + \tilde{U_2}H(t)$.

This gives,

(3.8) $U^T - \tilde{U_1} - \tilde{U_2} \simeq G^T$,

where $\tilde{U_1}$ and $\tilde{U_2}$ are functions of U and (3.8) is a system of linear equations. Solving this system of linear equations and substituting the obtained unknown vector U (3.1), we get the solution of (1.1).

4 Convergence and error analysis

The convergence and error analysis of the method presented for solving stochastic integral equations is studied.

***Theorem* 4.1** *Let $g(t) \in L^2[0,1)$ be any arbitrary function such that $|g'(t)| < \epsilon$, and $e_N(t) = g(t) - \sum_{i=0}^{N-1} g_ih_i(t)$, then*

(4.1) $\|e_N(t)\|_2 \leq \dfrac{\epsilon}{\sqrt{3}N}$.

Proof. By the definition,

(4.2) $\|e_N(t)\|_2^2 = \int_0^1 \left(\sum_{i=N}^{\infty} g_ih_i(t)dt\right)^2 dt = \sum_{i=N}^{\infty} g_i^2$.

In equation (4.2), $i = 2^j + k$ and

$$g_i = \int_0^1 h_i(t)g(t)dt = 2^{\frac{j}{2}}\left(\int_{k2^{-j}}^{(k+\frac{1}{2})2^{-j}} g(t)dt - \int_{(k+\frac{1}{2})2^{-j}}^{(k+1)2^{-j}} g(t)dt\right).$$

Using the mean value theorem for integrals, there exist $\eta_{1j} \in \left(k2^{-j}, \left(\frac{k+1}{2}\right)2^{-j}\right)$ and $\eta_{2j} \in \left(\left(\frac{k+1}{2}\right)2^{-j}, (k+1)2^{-j}\right)$ such that

(4.3)
$$\begin{aligned} g_i = \int_0^1 h_i(t)g(t)dt &= 2^{\frac{j}{2}}\left(g(\eta_{1j})\int_{k2^{-j}}^{(k+\frac{1}{2})2^{-j}} dt - g(\eta_{2j})\int_{(k+\frac{1}{2})2^{-j}}^{(k+1)2^{-j}} dt\right) \\ &= 2^{\frac{j}{2}}\left(g(\eta_{1j})\left[\left(k+\frac{1}{2}\right)2^{-j} - k2^{-j}\right] - g(\eta_{2j})\left[(k+1)2^{-j} - \left(k+\frac{1}{2}\right)2^{-j}\right]\right) \\ &= 2^{-\frac{-j}{2}-1}\left(g(\eta_{1j}) - g(\eta_{2j})\right) \\ &= 2^{-\frac{-j}{2}-1}\left(\eta_{1j} - \eta_{2j}\right)g'(\eta_j), \;\; \eta_{1j} < \eta_j < \eta_{2j}. \end{aligned}$$

Equation (4.3) gives

(4.4)
$$\begin{aligned} \|e_N(t)\|_2^2 = \sum_{i=N}^{\infty} g_i^2 &= \sum_{j=J}^{\infty} 2^{-j-2}(\eta_{1j} - \eta_{2j})^2 \\ &\leq \sum_{j=J}^{\infty} 2^{-j-2}2^{-2j}\epsilon^2 \\ &= \frac{\epsilon^2}{4}\sum_{j=J}^{\infty} 2^{-3j} \\ &= \frac{\epsilon^2}{3}2^{-2J}. \end{aligned}$$

Therefore,

(4.5) $\|e_N(t)\|_2 \leq \dfrac{\epsilon}{\sqrt{3}N}$.

***Theorem* 4.2** *Let $g(s,t) \in L^2([0,1)\times[0,1))$ be any arbitrary function such that $|\frac{\partial^2 g}{\partial s \partial t}| < \epsilon$, and $e_N(s,t) = g(s,t) - \sum_{i=0}^{N-1}\sum_{j=0}^{N-1} g_{ij}h_i(s)h_j(t)$, then*

(4.6) $\|e_N(s,t)\|_2 \leq \dfrac{\epsilon}{3N^2}.$

Proof. By the definition,

(4.7) $\|e_N(s,t)\|_2^2 = \int_0^1 \left(\sum_{i=N}^{\infty}\sum_{j=N}^{\infty} g_{ij}h_i(s)h_j(t)\right)^2 dt = \sum_{i=N}^{\infty}\sum_{j=N}^{\infty} g_{ij}^2.$

In equation (4.7), $i = 2^j + k,\ j = 2^{j'} + k$, and

$$g_{ij} = \int_0^1\int_0^1 h_i(s)h_j(t)g(s,t)dsdt.$$

Using the mean value theorem for integrals, there exist η_j, η_{1j}, η_{2j}, $\eta_{j'}$, $\eta_{1j'}$, and $\eta_{2j'}$ such that

(4.8)
$$\begin{aligned} g_{ij} &= \int_0^1 h_i(s)\left(\int_0^1 h_i(t)g(s,t)dt\right)ds \\ &= \int_0^1 h_i(s)\left[2^{\frac{-j'}{1}-1}(\eta_{1j'}-\eta_{2j'})\tfrac{\partial g(s,\eta_{j'})}{\partial t}\right]ds \\ &= 2^{\frac{-j'}{1}-1}(\eta_{1j'}-\eta_{2j'})\int_0^1 \tfrac{\partial g(s,\eta_{j'})}{\partial t}h_i(s)ds \\ &= 2^{\frac{-j}{2}-\frac{j'}{2}-2}(\eta_{1j'}-\eta_{2j'})(\eta_{1j}-\eta_{2j})\frac{\partial^2 g(\eta_j,\eta_{j'})}{\partial t\partial s}. \end{aligned}$$

Equation (4.8) gives

(4.9)
$$\begin{aligned} \|e_N(s,t)\|_2^2 = \sum_{i=N}^{\infty}\sum_{j=N}^{\infty} g_{ij}^2 &= \sum_{j=J}^{\infty}\sum_{j'=J}^{\infty} 2^{-j-j'-4}(\eta_{1j'}-\eta_{2j'})^2(\eta_{1j}-\eta_{2j})^2\left|\tfrac{\partial^2 g(\eta_j,\eta_{j'})}{\partial t\partial s}\right|^2 \\ &\leq \sum_{j=J}^{\infty}\sum_{j'=J}^{\infty} \epsilon^2 2^{-3j-3j'-4}. \end{aligned}$$

From equation (4.4), we get

(4.10) $\|e_N(s,t)\|_2^2 \leq N^2 \sum_{j=J}^{\infty} 2^{-3j-2}\sum_{j'=J}^{\infty} 2^{-3j'-2} = \frac{\epsilon^2}{(3N^2)^2}.$

Therefore

$$\|e_N(s,t)\|_2 \leq \frac{\epsilon}{3N^2}.$$

***Theorem* 4.3** *Let $U(t)$ and $U_N(t)$ be the exact and approximate solution of* (1.1)*. Let us assume that*

1. $\|U(t)\| \leq \delta,\ t \in [0,1)$,
2. $\|k_i(s,t)\| \leq D_i,\ i = 1,2$,
3. $(D_1 + \xi_1) + \|B(t)\|_\infty (D_2 + \xi_2)$,

then,

$$\|U(t) - U_N(t)\|_2 \leq \frac{\mu_N + \xi_{1N} + \|B(t)\|_\infty \xi_{2N}}{1 - (D_1 + \xi_1) - \|B(t)\|_\infty (D_2 + \xi_{2N})},$$

where

$$\mu_N = \sup_{t\in[0,1)} \frac{g'(t)}{\sqrt{3}N},$$

$$\xi_i = \frac{1}{3N^2}\sup_{s,t\in[0,1)}\left|\frac{\partial^2 k_i(s,t)}{\partial s\partial t}\right|,\ i = 1,2.$$

Proof. *From equation* (1.1)*, we have*

$$\begin{aligned} U(t) - U_N(t) = g(t) - g_N(t) &+ \int_0^t (k_1(s,t)U(s) - k_{1N}(s,t)U_N(s)ds) \\ &+ \int_0^t (k_2(s,t)U(s) - k_{2N}(s,t)U_N(s))\,dB(s). \end{aligned}$$

By using mean value theorem we have,

(4.11)
$$\begin{aligned} \|U(t) - U_N(t)\| \leq \|g(t) - g_N(t)\| &+ t\|k_1(s,t)U(s) - k_{1N}(s,t)U_N(s)\| \\ &+ B(t)\|k_2(s,t)U(s) - k_{2N}(s,t)U_N(s)\|. \end{aligned}$$

*Using **Theorem** 4.1 and **Theorem** 4.2, we have*

(4.12)

$$\|k_i(s,t)U(s) - k_{iN}(s,t)U_N(s)\| \le \|k_i(s,t)\| \, \||U(t) - U_N(t)\| + \|k_i(s,t) - k_{iN}(s,t)\| \, \||U(t)\| + \|k_i(s,t) - k_{iN}(s,t)\| \, \||U(t) - U_N(t)\|.$$

Substituting (4.12) *in* (4.11)*, we get*

(4.13)

$$\|U(t) - U_N(t)\| \le \mu_N + t\left[(D_1 + \xi_{1N})\|U(t) - U_N(t)\| + \delta\xi_{1N}\right] + B(t)\left[(D_2 + \xi_{2N})\|U(t) - U_N(t)\| + \delta\xi_{2N}\right].$$

Using the assumption (3)*, we get*

$$\|U(t) - U_N(t)\|_2 \le \frac{\mu_N + \xi_{1N} + \|B(t)\|_\infty \xi_{2N}}{1 - (D_1 + \xi_1) - \|B(t)\|_\infty (D_2 + \xi_{2N})}.$$

5 Numerical Experiments

Here some examples are presented in order to show the efficiency of the method presented.

***Test Problem* 5.1** *Consider the stochastic integral equation,*

(5.1) $U(t) = 1 + \int_0^t sin(s)U(s)dB(s),$

where $U(t)$ is the unknown stochastic process defined on the probability space (Ω, F, P), and $B(t)$ is the Brownian motion process. Exact solution of equation (5.1) is

(5.2) $U(t) = exp\left[\frac{-1}{4}(t - cos(t)sin(t)) + \int_0^t sin(s)dB(s)\right].$

Method of Implementation

For $N = 4$.

Comparing (5.1)with equation (1.1), we get

(5.3) $g(t) = 1,$

(5.4) $k_1(s,t) = 0,$

and

(5.5) $k_2(s,t) = sin(s).$

Approximating equations (5.3), (5.4), and (5.5) using Haar wavelets, we obtain

$$G = [\ 1 \quad 0 \quad 0 \quad 0\ ],$$

$$K_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \ and\ K_2 = \begin{pmatrix} 0.4609 & 0 & 0 & 0 \\ -0.2154 & 0 & 0 & 0 \\ -0.1208 & 0 & 0 & 0 \\ -0.0912 & 0 & 0 & 0 \end{pmatrix}.$$

Let our assumed solution be U and approximating this using Haar wavelets, we get

$$U(t) \simeq U^T H(t) = UH^T(t).$$

Substituting the obtained vector G, matrices K_1 and K_2 and the approximated unknown solution U in equation (5.1)and by the use of operational matrix of integration of Haar wavelets and the stochastic operational matrix of integration Haar wavelets, we obtain the unknown vector U as

$$U = [\ 0.90697 \quad 0.043482 \quad 0.024383 \quad 0.018413\ ].$$

Substituting this in $U(t) \simeq U^T H(t) = UH^T(t)$,we obtain the solution as

$$U(t) = [\ 0.9748 \quad 0.9261 \quad 0.8819 \quad 0.8451\ ].$$

The exact and approximate solutions of ***Test Problem* 5.1** for $N = 4$ and $N = 8$ are shown in ***Table* 5.1**, maximum absolute error (E_{max}) for different values of N are shown in ***Table* 5.2** and the graphs of absolute errors for different values of N are shown in ***Figure* 5.1**.

***Table* 5.1:** Exact solution, approximate solution and absolute errors for ***Test Problem* 5.1**.

	$N = 4$			$N = 8$		
t	Exact Solution	Approximate Solution	Absolute error	Exact Solution	Approximate Solution	Absolute error
0	1.0000	0.9256	0.0744	1.0000	1.0440	0.0440
0.1	0.9996	0.9799	0.0197	1.0024	1.0107	0.0083
0.2	0.9965	0.9602	0.0363	1.0070	1.0213	0.0143
0.3	0.9925	0.9407	0.0518	1.0142	1.0318	0.0176
0.4	0.9864	0.9217	0.0647	1.0225	1.0418	0.0193
0.5	0.9737	0.9040	0.0697	1.0310	1.0515	0.0204
0.6	0.9610	0.8863	0.0747	1.0390	1.0606	0.0216
0.7	0.9403	0.8709	0.0695	1.0459	1.0692	0.0233
0.8	0.9169	0.8561	0.0608	1.0510	1.0771	0.0261
0.9	1.4098	1.3190	0.0907	1.0536	1.0841	0.0306

***Table* 5.2:** Absolute errors for different values of N of ***Test Problem* 5.1**.

N	E_{max}
4	0.0907
8	0.0440
16	0.0194

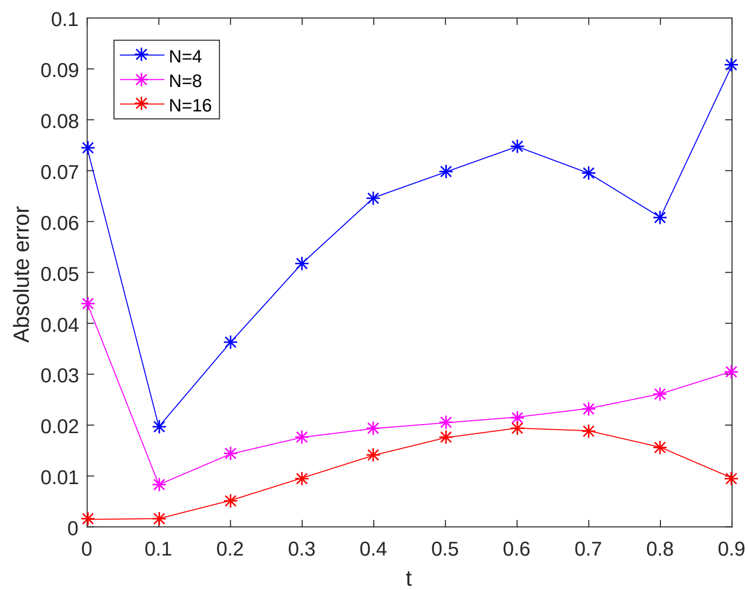

***Figure* 5.1:** Absolute errors for different values of N of ***Test Problem* 5.1**.

***Test Problem* 5.2** *Consider the stochastic integral equation*

$$(5.6)\quad U(t) = \frac{1}{12} + \int_0^t cos(s)U(s)ds + \int_0^t sin(s)U(s)dB(s),$$

where $U(t)$ *is the unknown stochastic process defined on the probability space* (Ω, F, P)*, and* $B(t)$ *is the Brownian motion process. Exact solution of* (5.6) *is*

$$(5.7)\quad U(t) = \frac{1}{12}exp\left[\frac{-t}{4} + sin(t) + \frac{sin(2t)}{8} + \int_0^t sin(s)dB(s)\right].$$

Implementation is shown in ***Test Problem* 5.1**. The exact as well as approximate solutions of ***Test Problem* 5.2** for $N = 4$ and $N = 8$ are shown in ***Table* 5.3**, maximum absolute error (E_{max}) for different values of N are shown in ***Table* 5.4** and the graphs of absolute errors for different values of N are shown in ***Figure* 5.2**.

Table **5.3:** Exact solution, approximate solution and absolute errors for ***Test Problem* 5.2**.

	$N = 4$			$N = 8$		
t	Exact Solution	Approximate Solution	Absolute error	Exact Solution	Approximate Solution	Absolute error
0	0.0833	0.0759	0.0075	0.0833	0.0729	0.0105
0.1	0.0921	0.0636	0.0285	0.0918	0.0729	0.0190
0.2	0.1017	0.0655	0.0362	0.1004	0.0782	0.0223
0.3	0.1116	0.0747	0.0368	0.1089	0.0710	0.0379
0.4	0.1214	0.0801	0.0413	0.1167	0.0775	0.0391
0.5	0.1311	0.0739	0.0572	0.1235	0.0710	0.0525
0.6	0.1409	0.0677	0.0732	0.1291	0.0647	0.0644
0.7	0.1487	0.0732	0.0756	0.1331	0.0720	0.0611
0.8	0.1560	0.0826	0.0734	0.1350	0.0649	0.0701
0.9	0.2133	0.1174	0.0959	0.1346	0.0711	0.0635

Table **5.4:** Absolute errors for different values of N of test problem ***Test Problem* 5.2**.

N	E_{max}
4	0.0959
8	0.0701
16	0.0411

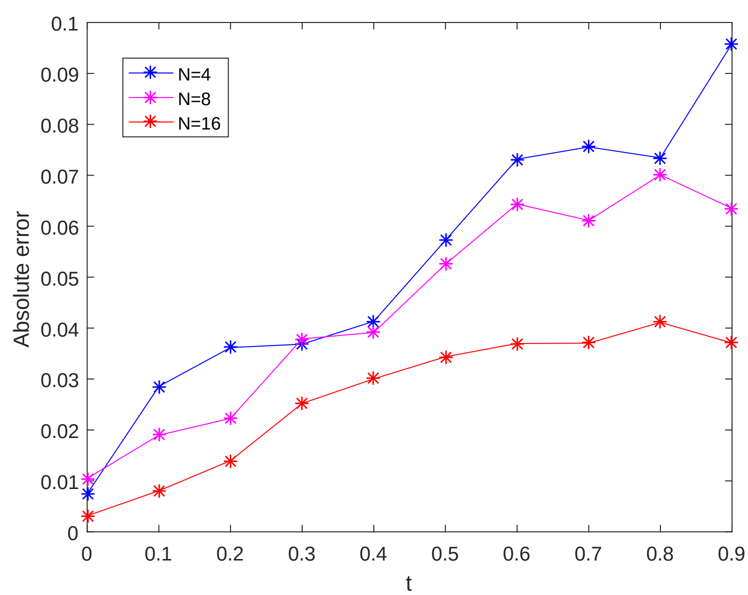

Figure **5.2:** Absolute errors for different values of N of ***Test Problem* 5.2**.

***Test Problem* 5.3** *Consider the stochastic integral equation*

$$(5.8)\quad U(t) = \frac{1}{10} + \int_0^t ln(1+s)U(s)ds + \int_0^t sU(s)dB(s).$$

where $U(t)$ *is the unknown stochastic process defined on the probability space* (Ω, F, P)*, and* $B(t)$ *is the Brownian motion process. Exact solution of* (5.8) *is*

$$(5.9)\quad U(t) = \frac{1}{10}exp\left[(1+t)ln(1+t) - t - \frac{t^3}{6} + \int_0^t sdB(s)\right].$$

Implementation is shown in ***Test Problem* 5.1**. The exact as well as approximate solutions of ***Test Problem* 5.3** for $N = 4$ and $N = 8$ are shown in ***Table* 5.5**, maximum absolute error (E_{max}) for different values of N are shown in ***Table* 5.6** and the graphs of absolute errors (E_{max}) *for different values of* N are shown in ***Figure* 5.3**.

***Table* 5.5:** Exact solution, approximate solution and absolute errors for ***Test Problem* 5.3**.

	$N=4$			$N=8$		
t	Exact Solution	Approximate Solution	Absolute error	Exact Solution	Approximate Solution	Absolute error
0	0.0833	0.0750	0.0083	0.1000	0.0806	0.0194
0.1	0.0997	0.0907	0.0090	0.0999	0.0949	0.0051
0.2	0.0985	0.0847	0.0137	0.0997	0.0907	0.0090
0.3	0.0969	0.0798	0.0171	0.0992	0.0845	0.0147
0.4	0.0948	0.0736	0.0211	0.0981	0.0836	0.0145
0.5	0.0909	0.0636	0.0274	0.0963	0.0759	0.0205
0.6	0.0871	0.0535	0.0336	0.0939	0.0694	0.0246
0.7	0.0816	0.0536	0.0281	0.0907	0.0709	0.0198
0.8	0.0756	0.0571	0.0186	0.0867	0.0615	0.0251
0.9	0.1237	0.0932	0.0305	0.0817	0.0670	0.0147

***Table* 5.6:** Absolute errors for different values of N of ***Test Problem* 5.3**.

N	E_{max}
4	0.0336
8	0.0251
16	0.0074

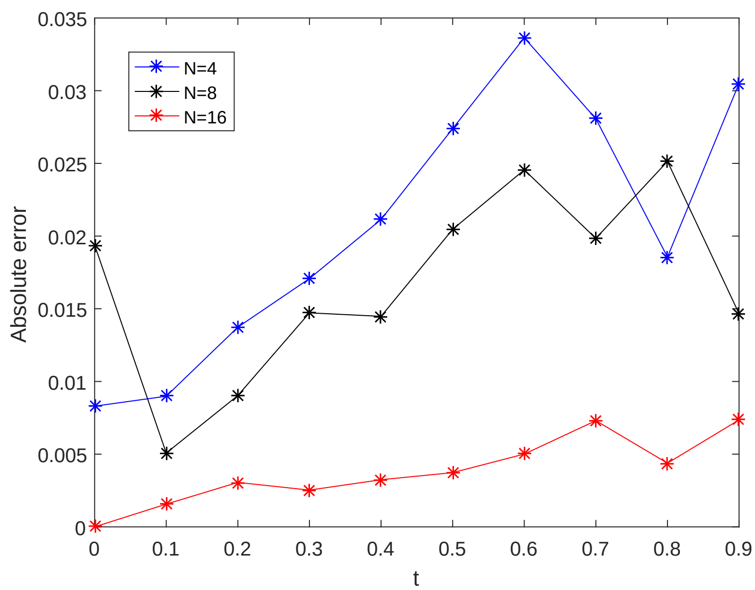

***Figure* 5.3:** Absolute errors for different values of N of ***Test Problem* 5.3**.

6 Conclusion

In this article, using Haar wavelets a new *HWSOMI* is developed to solve stochastic integral equations. From tables and figures we can see that the solutions obtained by proposed method are in good agreement with that of exact solutions. Hence, the investigated method is efficient and convenient for solving stochastic integral equations.

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