

FUZZY PREOPEN SETS AND FUZZY PRE-CONTINUITY

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Abstract

In the present paper, we introduce fuzzy preopen (closed) sets and fuzzy pre-continuity in Sostak fuzzy topological space. Also we investigate their significant characteristic properties.

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1 Introduction

The concept of fuzzy sets was introduced by Zadeh [9] and later Chang [2] defined fuzzy topological spaces. Sostak [8] introduced a new fuzzy topological space exploiting the idea of partial openness of fuzzy sets. This generalized fuzzy topological space was later rephrased by Chattopadhyay et.al. [3]. Several mathematicians have worked on this space (see [4], [5]).

The concepts of fuzzy preopen sets, fuzzy strong preopen sets and strong pre continuity (see [6], [7]) have been introduced in case of classical fuzzy topological spaces introduced by Chang [2]. In the present paper, we introduce fuzzy preopen (closed) sets and fuzzy pre continuity in the Sostak fuzzy topological space redefined by Chattopadhyay [3]. Further we establish their significant properties.

2 Preliminaries

Let X be a non-empty set and $I \equiv [0, 1]$ be the unit closed interval of real line. Let I^X denote the set of all fuzzy sets on X . A fuzzy set A on X is a mapping $A : X \rightarrow I$, where for any $x \in X$, $A(x)$ denotes the degree of membership of element x in fuzzy set A . The null fuzzy set 0 and whole fuzzy set 1 are the constant mappings from X to $\{0\}$ and $\{1\}$ respectively.

A family τ of fuzzy sets on X is called a fuzzy topology (see [2]) on X if (i) 0 and 1 belong to τ , (ii) Any union of members of τ is in τ , (iii) a finite intersection of members of τ is in τ . The system consisting of X equipped with fuzzy topology τ defined on it, is called a fuzzy topological space and is denoted as (X, τ) . Now we define the So-fuzzy topological space (see [3], [8]).

A So-fuzzy topology on a non-empty set X is a family τ of fuzzy sets on X satisfying the following axioms with respect to a mapping $\tau : I^X \rightarrow I$ such that

1. $\tau(0) = \tau(1) = 1$;
2. $\tau(A \cap B) \geq \tau(A) \wedge \tau(B)$; for any $A, B \in I^X$;
3. $\tau(\cup_{i \in J} A_i) \geq \wedge_{i \in J} \tau(A_i)$, for any arbitrary family $\{A_i : i \in J\} \subseteq I^X$.

The system (X, τ) is called So-fuzzy topological space and the real number $\tau(A)$ is called the degree (or grade) of openness of fuzzy set A . We note that

Proposition 2.1 *Let X be a non-empty set. Then the map $\tau : I^X \rightarrow I$ given by $\tau(0) = 1$ and $\tau(A) = \inf\{A(x) : x \in \text{supp}A\}$, if $A \neq 0$, satisfies the axioms of gradation of openness.*

If (X, τ) is a So-fuzzy topological space, then we observe that (see [2]) for any $\rho \in [0, 1]$, the family $\tau_\rho \equiv \{A \in I^X : \tau(A) \geq \rho\}$ is actually a fuzzy topology in sense of Chang [2] and it is called ρ -level fuzzy topology on X with respect to the gradation of openness τ . All fuzzy sets belonging to τ_ρ are called fuzzy- ρ -open sets and their complements are called fuzzy- ρ -closed sets.

For any fuzzy set A , the interior and closure of A with respect to τ_ρ are defined as follows:

$$Int_\rho(A) = \cup\{G \in I^X : G \subseteq A \text{ and } G \in \tau_\rho\}$$

$$Cl_\rho(A) = \cap\{K \in I^X : A \subseteq K \text{ and } K^c \in \tau_\rho\}$$

3 Fuzzy- ρ -Pre Open (Closed) Sets

In this section, we define fuzzy- ρ -pre open sets and fuzzy- ρ -pre closed sets in So-fuzzy topological space and investigate their properties.

Definition 3.1 Let (X, τ) be a So-fuzzy topological space and $A \in I^X$ be a fuzzy set. Then for any $\rho \in I$, a fuzzy set A is said to be a

- (i) Fuzzy- ρ -pre open set in X iff $A \subseteq \text{Int}_\rho(\text{Cl}_\rho(A))$,
- (ii) Fuzzy- ρ -pre closed set in X iff $A \supseteq \text{Cl}_\rho(\text{Int}_\rho(A))$.

Clearly fuzzy sets 0 and 1 are both trivially fuzzy ρ -pre open as well as fuzzy ρ -pre closed sets in X .

Remark 3.1 It is clear that every fuzzy- ρ -open (closed) set is a fuzzy- ρ -pre open (closed) set, but converse of these may not be true in general.

Example 3.1 Let $X = \{a, b\}$ and $A, B, C \in I^X$ be fuzzy sets defined as follows:

$$A = \{(a, 0.6), (b, 0.3)\}; \quad B = \{(a, 0.4), (b, 0.2)\}; \quad C = \{(a, 0.8), (b, 0.5)\}.$$

Define a map $\tau : I^X \rightarrow I$ as follows:

$$\tau(F) = \begin{cases} 1, & \text{if } F = 0, 1 \\ 0.3, & \text{if } F = A \\ 0.2, & \text{if } F = B \\ 0, & \text{otherwise.} \end{cases}$$

Suppose $\rho = 0.1$. We see that fuzzy set C is a fuzzy- ρ -pre open set because $\text{Int}_\rho(\text{Cl}_\rho(C)) = 1 \supseteq C$. But it is not a fuzzy- ρ -open set (because $\tau(C) = 0 \not\geq 0.1$).

Theorem 3.1 Let (X, τ) be a So-fuzzy topological space. Then for any $\rho \in I$,

- (a) Any union of fuzzy- ρ -pre open sets is a fuzzy- ρ -pre open set;
- (b) Any intersection of fuzzy- ρ -pre closed sets is a fuzzy- ρ -pre closed set.

Proof. (a) Let $\{A_i : i \in J\}$ be an arbitrary collection of fuzzy- ρ -pre open sets in So-fuzzy topological space (X, τ) . Then for each $i \in J$, we have $A_i \subseteq \text{Int}_\rho(\text{Cl}_\rho(A_i))$. Hence

$$\bigcup_{i \in J} A_i \subseteq \bigcup_{i \in J} \text{Int}_\rho(\text{Cl}_\rho(A_i)) \subseteq \text{Int}_\rho(\bigcup_{i \in J} \text{Cl}_\rho(A_i)) \subseteq \text{Int}_\rho(\text{Cl}_\rho(\bigcup_{i \in J} A_i)).$$

Thus $\bigcup_{i \in J} A_i$ is a fuzzy- ρ -pre open set. We can prove (b) similarly.

Definition 3.2 Let (X, τ) be a So-fuzzy topological space and $A \in I^X$ be a fuzzy set. Then for each $\rho \in I$, fuzzy- ρ -pre interior and fuzzy- ρ -pre closure of fuzzy set A denoted as $P\text{-int}_\rho(A)$ and $P\text{-cl}_\rho(A)$ are defined as follows:

$$P\text{-int}_\rho(A) = \bigcup \{G \in I^X : G \subseteq A \text{ and } G \text{ is a fuzzy-}\rho\text{-pre open set in } X\},$$

$$P\text{-cl}_\rho(A) = \bigcap \{K \in I^X : K \supseteq A \text{ and } K \text{ is a fuzzy-}\rho\text{-pre closed set in } X\}.$$

Theorem 3.2 Let (X, τ) be a So-fuzzy topological space and $A \in I^X$ be a fuzzy set. Then for any $\rho \in I$,

- (i) $P\text{-cl}_\rho(1 - A) = 1 - P\text{-int}_\rho(A)$,
- (ii) $P\text{-int}_\rho(1 - A) = 1 - P\text{-cl}_\rho(A)$.

Proof. (i) Suppose $\{G_i\}_{i \in J}$ is the family of all fuzzy- ρ -preopen sets in X contained in A . Then

$$P\text{-int}_\rho(A) = \bigcup_{i \in J} G_i = 1 - \bigcap_{i \in J} G_i^c.$$

Since $G_i \subseteq A$, we have $G_i^c \supseteq A^c$, $\forall i \in J$. Thus $\{G_i^c\}_{i \in J}$ is the collection of all fuzzy- ρ -preclosed sets containing A^c . Hence $\bigcap_{i \in J} G_i^c = P\text{-cl}_\rho(A^c) = P\text{-cl}_\rho(1 - A)$. Thus $P\text{-int}_\rho(A) = 1 - P\text{-cl}_\rho(1 - A)$. Hence $P\text{-cl}_\rho(1 - A) = 1 - P\text{-int}_\rho(A)$.

Proof. (ii) It can be proved in a similar manner.

Theorem 3.3 Let (X, τ) be a So-fuzzy topological space. Then for any $\rho \in I$, a fuzzy set $A \in I^X$ is a

- (a) Fuzzy- ρ -pre open set iff $P\text{-int}_\rho(A) = A$;
- (b) Fuzzy- ρ -pre closed set iff $P\text{-cl}_\rho(A) = A$.

Proof. (a) Let A be fuzzy- ρ -pre open set in X . Let $\{G_i\}_{i \in J}$ be the family of all fuzzy ρ -pre open sets contained in A . Since each $G_i \subseteq A$, $i \in J$, we have $\cup_{i \in J} G_i \subseteq A$. Therefore

$$(3.1) \quad P - \text{int}_\rho = \cup_{i \in J} \{G_i \in I^X : G_i \subseteq A \text{ and } G_i \text{ is a fuzzy } \rho - \text{preopen set}\} \subseteq A.$$

Since $A \subseteq A$ and A is a fuzzy- ρ -preopen set in X , hence $A \in \{G_i\}_{i \in J}$. Therefore

$$(3.2) \quad A \subseteq \cup_{i \in J} G_i \equiv P - \text{int}_\rho(A).$$

From equations (3.3.1) and (3.3.2), $A = P - \text{int}_\rho(A)$.

Conversely; suppose A is a fuzzy set in So-fuzzy topological space (X, τ) such that $A = P - \text{int}_\rho(A)$. Then

$$(3.3) \quad A = P - \text{int}_\rho(A) = \cup \{G_i \in I^X : G_i \subseteq A \text{ and } G_i \text{ is a fuzzy } \rho - \text{pre open set}\}.$$

Since any union of fuzzy- ρ -preopen sets is a fuzzy- ρ -preopen set, in view of (3.3.3), set A is a fuzzy- ρ -pre open set in X .

Proof. (b) This can be proved in a similar manner.

Theorem 3.4 Let (X, τ) be a So-fuzzy topological space. Then for any $\rho \in I$, the following properties hold for fuzzy- ρ -pre closure:

- (i) $P - \text{cl}_\rho(0) = 0$;
- (ii) $P - \text{cl}_\rho(A)$ is a fuzzy- ρ -pre closed set in X ;
- (iii) $P - \text{cl}_\rho(A) \subseteq P - \text{cl}_\rho(B)$, if $A \subseteq B$;
- (iv) $P - \text{cl}_\rho(P - \text{cl}_\rho(A)) = P - \text{cl}_\rho(A)$;
- (v) $P - \text{cl}_\rho(A \cup B) \supseteq P - \text{cl}_\rho(A) \cup P - \text{cl}_\rho(B)$;
- (vi) $P - \text{cl}_\rho(A \cap B) \subseteq P - \text{cl}_\rho(A) \cap P - \text{cl}_\rho(B)$.

Proof. It is easy to prove.

Theorem 3.5 Let (X, τ) be a So-fuzzy topological space and $A, B \in I^X$ be fuzzy sets. Then for any $\rho \in I$,

- (i) $P - \text{int}_\rho(1) = 1$;
- (ii) $P - \text{int}_\rho(A)$ is a fuzzy- ρ -pre open set in X ;
- (iii) $P - \text{int}_\rho(A) \subseteq P - \text{int}_\rho(B)$, if $A \subseteq B$;
- (iv) $P - \text{int}_\rho(P - \text{int}_\rho(A)) = P - \text{int}_\rho(A)$;
- (v) $P - \text{int}_\rho(A \cup B) \supseteq P - \text{int}_\rho(A) \cup P - \text{int}_\rho(B)$;
- (vi) $P - \text{int}_\rho(A \cap B) \subseteq P - \text{int}_\rho(A) \cap P - \text{int}_\rho(B)$.

4 Fuzzy- ρ -Pre Continuous Map

In this section, we define a fuzzy- ρ -pre continuous map from one So-fuzzy topological space to another and investigate its characteristic properties. We know fuzzy- ρ -continuous map is defined (see [3]) as follows:

Definition 4.1 Let (X, τ) and (Y, σ) be two So-fuzzy topological spaces. A map $f : X \rightarrow Y$ is said to be a fuzzy- ρ -continuous map if $\tau(f^{-1}(B)) \geq \sigma(B)$, for each fuzzy set $B \in I^Y$ such that $\sigma(B) \geq \rho$.

Now we define fuzzy- ρ -pre continuous map as follow:

Definition 4.2 Let (X, τ) and (Y, σ) be two So-fuzzy topological spaces. A map f from X to Y is called a fuzzy- ρ -pre continuous map iff $f^{-1}(B)$ is a fuzzy- ρ -pre open set for any fuzzy set $B \in I^Y$ such that $\sigma(B) \geq \rho$.

Remark 4.1 It is obvious that every fuzzy- ρ -continuous map is a fuzzy- ρ -pre continuous map, but converse may not be true.

Example 4.1 Let $X = \{a, b\}$, $Y = \{u, v\}$ and $A, B \in I^X$, $C \in I^Y$ be fuzzy sets defined as follows:

$$A = \{(a, 0.7), (b, 0.2)\}; \quad B = \{(a, 0.5), (b, 0.6)\}; \quad C = \{(u, 0.7), (v, 0.6)\};$$

$$D = \{(a, 0.5), (b, 0.2)\}; \quad E = \{(u, 0.8), (v, 0.7)\}.$$

We define fuzzy topologies $\tau : I^X \rightarrow I$ and $\sigma : I^Y \rightarrow I$ as follows:

$$\tau(F) = \begin{cases} 1, & \text{if } F = 0, 1 \\ 0.2, & \text{if } F = A, D \\ 0.5, & \text{if } F = B \\ 0.6 & \text{if } F = C \\ 0, & \text{otherwise,} \end{cases}$$

$$\sigma(F) = \begin{cases} 1, & \text{if } F = 0, 1 \\ 0.7, & \text{if } F = E \\ 0, & \text{otherwise.} \end{cases}$$

Consider a map $f : (X, \tau) \rightarrow (Y, \sigma)$ defined as $f(A) = u$, $f(b) = v$. Suppose $\rho = 0.1$. We see that $f^{-1}(E) \subseteq \text{Int}_\rho(\text{Cl}_\rho(f^{-1}(E)))$. Hence $f^{-1}(E)$ is a fuzzy- ρ -pre open set. Similarly $f^{-1}(0) \equiv 0$ and $f^{-1}(1) \equiv 1$ are also fuzzy- ρ -pre open sets. Thus f is a fuzzy- ρ -pre continuous map. But f is not a fuzzy- ρ -continuous map because $f^{-1}(E)$ is not a fuzzy- ρ -open set.

Theorem 4.1 Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be a map from one So-fuzzy topological space to another such that $\tau^*(f^{-1}(B)) \geq \rho$, for each $B \in I^Y$ with $\sigma^*(B) \geq \rho$, then f is a fuzzy- ρ -pre continuous map.

Proof. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be a map such that $\tau^*(f^{-1}(B)) \geq \rho$, for each $B \in I^Y$ for which $\sigma^*(B) \geq \rho$. Since $f^{-1}(B) \in I^X$ and $\tau^*(f^{-1}(B)) = \tau((f^{-1}(B))^c) = \tau(f^{-1}(B^c)) \geq \rho$, we conclude that $f^{-1}(B^c)$ is a fuzzy- ρ -open set in X . Since every fuzzy ρ -open set is a fuzzy ρ -pre open set, $f^{-1}(B^c)$ is a fuzzy- ρ -pre open set in X . Further $\sigma(B^c) = \sigma^*(B) \geq \rho$. Thus $f^{-1}(B^c)$ is a fuzzy- ρ -pre open set in X for each $B^c \in I^Y$ such that $\sigma(B^c) \geq \rho$. Therefore f is a fuzzy- ρ -pre continuous map.

Theorem 4.2 Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be a map from one So-fuzzy topological space to another. Then for any $\rho \in I$, following statements are equivalent:

- (a) f is a fuzzy- ρ -pre continuous map;
- (b) $f^{-1}(B)$ is a fuzzy- ρ -pre closed set for each fuzzy- ρ -closed set B in Y ;
- (c) $\text{Cl}_\rho(\text{Int}_\rho(f^{-1}(B))) \subseteq f^{-1}(\text{Cl}_\rho(B))$, for each fuzzy set B in Y ;
- (d) $f(\text{Cl}_\rho(\text{Int}_\rho(A))) \subseteq \text{Cl}_\rho(f(A))$, for each fuzzy set A in X .

Proof. Let (X, τ) and (Y, σ) be two So-fuzzy topological spaces. We will prove this theorem in following steps:

(i) (a)arrow(b): Let $f : X \rightarrow Y$ be a fuzzy- ρ -pre continuous map for any $\rho \in I$. Let B be a fuzzy- ρ -closed set in Y . Then B^c is a fuzzy- ρ -open set in Y so that $\sigma(B^c) \geq \rho$. Since f is a fuzzy ρ -continuous map, we find that $f^{-1}(B^c)$ is a fuzzy- ρ -pre open set in X . Therefore $(f^{-1}(B^c))^c = f^{-1}(B)$ is a fuzzy- ρ -pre closed set in X . Similarly we can prove (b)arrow(a).

(ii) (b)arrow(c): Let B be a fuzzy set in Y , then $\text{Cl}_\rho(B)$ is a fuzzy- ρ -closed set in Y and hence by (b), $f^{-1}(\text{Cl}_\rho(B))$ is a fuzzy- ρ -pre closed set in X . Therefore by definition, $f^{-1}(\text{Cl}_\rho(B)) \supseteq \text{Cl}_\rho(\text{Int}_\rho(f^{-1}(\text{Cl}_\rho(B)))) \supseteq \text{Cl}_\rho(\text{Int}_\rho(f^{-1}(B)))$. Thus $\text{Cl}_\rho(\text{Int}_\rho(f^{-1}(B))) \subseteq f^{-1}(\text{Cl}_\rho(B))$.

(iii) (c)arrow(d): Let $A \in I^X$ be any fuzzy set, then $f(A) \in I^Y$. Now by (c), $\text{Cl}_\rho(\text{Int}_\rho(f^{-1}(f(A)))) \subseteq f^{-1}(\text{Cl}_\rho(f(A)))$. It implies that $\text{Cl}_\rho(\text{Int}_\rho(A)) \subseteq f^{-1}(\text{Cl}_\rho(f(A)))$. Hence $f(\text{Cl}_\rho(\text{Int}_\rho(A))) \subseteq f(f^{-1}(\text{Cl}_\rho(f(A)))) \subseteq \text{Cl}_\rho(f(A))$.

(iv) (d)arrow(b): can be proved easily.

Theorem 4.3 Let (X, τ) , (Y, σ) and (Z, δ) be three So-fuzzy topological spaces and let $\rho \in I$ be any real number. If $f : X \rightarrow Y$ is a fuzzy- ρ -pre continuous map and $g : Y \rightarrow Z$ is a fuzzy- ρ -continuous map, then $g \circ f : X \rightarrow Z$ is a fuzzy- ρ -pre continuous map.

Proof. Let C be a fuzzy- ρ -open set in Z so that $\delta(C) \geq \rho$, then $\sigma(g^{-1}(C)) \geq \delta(C) \geq \rho$. Thus by hypothesis $g^{-1}(C)$ is a fuzzy- ρ -open set in Y . Since f is a fuzzy- ρ -pre continuous map, we get that $f^{-1}(g^{-1}(C))$ is a fuzzy- ρ -pre open set in X . Now $f^{-1}(g^{-1}(C)) = (g \circ f)^{-1}(C)$. Hence $(g \circ f)^{-1}(C)$ is a fuzzy- ρ -pre open set in X . Now $g \circ f : (X, \tau) \rightarrow (Z, \delta)$ is a map and we have derived that for any fuzzy- ρ -open set C in Z , fuzzy set $(g \circ f)^{-1}(C)$ is a fuzzy- ρ -pre open set in X . Hence $(g \circ f)$ is a fuzzy- ρ -pre continuous map.

5 Conclusion

In the present paper, we have defined fuzzy pre open (closed) sets and fuzzy pre-continuity in Sostak fuzzy topological space. The concept is introduced as an extension of concepts of fuzzy preopen sets introduced in [6]. Several significant results have been obtained.

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