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(Dedicated to Honor Dr. R. C. Singh Chandel on His 75^{th} Birth Anniversary Celebrations)

HYERS - ULAM STABILITY OF FIRST AND SECOND ORDER PARTIAL DIFFERENTIAL EQUATIONS

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Abstract

In this paper, we prove the Hyers-Ulam (HU) stability of the first and second order partial differential equations: $u_x(x,t) + K(x, u(x,t)) = 0$ and $u_{xx}(x,t) + F(x,u)u_x(x,t) + H(x,u) = 0$ respectively.
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1 Introduction

Hyers Ulam *(HU)* stability of differential equation has drawn much attention since Ulam's [16] presentation of the problem on stability of group homomorphism in 1940 and Hyers' [5] partial solution to it in 1941. For ordinary differential equations one can refer [3, 15, 6, 7] and [8, 9] for partial differential equations. Its various extensions have been named with additional word. One such extension is Hyers Ulam Rassias *(HUR)* stability. *HUR* stability for linear differential operators of n^{th} order with non-constant coefficients is studied in [10] and [11]. *HUR* stability for special types of non-linear equations has been studied in [1, 2, 12, 13, 14]. In 2011, Gordji et al. [4], proved the *HUR* stability of non-linear partial differential equations by using Banach's Contraction Principle. In this paper, we prove the *HU* stability of first and second order partial differential equations:

(1.1) $u_x(x,t) + K(x, u(x,t)) = 0,$

where $K : J \times \mathbb{R} \to \mathbb{R}$ is a continuous function, $u(x,t) \in C^1(J \times J)$, $J = [a, b]$ be a closed interval and

(1.2) $u_{xx}(x,t) + F(x,u)u_x(x,t) + H(x,u) = 0,$

where $F, H : J \times \mathbb{R} \to \mathbb{R}$ are continuous functions. Here $u(x,t) \in C^2(J \times J)$.

First we define HU stability.

***Definition* 1.1** *The equation* (1.1) *is said to be HU stable if the following holds:*
Let $\epsilon \geq 0$. Assume that , for any function $u(x,t) \in C^1$ satisfying the differential inequality

(1.3) $|u_x(x,t) + K(x, u(x,t))| \leq \epsilon, \ \forall x, t \in J,$

there exists a solution $u_0(x,t) \in C^1$ of equation (1.1) *and $M(\epsilon) > 0$ such that*

$$u(x,t) - u_0(x,t)| \leq M(\epsilon), \ \forall (x,t) \in J \times J.$$

Similarly we can define HU stability for equation (1.2).

We need the following result.

***Theorem* 1.1** *(Banach Contraction Principle)* [4] *: Let (X, d) be a complete metric space and $T : X \to X$ be a contraction , that is, there exists $\alpha \in (0, 1)$ such that $d(Tx, Ty) \leq \alpha d(x, y), \forall x, y \in X$. Then $\exists$ a unique $a \in X$ such that $Ta = a$. Moreover, $a = \lim_{n\to\infty} T^n x$ and $d(a, x) \leq \frac{1}{(1-\alpha)} d(x, Tx), \forall x \in X$.*

2 Main Results

In this section we prove the *HU* stability of first and second order partial differential equations (1.1) and (1.2) respectively.

Theorem 2.1 *Let $x_0 \in J$ and $K : J \times R \to R$ be a continuous function such that*

(2.1) $|K(x, v(x,t)) - K(x, w(x,t))| \leq \lambda |v(x,t) - w(x,t)|, \forall x, t \in J,$

where $\lambda > 0, \lambda \in R$ and $v(x,t), w(x,t) \in C^1$.

Let

(2.2) $M_1 = \sup_{x \in J} \left| \int_{x_0}^{x} ds \right|,$

with $0 < \lambda M_1 < 1$. Let $u(x,t) \in C^1$ satisfy

(2.3) $|u_x(x,t) + K(x, u(x,t))| \leq \epsilon, \forall x, t \in J,$

then there exists a unique function $u_0(x,t) \in C^1$, such that

$$\left| \frac{\partial u_0}{\partial x}(x,t) + K(x, u_0(x,t)) \right| < \epsilon \quad \text{and } |u(x,t) - u_0(x,t)| \leq \frac{M_1}{1 - \lambda M_1} \epsilon.$$

Proof. Consider the differential equation

(2.4) $u_x(x,t) + K(x, u(x,t)) = 0, \ \forall x, t \in J.$

We define a metric d and an operator P on C^1, respectively by

(2.5) $d(\zeta, \eta) = \sup_{x,t \in J} |\zeta(x,t) - \eta(x,t)|$

and

$$(P\zeta)(x,t) = u(x_0, t) - \int_{x_0}^{x} K(s, \zeta(s,t)) ds, \forall \zeta \in C^1.$$

Consider,

$$\begin{aligned} d(P\zeta, P\eta) &= \sup_{x,t \in J} |(P\zeta)(x,t) - (P\eta)(x,t)| \\ &= \sup_{x,t \in J} \left| - \int_{x_0}^{x} K(s, \zeta(s,t)) ds + \int_{x_0}^{x} K(s, \eta(s,t)) ds \right| \\ &= \sup_{x,t \in J} \left| \int_{x_0}^{x} K(s, \zeta(s,t)) ds - \int_{x_0}^{x} K(s, \eta(s,t)) ds \right| \\ &\leq \sup_{x,t \in J} \left| \int_{x_0}^{x} |K(s, \zeta(s,t)) - K(s, \eta(s,t))| ds \right| \\ &\leq \sup_{x,t \in J} \left| \int_{x_0}^{x} \lambda |\zeta(s,t)) - \eta(s,t)| ds \right| \quad \text{(by equation (2.1))} \\ &\leq \lambda \sup_{x,t \in J} \left| \int_{x_0}^{x} \sup_{s,t \in J} |\zeta(s,t)) - \eta(s,t)| ds \right| \\ &\leq \lambda d(\zeta, \eta) \times \sup_{x \in J} \left| \int_{x_0}^{x} ds \right| \\ &\leq \lambda d(\zeta, \eta) \times M_1 \quad \text{(by equation (2.2)).} \end{aligned}$$

Then by using Banach Contraction Principle, there exists a unique $u_0(x,t) \in C^1$ such that $Pu_0(x,t) = u_0(x,t)$. Thus $u_0(x,t)$ satisfy $u(x_0,t) - \int_{x_0}^{x} K(s, u_0(s,t)) ds = u_0(x,t)$ and

(2.6) $d(u_0, u) \leq \frac{1}{1 - \lambda M_1} d(u, Pu).$

Now by inequality (2.3) we get,

$$-\epsilon \leq \frac{\partial u}{\partial x}(x,t) + K(x, u(x,t)) \leq \epsilon, \forall x, t \in J.$$

Integrating from x_0 to x we get,

$$-\epsilon \int_{x_0}^{x} ds \leq \int_{x_0}^{x} \left\{ \frac{\partial u}{\partial s}(s,t) + K(s, u(s,t)) \right\} ds \leq \epsilon \int_{x_0}^{x} ds,$$

$$\Rightarrow -\epsilon \int_{x_0}^{x} ds \leq \{u(x,t) - u(x_0,t) + \int_{x_0}^{x} K(s, u(s,t))\} ds \leq \epsilon \int_{x_0}^{x} ds.$$

$$\Rightarrow -\epsilon \sup_{x\in J}\int_{x_0}^{x} ds \leq -\epsilon \int_{x_0}^{x} ds \leq \{u(x,t) - u(x_0,t) + \int_{x_0}^{x} K(s,u(s,t))\}ds \leq \epsilon \int_{x_0}^{x} ds \leq \sup_{x\in J}\int_{x_0}^{x} ds.$$

$$\Rightarrow -\epsilon \sup_{x\in J}|\int_{x_0}^{x} ds| \leq -\epsilon \sup_{x\in J}\int_{x_0}^{x} ds \leq -\epsilon \int_{x_0}^{x} ds \leq \{u(x,t) - u(x_0,t) + \int_{x_0}^{x} K(s,u(s,t))\}ds$$

$$\leq \epsilon \int_{x_0}^{x} ds \leq \sup_{x\in J}\int_{x_0}^{x} ds \leq \epsilon \sup_{x\in J}|\int_{x_0}^{x} ds|.$$

$$\Rightarrow -\epsilon M_1 \leq \{u(x,t) - u(x_0,t) + \int_{x}^{x_0} K(s,u(s,t))\}ds \leq \epsilon M_1.$$

$$\Rightarrow |u(x,t) - u(x_0,t) + \int_{x}^{x_0} K(s,u(s,t))ds| \leq \epsilon M_1.$$

$$\Rightarrow |u(x,t) - (Pu)(x,t)| \leq \epsilon M_1.$$

$$\Rightarrow \sup_{x,t\in J}|u(x,t) - (Pu)(x,t)| \leq \epsilon M_1.$$

$$\Rightarrow \quad d(u,Pu) \leq \epsilon M_1.$$

Using this inequality and equation (2.6) we get,

$$\begin{aligned}|u(x,t) - u_0(x,t)| &= |u_0(x,t) - u(x,t)| \\ &\leq \sup_{x,t\in J}|u_0(x,t) - u(x,t)| \\ &= d(u_0(x,t), u(x,t)) \\ &\leq \frac{1}{1-\lambda M_1} d(u,Pu) \\ &\leq \frac{M_1}{1-\lambda M_1}\epsilon.\end{aligned}$$

We now prove the *HU* stability of equation (1.2).

Theorem 2.2 *Let $x_0 \in J$ and $F, H : J \times \mathbb{R} \to \mathbb{R}$ be a continuous functions such that*

(2.7) $|F(x,v(x,t))v_x(x,t) - F(x,w(x,t))w_x(x,t)| \leq \lambda_1|v(x,t) - w(x,t)|,$

and

(2.8) $|H(x,v(x,t)) - H(x,w(x,t))| \leq \lambda_2|v(x,t) - w(x,t)|, \ \forall x,t \in J,$

where $\lambda_1, \lambda_2 > 0, \lambda_1, \lambda_2 \in \mathbb{R}$ and $v(x,t), w(x,t) \in C^2(J\times J)$.

Let

(2.9) $M_2 = \sup_{x,y\in J}\left|\int_{x_0}^{x}\int_{x_0}^{y} dsdy\right|,$

with $0 < \{\lambda_1 + \lambda_2\}M_2 < 1$. If $u(x,t) \in C^2(J\times J)$ satisfy

(2.10) $|u_{xx}(x,t) + F(x,u)u_x(x,t) + H(x,u)| \leq \epsilon, \ \forall x,t \in J,$

then there exists, a unique function, $u_0(x,t) \in C^2(J\times J)$,
such that $\frac{\partial^2 u_0}{\partial x^2}(x,t) + F(x,u_0(x,t))\frac{\partial u_0}{\partial x}(x,t) + H(x,u_0(x,t)) = 0$ and $|u(x,t) - u_0(x,t)| \leq \frac{M_2}{1-\{\lambda_1+\lambda_2\}M_2}\epsilon$.

Proof. Consider the differential equation

(2.11) $$\frac{\partial^2 u}{\partial x^2}(x,t) + F(x,u(x,t))\frac{\partial u}{\partial x}(x,t) + H(x,u(x,t)) = 0, \forall x,t \in J.$$

We define a metric d and an operator P on $C^2(J\times J)$, respectively by

(2.12) $d(\zeta,\eta) = \sup_{x,t\in J}|\zeta(x,t) - \eta(x,t)|,$

and

$$(P\zeta)(x,t) = u(x_0,t) - \int_{x_0}^{x}\int_{x_0}^{y} F(s,\zeta(s,t))\zeta_s(s,t)dsdy - \int_{x_0}^{x}\int_{x_0}^{y} H(s,\zeta(s,t))dsdy$$

$\forall \ \zeta \in C^2(J\times J)$.

Consider

$$d(P\zeta,P\eta) = \sup_{x,t\in J}|(P\zeta)(x,t) - (P\eta)(x,t)|$$

$$= \sup_{x,t\in J} \Big| - \int_{x_0}^{x}\int_{x_0}^{y} F(s,\zeta(s,t))\zeta_s(s,t)dsdy - \int_{x_0}^{x}\int_{x_0}^{y} H(s,\zeta(s,t))dsdy$$
$$+ \int_{x_0}^{x}\int_{x_0}^{y} F(s,\eta(s,t))\eta_s(s,t)dsdy + \int_{x_0}^{x}\int_{x_0}^{y} H(s,\eta(s,t))dsdy\Big|$$
$$= \sup_{x,t\in J} \Big| \int_{x_0}^{x}\int_{x_0}^{y} F(s,\zeta(s,t))\zeta_s(s,t)dsdy + \int_{x_0}^{x}\int_{x_0}^{y} H(s,\zeta(s,t))dsdy$$
$$- \int_{x_0}^{x}\int_{x_0}^{y} F(s,\eta(s,t))\eta_s(s,t)dsdy - \int_{x_0}^{x}\int_{x_0}^{y} H(s,\eta(s,t))dsdy\Big|$$
$$= \sup_{x,t\in J} \Big| \int_{x_0}^{x}\int_{x_0}^{y} F(s,\zeta(s,t))\zeta_s(s,t)dsdy - \int_{x_0}^{x}\int_{x_0}^{y} F(s,\eta(s,t))\eta_s(s,t)dsdy$$
$$+ \int_{x_0}^{x}\int_{x_0}^{y} H(s,\zeta(s,t))dsdy - \int_{x_0}^{x}\int_{x_0}^{y} H(s,\eta(s,t))dsdy\Big|$$
$$\leq \sup_{x,t\in J} \Big| \int_{x_0}^{x}\int_{x_0}^{y} F(s,\zeta(s,t))\zeta_s(s,t)dsdy - \int_{x_0}^{x}\int_{x_0}^{y} F(s,\eta(s,t))\eta_s(s,t)dsdy\Big|$$
$$+ \sup_{x,t\in J} \Big| \int_{x_0}^{x}\int_{x_0}^{y} H(s,\zeta(s,t))dsdy - \int_{x_0}^{x}\int_{x_0}^{y} H(s,\eta(s,t))dsdy\Big|$$
$$\leq \sup_{x,t\in J} \Big| \int_{x_0}^{x}\int_{x_0}^{y} |F(s,\zeta(s,t))\zeta_s(s,t) - F(s,\eta(s,t))\eta_s(s,t)|dsdy\Big|$$
$$+ \sup_{x,t\in J} \Big| \int_{x_0}^{x}\int_{x_0}^{y} |H(s,\zeta(s,t)) - H(s,\eta(s,t))|dsdy\Big|$$
$$\leq \sup_{x,t\in J} \Big| \int_{x_0}^{x}\int_{x_0}^{y} \lambda_1|\zeta(s,t)) - \eta(s,t)|dsdy\Big| + \sup_{x,t\in J} \Big| \int_{x_0}^{x}\int_{x_0}^{y} \lambda_2|\zeta(s,t)) - \eta(s,t)|dsdy\Big|$$

(by equation (2.7) and (2.8))

$$\leq \lambda_1 \sup_{x,t\in J} \Big| \int_{x_0}^{x}\int_{x_0}^{y} \sup_{s,t\in J}|\zeta(s,t)) - \eta(s,t)|dsdy\Big| +$$
$$\lambda_2 \sup_{x,t\in J} \Big| \int_{x_0}^{x}\int_{x_0}^{y} \sup_{s,t\in J}|\zeta(s,t)) - \eta(s,t)|dsdy\Big|$$
$$\leq \lambda_1 \sup_{x,t\in J} \Big| \int_{x_0}^{x}\int_{x_0}^{y} d(\zeta,\eta)dsdy\Big| + \lambda_2 \sup_{x,t\in J} \Big| \int_{x_0}^{x}\int_{x_0}^{y} d(\zeta,\eta)dsdy\Big|$$
$$\leq \{\lambda_1 + \lambda_2\}d(\zeta,\eta) \sup_{x,t\in J} \Big| \int_{x_0}^{x}\int_{x_0}^{y} dsdy\Big|$$
$$\leq \{\lambda_1 + \lambda_2\}d(\zeta,\eta)M_2 \qquad \text{(by equation (2.9))}$$
$$\leq \{\lambda_1 + \lambda_2\}M_2 \times d(\zeta,\eta).$$

Therefore by using ***Theorem* 1.1** , there exists, a unique, $u_0(x,t) \in C^2(J \times J)$ such that $Pu_0(x,t) = u_0(x,t)$. Thus $u_0(x,t)$ satisfies

(2.13) $u(x_0,t) - \int_{x_0}^{x}\int_{x_0}^{y} F(s,u_0(s,t))u_s(s,t))dsdy - \int_{x_0}^{x}\int_{x_0}^{y} H(s,u_0(s,t))dsdy = u_0(x,t)$

and

$$d(u_0,u) \leq \frac{1}{1-(\lambda_1+\lambda_2)M_2} d(u,Pu).$$

Now by inequality (2.10) we get,

$$-\epsilon \leq \frac{\partial^2 u}{\partial x^2}(x,t) + F(x,u)\frac{\partial u}{\partial x}(x,t) + H(x,u) \leq \epsilon, \quad \forall x,t \in J.$$

Integrating from x_0 to x we derive,

$$-\epsilon \int_{x_0}^{x} ds \leq \tfrac{\partial u}{\partial x}(x,t) - \tfrac{\partial u}{\partial x}(x_0,t) + \int_{x_0}^{x} F(s,u(s,t))u_s(s,t)ds + \int_{x_0}^{x} H(s,u(s,t))ds \leq \epsilon \int_{x_0}^{x} ds.$$

Again integrating from x_0 to x we obtain,

$$-\epsilon \int_{x_0}^{x}\int_{x_0}^{y} dsdy \leq u(x,t) - u(x_0,t) - [u(x_0,t) - u(x_0,t)] + \int_{x_0}^{x}\int_{x_0}^{y} F(s,u(s,t))u_s(s,t)dsdy$$
$$+ \int_{x_0}^{x}\int_{x_0}^{y} H(s,u(s,t))dsdy \leq \epsilon \int_{x_0}^{x}\int_{x_0}^{y} dsdy.$$

By using the equation (2.12) we get,

$$-\epsilon \int_{x_0}^{x}\int_{x_0}^{y} dsdy \leq u(x,t) - (Pu)(x,t) \leq \epsilon \int_{x_0}^{x}\int_{x_0}^{y} dsdy.$$
$$\Rightarrow -\epsilon \sup_{x,y\in J} \int_{x_0}^{x}\int_{x_0}^{y} dsdy \leq -\epsilon \int_{x_0}^{x}\int_{x_0}^{y} dsdy \leq u(x,t) - (Pu)(x,t) \leq \epsilon \int_{x_0}^{x}\int_{x_0}^{y} dsdy \leq \epsilon \sup_{x,y\in J} \int_{x_0}^{x}\int_{x_0}^{y} dsdy.$$
$$\Rightarrow -\epsilon \sup_{x,y\in J} \Big| \int_{x_0}^{x}\int_{x_0}^{y} dsdy\Big| \leq -\epsilon \sup_{x,y\in J} \int_{x_0}^{x}\int_{x_0}^{y} dsdy \leq -\epsilon \int_{x_0}^{x}\int_{x_0}^{y} dsdy \leq u(x,t) - (Pu)(x,t)$$

$$\leq \epsilon \int_{x_0}^{x}\int_{x_0}^{y} dsdy \leq \epsilon \sup_{x,y\in J} \int_{x_0}^{x}\int_{x_0}^{y} dsdy \leq \epsilon \sup_{x,y\in J}\left| \int_{x_0}^{x}\int_{x_0}^{y} dsdy\right|.$$

$$\Rightarrow -\epsilon \sup_{x,y\in J}\left| \int_{x_0}^{x}\int_{x_0}^{y} dsdy\right| \leq u(x,t)-(Pu)(x,t) \leq \epsilon \sup_{x,y\in J}\left| \int_{x_0}^{x}\int_{x_0}^{y} dsdy\right|.$$

$$\Rightarrow -\epsilon M_2 \leq u(x,t)-(Pu)(x,t) \leq \epsilon M_2.$$

$$\Rightarrow |u(x,t)-(Pu)(x,t)| \leq \epsilon M_2.$$

$$\Rightarrow \sup_{x,t\in J}|u(x,t)-(Pu)(x,t)| \leq \epsilon M_2.$$

$$\Rightarrow d(u,Pu) \leq \epsilon M_2,$$

which with equation (2.13) yields

$$\begin{aligned} |u(x,t)-u_0(x,t)| &= |u_0(x,t)-u(x,t)| \\ &\leq \sup_{x,t\in J}|u_0(x,t)-u(x,t)| \\ &= d(u_0(x,t),u(x,t)), \\ &\leq \frac{1}{1-\{\lambda_1+\lambda_2\}M_2} d(u,Pu), \\ &\leq \frac{M_2}{1-\{\lambda_1+\lambda_2\}M_2}\epsilon. \end{aligned}$$

Hence the result.

3 Conclusion

In this paper, we have proved the Hyers - Ulam stability of first and second order partial differential equations (1.1) and (1.2) respectively by employing Banach's Contraction Principle.

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