

## CERTAIN QUADRUPLE SERIES EQUATIONS INVOLVING LAGUERRE POLYNOMIALS

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### Abstract

Srivastava ([13], [15]) has solved dual series equations involving Bateman- $k$  functions and Jacobi polynomials. Srivastava [16] has obtained more results for the Konhauser-biorhogonal set. Lowndes ([3], [4]), Srivastava [12], Lowndes and Srivastava [5], Srivastava[14], Srivastava and Panda [17] have obtained the solution of dual series equations involving Jacobi and Laguerre polynomials and also solved triple series equations involving Laguerre polynomials. Singh, Rokne and Dhaliwal [10] have find out the solution of triple series equations involving Laguerre polynomials in a closed form. Kuldeep Narain ([7], [8]), Rajnesh Krishnan Mudaliar and Kuldeep Narain [6] have solved Certain dual and quadruple series equations involving generalized Laguerre polynomials and Jacobi polynomials as kernels. In the present paper, an exact solution has been obtained for the quadruple series equations involving Laguerre polynomials by Noble [9] modified multiplying factor technique.

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## 1 Introduction

Earlier Srivastava ([13], [15]) has solved dual series equations involving Bateman- $k$  functions and Jacobi polynomials. Srivastava [16] has obtained more results like generating functions, bilinear generating functions, recurrence relations, some expansions of functions for the Konhauser-biorhogonal set and general result for the dual series equations involving generalized Laguerre polynomials by putting  $k = 1$  in (3.10) and (3.11) in [15,p.645]. Lowndes ([3], [4]), Srivastava [12], Lowndes and Srivastava [5], Srivastava[14], Srivastava and Panda [17] have obtained the solution of dual series equations involving Jacobi and Laguerre polynomials and also solved triple series equations involving Laguerre polynomials. Singh, Rokne and Dhaliwal [10] have find out the solution of triple series equations involving Laguerre polynomials in a closed form. Kuldeep Narain ([7], [8]), Rajnesh Krishnan Mudaliar and Kuldeep Narain [6] have solved Certain dual and quadruple series equations involving generalized Laguerre polynomials and Jacobi polynomials as kernels. In this paper, we have obtained the solution of the following quadruple series equations:

$$(1.1) \sum_{n=0}^{\infty} \frac{A_n}{\Gamma(\alpha+n+p+1)} L_{n+p}^{\alpha}(x) = \phi_1(x), 0 \leq x < a,$$

$$(1.2) \sum_{n=0}^{\infty} \frac{A_n}{\Gamma(\alpha+\beta+n+p)} L_{n+p}^{\alpha}(x) = \phi_2(x), a < x < b,$$

$$(1.3) \sum_{n=0}^{\infty} \frac{A_n}{\Gamma(\alpha+n+p+1)} L_{n+p}^{\alpha}(x) = \phi_3(x), b < x < c,$$

$$(1.4) \sum_{n=0}^{\infty} \frac{A_n}{\Gamma(\alpha+\beta+n+p)} L_{n+p}^{\alpha}(x) = \phi_4(x), c < x < \infty,$$

where  $0 < \beta + m$ ,  $0 < \alpha + \beta < \alpha + 1$ ,  $p$  and  $m$  are non-negative integer.

$$(1.5) L_{n+p}^{\alpha}(x) = \binom{\alpha+n+p}{n+p}, F_1[-n-p; \alpha+1; x]$$

is the Laguerre Polynomial,  $\phi_1(x)$ ,  $\phi_2(x)$ ,  $\phi_3$  and  $\phi_4(x)$  are prescribed functions.

The solution presented in this paper is obtained by employing a multiplying factor technique similar to that used by Noble [9] or Lowndes ([3],[4]).

## 2 Preliminaries

The results, which will be required in the course of analysis, are given below for ready reference. From Erdélyi [2], it can be deduced that

$$(2.1) \int_0^y x^\alpha (y-x)^{\beta+m-1} L_{n+p}^\alpha(x) dx = \frac{\Gamma(\beta+m)\Gamma(\alpha+n+p+1)}{\Gamma(\alpha+\beta+m+n+p+2)} y^{\alpha+\beta+m} L_{n+p}^{\alpha+\beta+m}(y)$$

where  $0 < y < d$ ,  $-1 < k$ ,  $0 < \beta + m$ , and

$$(2.2) \int_y^\infty e^{-x} (x-y)^{-\beta} L_{n+p}^\alpha(x) dx = \Gamma(1-\beta) e^{-y} L_{n+p}^{\alpha+\beta-1}(y)$$

where  $d < y < \infty$ ,  $\alpha + 1 > \alpha + \beta > 0$ .

From Erdélyi [2], we derive the following orthogonality relation for the Laguerre polynomial:

$$(2.3) \int_y^\infty e^{-x} x^\alpha L_m^\alpha(x) L_n^\alpha(x) dx = \frac{\Gamma(\alpha+n+1)}{n!} \delta_{m,n}$$

where  $\alpha > -1$  and  $\delta_{m,n}$  is the Kronecker delta.

The differentiation formula:

$$(2.4) \frac{d^{m+1}}{dx^{m+1}} \{x^{\alpha+m+1} L_n^{\alpha+m+1}(x)\} = \frac{\Gamma(\alpha+m+n+2)}{\Gamma(\alpha+n+1)} x^\alpha L_n^\alpha(x)$$

follows from Erdélyi [1].

The analysis here is formal and no attempt has been made to justify the various limiting process.

Making an appeal to the results due to Lowndes ([3], p.123, p.126 eqns (5), (20)), he easily derived ([4], p.168, eqn. (10))

$$(2.5) S(r, x) = (r, x)^\alpha \sum_{n=0}^\infty \frac{\Gamma(n+1)\Gamma(\alpha+\beta+n)}{\{\Gamma(\alpha+1+n)\}^2} L_n(\alpha, x) L_n(\alpha, r),$$

$$(2.6) = \frac{1}{\{\Gamma(1-\beta)\}^2} \int_0^1 n(y) (r-y)^{-\beta} (x-y)^{-\beta} dy,$$

$$(2.7) = -\frac{1}{\{\Gamma(1-\beta)\}^2} S_t(r, x),$$

where  $\beta < 1$ ,  $\alpha + \beta > 0$ ,  $n(y) = e^y y^{\alpha+\beta-1}$  and  $t = \min(r, x)$  of  $f(x)$  and  $f'(x)$  are continuous in  $a \leq x \leq b$  and if  $0 < \sigma < 1$ , then the solutions of the Abel integral equations

$$(2.8) f(x) = \int_0^x \frac{F(y)}{(x-y)^\sigma} dy,$$

$$(2.9) f(x) = \int_x^b \frac{F(y)}{(y-x)^\sigma} dy,$$

are given by

$$(2.10) F(y) = \frac{\sin \sigma \pi}{\pi} \frac{d}{dy} \int_a^y \frac{f(x)}{(x-y)^{1-\sigma}} dx$$

and

$$(2.11) F(y) = \frac{\sin \sigma \pi}{\pi} \frac{d}{dy} \int_y^b \frac{f(x)}{(x-y)^{1-\sigma}} dx$$

respectively.

## 3 Solution of the problem

Multiply equation (1.1) by  $x^\alpha (y-x)^{\beta+m-1}$ , integrate with respect to  $x$  over  $(0, y)$  and then use (2.1) to obtain

$$(3.1) \sum_{n=0}^\infty \frac{A_n}{\Gamma(\alpha+\beta+m+n+p+2)} y^{\alpha+\beta+m} L_{n+p}^{\alpha+\beta+m} = \frac{1}{\Gamma(\beta+m)} \int_0^y x^\alpha (y-x)^{\beta+m-1} \phi_1(x) dx,$$

where  $0 < y < a$ ,  $-1 < \alpha$ ,  $0 < \beta + m$  and  $m$  is a non-negative integer.

Differentiate (3.1)  $(m+1)$  times with respect to  $y$  and use (2.4) to find

$$(3.2) \sum_{n=0}^\infty \frac{A_n}{\Gamma(\alpha+\beta+n+p)} L_{n+p}^{\alpha+\beta-1}(y) = \frac{y^{1-n-\beta}}{\Gamma(\beta+m)} \frac{d^{m+1}}{dy^{m+1}} \int_0^y x^\alpha (y-x)^{\beta+m-1} \phi_1(x) dx,$$

where  $0 < y < a$ ,  $-1 < \alpha$ ,  $0 < \beta + m$  and  $m$  is a non-negative integer. Again multiply (1.2) by  $e^{-x} (x-y)^{-\beta}$ , integrate with respect to  $x$  over  $(y, \infty)$ , then use (2.2) to find

$$(3.3) \sum_{n=0}^\infty \frac{A_n}{\Gamma(\alpha+\beta+n+p)} L_{n+p}^{\alpha+\beta-1}(y) = \frac{e^{-y}}{\Gamma(1-\beta)} \int_y^b (x-y)^{-\beta} e^{-x} \phi_2(x) dx,$$

where  $a < y < b$ ,  $\beta < 1$  and  $0 < \alpha + \beta$ .

Now, multiply equation (1.3) by  $x^\alpha (y-x)^{\beta+m-1}$ , integrate with respect to  $x$  over  $(a, y)$ , then use (2.1) to get

$$(3.4) \sum_{n=0}^\infty \frac{A_n}{\Gamma(\alpha+\beta+m+n+p+2)} y^{\alpha+\beta+m} L_{n+p}^{\alpha+\beta+m}(y) = \frac{1}{\Gamma(\beta+m)} \int_b^y x^\alpha (y-x)^{\beta+m-1} \phi_3(x) dx,$$

where  $b < y < c$ ,  $-1 < \alpha$ ,  $0 < \beta + m$  and  $m$  is a non-negative integer.

Differentiate (3.4),  $(m + 1)$  times with respect to  $y$  and use (2.4) to find

$$(3.5) \sum_{n=0}^{\infty} \frac{A_n}{\Gamma(\alpha+\beta+n+p)} L_{n+p}^{\alpha+\beta-1}(y) = \frac{y^{1-\alpha-\beta}}{\Gamma(\beta+\alpha+m)} \frac{d^{m+1}}{dy^{m+1}} \int_b^y x^\alpha (y-x)^{\beta+m-1} \phi_3(x) dx,$$

where  $b < y < c$ ,  $-1 < \alpha$ ,  $0 < \beta + m$  and  $m$  is a non-negative integer.

Now multiply (1.4) by  $e^{-x}(x-y)^{-\beta}$ , integrate with respect to  $x$  over  $(y, \infty)$ , then use (2.2) to get

$$(3.6) \sum_{n=0}^{\infty} \frac{A_n}{\Gamma(\alpha+\beta+n+p)} L_{n+p}^{\alpha+\beta-1}(y) = \frac{e^{-y}}{\Gamma(1-\beta)} \int_y^{\infty} (x-y)^{-\beta} e^{-x} \phi_4(x) dx,$$

where  $c < y < \infty$ ,  $\beta < 1$  and  $0 < \alpha + \beta$ .

The left hand sides of equations (3.2), (3.3), (3.5), (3.6) are now identical and hence on using orthogonality relation (2.3), we obtain the solution of equations (1.1), (1.2), (1.3) and (1.4) in the form:

$$(3.7) A_n = \frac{(n+p)!}{\Gamma(\beta+m)} \left[ \int_0^a e^{-y} L_{n+p}^{\alpha+\beta-1}(y) F_1(y) dy + \int_b^c e^{-y} L_{n+p}^{\alpha+\beta-1}(y) F_3(y) dy \right] \\ + \frac{(n+p)!}{\Gamma(1-\beta)} \left[ \int_a^b y^{\alpha+\beta-1} L_{n+p}^{\alpha+\beta-1}(y) F_2(y) dy + \int_c^{\infty} e^{-y} L_{n+p}^{\alpha+\beta-1}(y) F_4(y) dy \right],$$

where

$$(3.8) F_1(y) = \frac{d^{m+1}}{dy^{m+1}} \int_0^y x^\alpha (y-x)^{\beta+m-1} \phi_1(x) dx,$$

$$(3.9) F_2(y) = \int_y^b (x-y)^{-\beta} e^{-x} \phi_2(x) dx,$$

$$(3.10) F_3(y) = \frac{d^{m+1}}{dy^{m+1}} \int_b^y x^\alpha (y-x)^{\beta+m-1} \phi_3(x) dx,$$

$$(3.11) F_4(y) = \int_y^{\infty} (x-y)^{-\beta} e^{-x} \phi_4(x) dx.$$

The solution of Lowndes equations

$$(3.12) \sum_{n=0}^{\infty} C_n \Gamma(\alpha + \beta + n) L_n^\alpha(x) = \phi_1(x), \quad 0 \leq x < a,$$

$$(3.13) \sum_{n=0}^{\infty} C_n \Gamma(\alpha + 1 + n) L_n^\alpha(x) = \phi_2(x), \quad a < x < b,$$

$$(3.14) \sum_{n=0}^{\infty} C_n \Gamma(\alpha + \beta + n) L_n^\alpha(x) = \phi_3(x), \quad b < x < c,$$

$$(3.15) \sum_{n=0}^{\infty} C_n \Gamma(\alpha + 1 + n) L_n^\alpha(x) = \phi_4(x), \quad c < x < \infty,$$

can be obtained by putting  $A_n = C_n \Gamma(\alpha + n + 1) \Gamma(\alpha + \beta + n)$  and  $p = 0$  in the solution (3.7).

## 4 Conclusion

The Laguerre polynomials have been applied by many authors like Lowndes ([3],[4]), Srivastava [12], Srivastava and Panda [17], Lowndes and Srivastava [5], Singh, Rokne and Dhaliwal [10], Kuldeep Narain ([7], [8]), Mudaliar and Kuldeep Narain [6] to solve dual, triple and quadruple series equations. The solution presented in this paper is obtained by employing a multiplying factor technique similar to that used by Noble [9] or Lowndes ([3], [4]). Thus we have obtained an exact solution for the quadruple series equations involving Laguerre polynomials by modified multiplying factor technique Noble [9].

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