***Jñānābha*, Vol. 50(2) (2020), 223-228**
(Dedicated to Honor Dr. R. C. Singh Chandel on His 75th Birth Anniversary Celebrations)

MATHEMATICAL MODELLING OF FOOD MANAGEMENT FOR WILD LIFE POPULATION WITH MILD ENVIRONMENTAL EFFECT

By

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(*Received : September* 11, 2020 *; Revised: October* 17, 2020*; Final Form : December* 20, 2020)

Abstract

In this paper we constructed a Mathematical Model of food management for animal species interacting with natural environment. The mathematical framework has been generated with generations as the time scale. Efforts have been made to mathematically formalize the environmental changes dependent food availability and consequent changes. The aim to develop a mathematical perspective to understand the environmental impact on wild life population and its food management. The carrying capacity of environment in terms of primary food has been assumed to be limited. The environmental changes lead to changes in availability of vegetation for animal species. A Mathematical model is constructed in terms of a system of nonlinear difference equations. The solutions have been worked out for some special cases. These solutions can be expressed as finite polynomials. Graphical patterns have been worked out as examples

1 Introduction

The environment is the source of the primary food production in the nature and therefore it has strong bearing upon the interplay of animal population along the food chain [6]. The environmental changes have led to drastic changes in food production and natural habitats of wildlife species which have given rise to the changes in the interactional patterns among the various trophic levels of the food chain ([4], [5]). Despite the inherent complexity of interactions between animal species and environment. Mathematical Models of single species interacting with natural environment has been a subject matter of interest for the mathematicians. The single species models have been primarily inspired by their simplicity of mathematical treatment and scope for further development [11]. We consider the growth of animal population is dependent on limited carrying capacity of the environment like water, vegetation and temperature. A mathematical model is constructed in terms of a system of nonlinear difference equations incorporating all the significant parameters. The mathematical framework has been generated with generations as the time scale ([7], [3]). It has been observed that the interactions between the species in various generation does not remain confined to those specific generations but decides the population patterns in all the successive generation in the form of a series ([12],[14], [17]).

Sustainable Development Goals (SDG) of United Nations has raised several challenges for development planners particularly in developing countries where the policy planners are not adequately capacitated to plan the strategies and monitor the outcomes. As SDGs are time bound their effective implementation will requires predictive methods ([15], [16], [18], [19]). The Mathematical Tools can prove handy to adequately plan the developmental interventions and project the outcomes ([8], [9], [10]). There has been a growing realization that Mathematical Modelling can be a viable tool for Sustainable Development Planning [11].

Earlier researchers ([1], [2], [10]) have developed and employed single species finite animal population models to understand patterns have growth of populations.

2 Mathematical Framework for Food Availability

Following factors will attend the growth of animal population surviving on the limited carrying capacity of the environment

1. The food availability per animal of single species for a specific generation,
2. the competition with other community member to access the food is avoided.
3. the migration of the species from the ecosystem,

4. the rate of supply of food availability and

5. the food competition will enhance the growth of species whereas it will tend to refer the growth of vegetative.

We assume that F_n denotes the food availability per animal for the n^{th} generation of the species. It has also been presumed that the food requirements are the same across the age groups which means that infants adults and elder species have same requirements and food accessing behaviour from the environment. The food availability for different generation will be different primarily because of the climate change, consumption of food by earlier generation and requirement of the food by environment. Hence change in food availability at n^{th} generation can be written as below:

$$\Delta F_n = \alpha_n F_{n-1} - \beta_n F_{n-1} - \Delta E_n,$$

where

α_n = Rate of growth of food availability i.e. because the food availability gets continuously regeneration in the environment. The growth has been calculated over unit area.

β_n = rate of consumption for the food per unite area.

ΔE_n = change in environmental conditions per unite area.

$$\Delta F_n = F_n - F_{n-1}.$$

Then

$$F_n = (\alpha_n F_{n-1} - \beta_n F_{n-1} - \Delta E_n) + F_{n-1},$$
$$F_n = (1 + \alpha_n - \beta_n)F_{n-1} - \Delta E_n.$$

Let

$$1 + \alpha_n - \beta_n = g_n \qquad \text{(effective growth rate).}$$

Then

$$F_n = g_n F_{n-1} - \Delta E_n. \tag{2.1}$$

The solution of the equation (2.1) can be given as

$$F_n = g_n g_{n-1} g_{n-2} \dots g_2 g_1 F_0 - g_n g_{n-1} g_{n-2} \dots g_4 g_3 g_2 \Delta E_1 - g_n g_{n-1} g_{n-2} \cdots g_5 g_4 g_3 \Delta E_2 - \dots$$
$$- g_n g_{n-1} g_{n-2} \Delta E_{n-3} - g_n g_{n-1} \Delta E_{n-2} - g_n \Delta E_{n-1} - \Delta E_n$$

or

$$F_n = \prod_{i=1}^{n} g_i F_0 - (\sum_{j=1}^{n-1} \prod_{i=j+1}^{n} g_i \Delta E_j) - \Delta E_n. \tag{2.2}$$

Assuming that the effective growth rate is uniform i.e.

If

$$g_1 = g_2 = g_3 = g_4 = \dots g_n = g.$$

Then

$$F_n = g^n F_0 - (\Delta E_n + g\Delta E_{n-1} + g^2 \Delta E_{n-2} + g^3 \Delta E_{n-3} + \dots + g^{n-1} \Delta E_1)$$
$$F_n = g^n F_0 - \sum_{r=0}^{n-1} (\Delta E)_{n-r} g^r$$
$$F_n = g^n F_0 - \sum_{r=0}^{n-1} M_{n-r} g^r, \tag{2.3}$$

where

$$M_{n-r} = (\Delta E)_{n-r}, \qquad r \neq n,$$
$$M_{n-r} = E_0, \qquad r = n.$$

The right hand side is a polynomial in g, which gives a very important conclusion that the production and consumption has cumulative impact across the generation. Any displacement in the pattern of production and consumption get magnified many times across the generation. Saxena, V. P. ([13]) has introduced classical polynomials for the study of finite animal population. We extend this concept in our problem and give some specific polynomials applications as given below

2.1 Hermite Polynomials

$H_n(x) = \sum_{m=0}^{[\frac{n}{2}]} \frac{(-1)^m n!(2x)^{n-2m}}{m!(n-2m)!}$.

Assuming that the environmental effect will follow the pattern of this polynomial.

Accordingly from the equation (2.3) we get,

(2.4) $\sum_{r=0}^{[n-1]} M_{n-r} g^r = \sum_{m=0}^{[\frac{n}{2}]} \frac{(-1)^m n!(2x)^{n-2m}}{m!(n-2m)!}$,

where

$$n = \frac{n}{2} + 1.$$

After comparing the coefficients of equation (2.4) we derive,

(2.5) $M_{n-r} = \dfrac{(-1)^m n!(2x)^{n-2m}}{m!(n-2m)!}$.

2.2 Laguerre Polynomials

$L_n(x) = \sum_{r=0}^{[n]} \frac{(-1)^r n!(x)^r}{(n-r)!(r!)^2}$.

Comparing with equation (2.3) we get,

(2.6) $\sum_{r=0}^{[n-1]} M_{n-r} g^r = \sum_{r=0}^{[n]} \frac{(-1)^r n!}{(n-r)!(r!)^2}$,

where $n = n + 1$.

After comparing the coefficients of the equation (2.6), we get

(2.7) $M_{n-r} = \dfrac{(-1)^r n!}{(n-r)!(r!)^2}$.

2.3 Jacobi Polynomials

$P_n^{(\alpha,\delta)}(1-\alpha) = \sum_{k=0}^{n} \frac{(-n)_k(1+\gamma+\delta+n)_k(1+\gamma)_n \alpha^k}{n!(1+\gamma)_k 2^k}$.

From equation (2.3) we get,

(2.8) $\sum_{r=0}^{[n-1]} M_{n-r} g^r = \sum_{k=0}^{n} \frac{(-n)_k(1+\gamma+\delta+n)_k(1+\gamma)_n \alpha^k}{n!(1+\gamma)_k 2^k}$,

where $n = n + 1$.

After comparing the coefficients of equation (2.8) we get,

(2.9) $M_{n-r} = \dfrac{(-n)_k(1+\gamma+\delta+n)_k(1+\gamma)_n}{n!(1+\gamma)_k 2^k}$.

2.4 Gauss Hypergeometric Function

(2.10) $2_{F_1}[-n, b; c; y] = \sum_{r=0}^{n} \frac{(-n)_r(b)_r y^r}{(c)_r r!}$.

Comparing with the equation (2.3) we get,

(2.11) $\sum_{r=0}^{[n-1]} M_{n-r} g^r = \sum_{r=0}^{n} \frac{(-n)_r(b)_r y^r}{(c)_r r!}$,

where $n = n + 1$.

After comparing the coefficients of equation (2.11) we get,

(2.12) $M_{n-r} = \dfrac{(-n)_r(b)_r}{(c)_r r!}$.

The evolution of species will be given as following equation

$$\Delta P_n = B_n F_n P_{n-1} - D_n P_{n-1} - M_n,$$

where

B_n = Birth rate of species which should be further proportional to food availability,

D_n = Natural Death rate of the species,

M_n = Migration of the species.

Therefore,

$$\Delta P_n = B_n(g_n F_{n-1} - \Delta E_n)P_{n-1} - D_n P_{n-1} - M_n$$

$$\Delta P_n = B_n g_n F_{n-1} P_{n-1} - B_n g_n P_{n-1}(\Delta E_n) - D_n P_{n-1} - M_n$$

$$\Delta P_n = P_n - P_{n-1}.$$

Hence

$$P_n = \Big[B_n g_n F_{n-1} - B_n g_n(\Delta E_n) - D_n\Big]P_{n-1} - M_n + P_{n-1},$$

$$P_n = \Big[B_n g_n F_{n-1} - B_n g_n(\Delta E_n) - D_n + 1\Big]P_{n-1} - M_n.$$

Let $B_n g_n F_{n-1} - B_n g_n(\Delta E_n) - D_n + 1 = Z_n$.
Then

(2.13) $P_n = Z_n P_{n-1} - M_n$

The Solution of the equation (2.13) can be given as

$$P_n = z_n z_{n-1} z_{n-2} \ldots z_3 z_2 z_1 P_0 - z_n z_{n-1} z_{n-2} \ldots z_4 z_3 z_2 M_1 - z_n z_{n-1} z_{n-2} \ldots$$
$$z_5 z_4 z_3 M_2 - \ldots - z_n z_{n-1} z_{n-2} M_{n-3} - z_n z_{n-1} M_{n-2} - z_n M_{n-1} - M_n,$$

or

(2.14) $P_n = \prod_{i=1}^{n} z_i P_0 - \Big(\sum_{j=1}^{n-1} \prod_{i=j+1}^{n} z_i M_j\Big) - M_n.$

Let us further assume that Z_n is uniform across the generation i.e.
If $z_1 = z_2 = z_3 = z_4 = \ldots z_n = z$
Then

$$P_n = z^n P_0 - \Big(M_n + zM_{n-1} + z^2 M_{n-2} + z^3 M_{n-3} + \ldots + z^{n-1} M_1\Big),$$

$$P_n = z^n P_0 - \sum_{r=0}^{n-1} M_{n-r} z^r.$$

Hence

(2.15) $P_n = z^n P_0 - \sum_{r=0}^{n-1} M_{n-r} z^r.$

The right hand side is a polynomial n in z.

3 Numerical Examples

If

$$\alpha_1 = \alpha_2 = \alpha = 0.3$$
$$\beta_1 = \beta_2 = \beta = 0.001$$
$$M_1 = M_2 = M = 10$$

The calculation of population have been shown in ***Figures* 3.1, 3.2** and **3.3** excluding the hunting or natural death of the species. This factor will significantly bring down the values of population figures as depicted in the figures. The inclusion of this factor will help us to fine-tune the population figures to acceptable values.

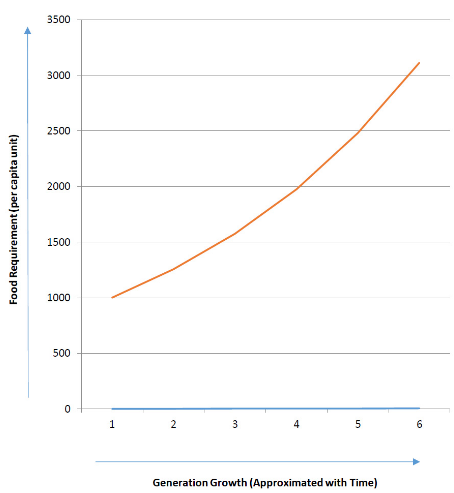

***Figure* 3.1:** Graph between food and generation for $\alpha = 0.3, \beta = 0.001$ and $M = 10$.

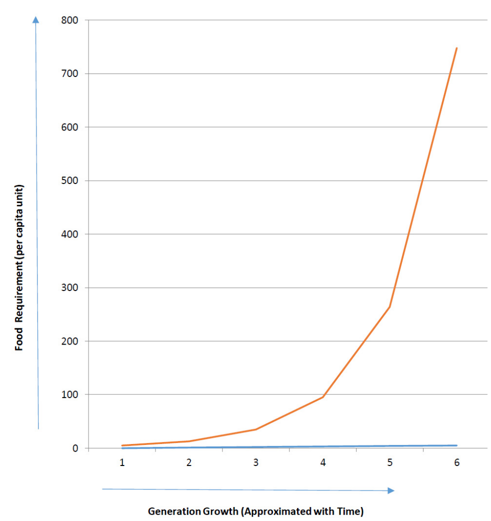

***Figure* 3.2:** Graph between food and generation for $\alpha = 0.2, \beta = 0.002$ and $M = 10$.

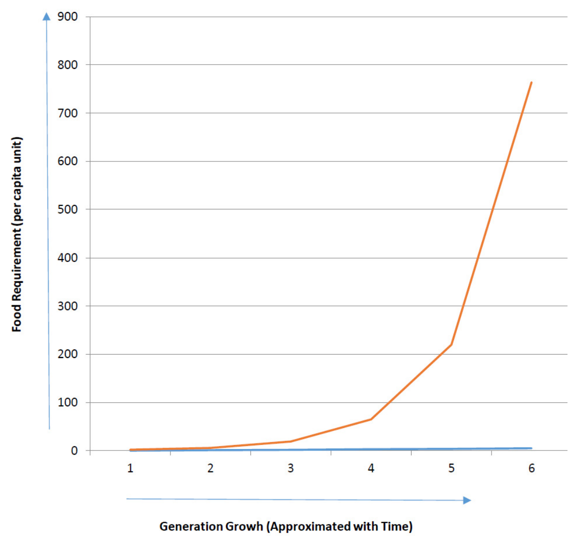

***Figure* 3.3:** Graph between food and generation for $\alpha = 0.1, \beta = 0.003$ and $M = 10$

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4 Conclusion and Way Forward

The study of animal species living in natural environment is quite useful to develop more realistic mathematical models in future. In this context this study is quite useful. The production of primary food, its consumption by animal species and likely impact of climatic change has been included in a realistic way in this model. There is however ample scope for further development of this mathematical model. Pursuits for Sustainable Development Goals require a better understanding of ecological dynamics and enhanced predictive capacity of ecological evolution. It will help us to take timely measures to mitigate the undesirable impact in ecological dynamics. This paper tries to develop a perspective to comprehend the impact of environment on evolution of wildlife species. Prey Predator Model has been revisited and the changing environmental factors have been included in the mathematical formalism.

Some of the important points which make the study useful for further studies are given below

4.1 The impact of adverse environmental impact on wild life evolution is a new trend. Our study is quite general and broad based which includes the impact of environmental impacts on population growth. Future studies can use different trends of environmental change and calculate the figures. The results will help in the food management of animal species under observation.

4.2 The study is unique in the sense that the time scale has been chosen in the terms of generation of species.

4.3 The traditional studies on ecological modelling have not used the application of polynomials. The properties of polynomials can be utilized to calculate the figures easily. It gives rise to new perspectives in population modelling. The future studies can enlarge these Mathematical treatments to make the model more realistic.

4.4 The study highlights the fact that effective growth rate of primary food in the environment will depend upon comparative rate of consumption, regeneration and environmental degradation. The population will undergo sharp decline if the rates of consumption are persistently higher than its regeneration. The rate of decline will increase over the generation.

4.5 The population of the species which survives on the primary food will ultimately depend upon a mix of the parameters consumption, regeneration, environmental degradation, natural birth/death rate of species and its consumption habits. Over the generations these parameters will have a complex dependence on these parameters which will be governed by a polynomial.

The future studies can employ different trends of environmental degradation to work out the evolution of the primary food production. These figures will help to calculate the trends of animal population. The studies can also explore the impact of more number of species in the evolution of primary food and animal population. Such efforts will help to proceed step by step towards more realistic models in ecology.

Acknowledgement. Authors are very much thankful to the Editors and Reviewers for their valuable comments to bring the paper in its present form.

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