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(Dedicated to Honor Dr. R. C. Singh Chandel on His 75^{th} Birth Anniversary Celebrations)

A NEW APPROACH TO GENERATE FORMULAE FOR PYTHAGOREANS TRIPLES, QUADRUPLES AND THEIR GENERALISATION TO *N*-TUPLES

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Abstract

In this paper, innovative methods have been devised to generate formulae for Pythagorean's Triples, Quadruples and these are finally generalised to generate Pythagorean's *n*-tuples. First method utilises formula for solution of a quadratic equation and generate two sets of Pythagorean's Triples. Second method determines universal identities that satisfy Pythagorean's Triples, Quadruples so on up to *n*-tuples. These methods are unprecedented, easy to derive at and hence are comprehensible to students and scholars alike.
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1 Introduction

Integers X, Y and Z are said to be Pythagorean's Triple if these satisfy the relation

(1.1) $X^2 + Y^2 = Z^2$.

When $X = Y$, in that event, Equation (1.1) yields Z equal to $\sqrt{2}.X$ which is an irrational quantity and can not be Pythagorean's triple. It is amply explained in basic books of number theory [3], [5]. On the other hand, when X and Y are unequal say $X = x, Y = x + a, Z = x + b$, a and b are real and rational quantities then algebraic Equation (1.1) takes the form

$$(x)^2 + (x + a)^2 = (x + b)^2.$$

Such algebraic equations have also been used [6] while deriving identities for Pythagorean's quadruples. On expansion

(1.2) $x^2 - 2x(b - a) - (b^2 - a^2) = 0$.

Equation (1.2) being a quadratic has two roots given by

(1.3) $x = (b - a) \pm \sqrt{(b - a)^2 + (b^2 - a^2)}$.

2 Theory and Concept

***Lemma* 2.1** *Integers $x, (x+a)$ and $(x+b)$ will be unnormalised Pythagorean's Triples if $x = (b-a) \pm \sqrt{(b-a) + (b^2 - a^2)}$ and a and b are rational quantities such that $\pm\sqrt{(b - a) + (b^2 - a^2)}$ is real, perfect square and hence rational. If $x, (x + a)$ and $(x + b)$ so determined are fractions (unnormalised) , then multiplying them with their lowest common multiplier LCM will give normalised Pythagorean's Triples in integer form.*

***Lemma* 2.2** *If real and rational quantities $x, (x + a)$ and $(x + b)$ have $x = (b - a) \pm \sqrt{(b - a) + (b^2 - a^2)} = p/q, a = p_1/q_1$, $b = p_2/q_2$ and p, q, p_1, q_1, p_2, q_2 are integers then $(p.q_1.q_2), (p.q_1.q_2 + p_1.q.q_2)$ and $(p.q_1.q_2 + p_2.q.q_1)$ will be Pythagorean's Triple.*

2.1 Values of x, a and b Satisfying Equation (1.3)

We will assume different values of a and b so that Equation (1.3) has real and rational solutions. That is term $\pm\sqrt{(b - a) + (b^2 - a^2)}$ should be a perfect square. After assumption of different values of a and b, values of x are calculated by Equation (1.3). Two cases then arise, one when x is found integer, two when x found is a fraction as p/q. If x is found to be an integer, no further operation is required,

2.1a. Normalisation: When x Found is a Fraction p/q.

If x is found to be a fraction of form p/q where p, q, a and b are integers then by Equation (1.1)

$$(p/q)^2 + (p/q + a)^2 = (p/q + b)^2.$$

On expansion and simplification

(2.1) $(p)^2 + (p + q.a)^2 = (p + q.b)^2$.

Since p, q, a and b are all integers then $p, (p + q.a)$and $(p + q.b)$ being sum and product of integers are always integers**.** Such explanation can be found in basic books of number theory [3], [5]. Equation (2.1) is obtained on multiplying$(p/q)^2 + (p/q + a)^2 = (p/q + b)^2$ with lowest common multiplier *LCM*. On putting different integer values of a and b, values of x are calculated from Equation (1.3) and if values of x are found to be fractions, these are normalised as discussed above. Explanation of *LCM* and conversion of fraction to integers are given in books on number theory [3], [5]. If x found is of form p/q , a of form p_1/q_1 and b of form p_2/q_2 then

$$(p/q)^2 + (p/q + p_1/q_1)^2 = (p/q + p_2/q_2)^2$$

or

(2.2) $(p.q_1.q_2)^2 + (p.q_1.q_2 + p_1.q.q_2)^2 = (p.q_1.q_2 + p_2.q.q_1)^2$.

Values of x calculated with different values of a and b are given in ***Table* 1.1** after normalisation. It is also worth mentioning if value of x is found to be negative and as its square will always be positive, therefore, negative value of x will not have any effect on Pythagorean's triple. In the ***Table* 1.1** , at one place, x is calculated negative $11/2$, after normalisation it is taken as positive 11 on account of the fact already discussed.

***Table* 2.1:** Pythagorean's Triples

x	a	b	Normalised x	Normalised $y = x + a$	Normalised $z = x + b$
3	1	2	3	4	5
5/4	7/4	2	5	12	13
8/3	7/3	3	8	15	17
7/6	17/6	3	7	24	25
12/5	23/5	5	12	35	37
9/16	31/16	2	9	40	41
28/5	17/5	5	28	45	53
−11/12	71/12	6	11	60	61
33/8	23/8	4	33	56	65
48/5	7/5	5	48	55	73
13/12	71/12	6	13	84	85
39/10	41/10	5	39	80	89
65/8	7/8	4	65	72	97
88/7	17/7	7	88	105	137
85/12	47/12	6	85	132	157
119/10	1/10	5	119	120	169
133/12	23/12	6	133	156	205
207/14	17/1	7	207	224	305

2.2 Pythagorean's Quadruples

Pythagorean's quadruples x, y, w and z are given by equation given below. Studies for derivation of identifies for Pythagorean's quadruples from the point of view of area of one face of tetrahedron that equals areas of three opposite faces has been done and also Wikipedia provides an analytic account of it [1], [9].

(2.3) $x^2 + y^2 + w^2 = z^2$.

Again by putting $y = x + a$, $z = x + b$ and $w = x + c$ where$x, a, b$and c are all rational quantities, In this regards, theory of representation of rational quantities in algebraic is given in basic books on number theory [3], [5]. Equation (2.3) transforms to $(x)^2 + (x + a)^2 + (x + b)^2 = (x + c)^2$ or

(2.4) $x^2 - x(c - b - a) - \frac{1}{2}(c^2 - b^2 - a^2) = 0$.

Equation (2.4) being quadratic has roots given by

$$(2.5)\quad x = \frac{1}{2}(c-b-a) \pm \frac{1}{2}\Big\{(c-b-a)^2 + 2.(c^2-b^2-a^2)\Big\}^{\frac{1}{2}}.$$

***Lemma* 2.3** *Integers x, y, z and w are Pythagorean's quadruples if*

$$x = \frac{1}{2}(c-b-a) \pm \frac{1}{2}\Big\{(c-b-a)^2 + 2.(c^2-b^2-a^2)\Big\}^{\frac{1}{2}}, y = (x+a), w = (x+b)$$

and $z = (x+c)$ where x, a, b and c are rational quantities such that quantity $\pm\frac{1}{2}\Big\{(c-b-a)^2 + 2(c^2-b^2-a^2)\Big\}^{\frac{1}{2}}$ is rational, in other words quantity $\pm\frac{1}{2}\Big\{(c-b-a)^2 + 2(c^2-b^2-a^2)\Big\}^{\frac{1}{2}}$ is real and a perfect square. If x, y, z and w so determined are fractions, then multiplying x, y, z and w with their lowest common multiplier LCM will give normalised Pythagorean's quadruples in integer form.

***Lemma* 2.4** *If x, y, z and w are rational quantities and x is is given by*

$$x = \frac{1}{2}(c-b-a) \pm \frac{1}{2}\Big\{(c-b-a)^2 + 2(c^2-b^2-a^2)\Big\}^{\frac{1}{2}}, y = (x+a), w = (x+b)$$

and $z = (x+c)$ where a, b and c are rational quantities such that quantity $\pm\frac{1}{2}\Big\{(c-b-a)^2 + 2.(c^2-b^2-a^2)\Big\}^{\frac{1}{2}}$ is rational, in other words quantity $\pm\frac{1}{2}\Big\{(c-b-a)^2 + 2(c^2-b^2-a^2)\Big\}^{\frac{1}{2}}$ is real and perfect square and if x is of form p/q, a of form p_1/q_1, b of form p_2/q_2 and c of form p_3/q_3 then $(p.q_1.q_2.q_3), (p.q_1.q_2.q_3 + p_1.q.q_2.q_3), (p.q_1.q_2.q_3 + p_2.q.q_1.q_3)and(p.q_1.q_2.q_3 + p_3.q.q_1.q_2)$ will be Pythagorean's Quadruples..

2.2a. Values of x, a and b Satisfying Equation (2.5)

Different values of$a, b$and c are assumed so that Equation (2.5) has rational solutions i.e. $\Big\{(c-b-a)^2+2(c^2-b^2-a^2)\Big\}^{\frac{1}{2}}$ is rational and is a perfect square. After assumption of values of a, b and c, values of x are calculated by equations (2.5). On calculations two cases arise, one when x is found integer, no further operation is required, two if x is found to be a fraction of the form p/q, it requires normalisation.

2.2b. Normalisation: When x on Calculation is a Fraction of the Form p/q

When x is a fraction say of kind p/q where p, q, a and b are integers then by Equation (2.3),

$$(p/q)^2 + (p/q+a)^2 + (p/q+b)^2 = (p/q+c)^2.$$

After normalisation by multiplying with LCM,

$$(2.6)\quad p^2 + (p+q.a)^2 + (p+q.b)^2 = (p+q.c)^2.$$

Sincep, q, a and b are all integers, therefore,$p, (p+q.a), (p+q.b)$ and $(p+q.c)$ being sum and product of integers are always integers. Such explanation is given in books [3], [5]. On putting different integer values of a, b and c, values of x are calculated from equation (2.5) and if values of x are found to be fractions, these are normalised as discussed above. If x calculated is of form p/q, a of form p_1/q_1, b of form $p_2/q_2$and c of the form p_3/q_3 then

$$(p/q)^2 + (p/q + p_1/q_1)^2 + (p/q + p_2/q_2)^2 = (p/q + p_3/q_3)^2.$$

After normalisation on multiplying with LCM,

$$(2.7)\quad (p.q_1.q_2.q_3)^2 + (p.q_1.q_2.q_3 + p_1.q.q_2.q_3)^2 + (p.q_1.q_2.q_3 + p_2.q.q_1.q_3)^2(p.q_1.q_2.q_3 + p_3.q.q_1.q_2)^2.$$

Values of x are calculated by putting different values of a and b in equations (2.5). These values of $x, a, b$and c are given in ***Table* 2.2**. It is is also worth mentioning that if value of x is found to be negative and as its square will always be positive, therefore, assuming this negative value of x or $(x+a)$ or$(x+b)$ or $(x+c)$ as positive will not have any effect on Pythagorean's quadruples. In the ***Table* 2.2***,* at some places, x is calculated negative, but after normalisation, it is taken as positive on account of the facts as have already been discussed.

Table 2.2: Pythagorean's Quadruples given by equation $x^2 + y^2 + z^2 = w^2$

x	a	b	c	Norm. x	Norm. $y = x + a$	Norm. $z = x + b$	Norm. $w = x + c$
1	1	1	2	1	2	2	3
1/4	7/4	3/4	2	1	8	4	9
1/8	11/8	11/8	2	1	12	12	17
2/3	7/3	4/3	3	2	9	6	11
4	0	−2	2	4	4	2	6
−7/8	11/8	11/8	2	7	4	4	9
−7/6	13/6	−13/6	3	7	6	6	11
7/2	23/2	23/2	3	7	30	30	43
9/4	−17/4	−21/4	2	9	8	12	17
9/8	−21/8	−29/8	2	9	12	20	25
−17/12	23/12	23/12	3	17	6	6	19
21/4	−5/4	−9/4	2	21	16	12	29
29/8	−15/8	−15/8	2	29	14	14	45
31/12	11/12	11/12	3	31	42	42	67
37/4	-17/4	−21/4	2	37	20	16	45
57/8	-21/8	−29/8	2	57	36	28	73

2.2c. Pythagorean Quadruples of the Form $x^2 + y^2 = z^2 + w^2$

Let $y = x + a, z = x + b$ and $w = x + c$ then $x^2 + (x + a)^2 = (x + b)^2 + (x + c)^2$.

Therefore,

$$x = \frac{b^2 + c^2 - a^2}{2(a - b - c)}, \tag{2.8}$$

where$x, a, b$and c are rationals. Amongst others, Wikipedia describes the method of generation of Pythagorean's triples [8]. Let x is of form p/q , a of form p_1/q_1 , b of form p_2/q_2 , c of form p_3/q_3 where $p, q, p_1 q_1, p_2, q_2, p_3 and q_3$ are all integers then after normalisation,

$$(p.q_1.q_2.q_3)^2 + (p.q_1.q_2.q_3 + p_1.q.q_2.q_3)^2 = (p.q_1.q_2.q_3 + p_2.q.q_1.q_3)^2 + (p.q_1.q_2.q_3 + p_3.q.q_1.q_2)^2. \tag{2.9}$$

Based on the above said formula, some Pythagorean's quadruples are given in ***Table* 2.3**.

Table 2.3: Pythagorean Quadruples of the Form $x^2 + y^2 = z^2 + w^2$

x	a	b	c	Normalised x	Normalised $y = x + a$	Normalised $z = x + b$	Normalised $w = x + c$
-7/6	1	2	2	7	1	5	5
-17/10	1	3	3	17	7	13	13
-31/4	1	4	4	31	17	25	25
-19/2	1	-2	4	19	17	23	11
-19/14	-1	2	4	19	33	9	7
19/6	1	2	-4	19	25	31	5
1/6	2	1	-2	1	13	7	11
-1/10	-2	1	2	1	21	9	19
7/2	2	3	-3	7	11	13	1
7/8	2	-3	-3	7	23	17	17
-1/3	3	-2	-1	1	8	7	4
-1/10	3	-3	1	1	31	29	11
1/14	3	-3	-1	1	43	41	13
-9/2	1	-1	3	9	7	11	13

2.2d. Pythagorean Quadruples of the Form $x^2+y^2+w^2 = v^2$ where $x^2+y^2 = z^2$ and $z^2+w^2 = v^2$

Pythagorean's triples of the form$x^2+y^2 = z^2$have already been formulated in paragraphs **1** to **2.1a** and are not repeated here. After finding value of z, value of c and d are to be found so that these may satisfy the equation $z^2 + w^2 = v^2$. This equation is same as $x^2 + y^2 = z^2$and values of z can be found by Equation (1.3). Values of x, y, z and w are given in ***Table* 2.4.**

***Table* 2.4:** Pythagorean Quadruples of the Form $x^2 + y^2 = z^2 + w^2$

x	y	$z = \left(x^2+y^2\right)^{\frac{1}{2}}$	w	$v = \left(z^2+w^2\right)^{\frac{1}{2}}$
3	4	5	12	13
5	12	13	84	85
9	12	15	8	17
13	84	85	132	157
15	36	39	80	89
28	96	100	105	145
33	56	65	420	425
33	56	65	72	97

2.2e. Pythagorean Quintuples of the Form $x^2 + y^2 + z^2 + w^2 = v^2$

Let $y = (x + a), z = (x + b), w = (x + c)$ and $v = (2x + d)$ [8] then $(x)^2 + (x + a)^2 + (x + b)^2 + (x + c)^2 = (2.x + d)^2$.

Therefore,

$$x = \frac{d^2 - a^2 - b^2 - c^2}{2(a + b + c - 2.d)}. \quad (2.10)$$

On putting different integer values of a, b, c and d, different values of x are obtained. From those values of x, values ofy, z, w and v are calculated and normalised. Wikipedia describes the method of generation of Pythagorean's quintuples [8]. ***Table* 2.5** gives Pythagorean's Quintuples based on the above said formula.

***Table* 2.5:** Pythagorean Pentagonal Numbers of the Form $x^2 + y^2 + z^2 + w^2 = v^2$

a	b	c	d	x	Norm.x	Norm.$y = x+a$	Norm.$z = x+b$	Norm.$w = x+c$	Norm.$v = 2x+d = (x^2+y^2+z^2+w^2)^{\frac{1}{2}}$
1	2	3	4	-1/2	1	1	3	5	6
1	3	4	5	1/4	1	5	13	17	22
1	2	3	-\4	1/14	1	15	29	43	54
1	2	-\3	4	-1/8	1	7	15	25	30
-1	2	3	4	-1/4	1	5	7	11	14
-1	-2	3	4	-1/8	1	9	17	23	30
-1	2	-3	4	-1/10	1	11	19	31	38
-1	2	3	-\4	1/12	1	11	25	37	46
1	-3	4	5	1/16	1	17	47	65	82
1	3	-\4	5	1/20	1	21	61	79	102

2.2f. Pythagorean *N*-Tuples of the Form $x_1^2 + x_2^2 + x_3^2 + \ldots + x_n^2 = y^2$

Let $x_2 = (x_1 + a_2)$, $x_3 = (x_1 + a_3)$, $x_4 = (x_1 + a_4), \ldots, x_n = (x_1 + a_n)$ and $y = (x_1 + a_1)$then putting these values in equation $x_1^2 + x_2^2 + x_3^2 + \ldots x_n^2 = y^2$ it will take the form

$$x_1^2 + (x_1 + a_2)^2 + (x_1 + a_3)^2 + (x_1 + a_4)^2 + \ldots + (x_1 + a_n)^2 = (x_1 + a_1)^2.$$

Methods of generating such generalised Pythagorean's n-tuples has been described by many mathematicians [4], [10], [1], [2]. On expansion and rearrangement,

$$(n-1)x_1^2 + 2x_1(a_2 + a_3 + a_4 + \ldots + a_n - a_1) + a_2^2 + a_3^2 + a_4^2 + \ldots a_n^2 - a_1^2 = 0$$

This is a quadratic in x_1 and its roots are

$$x_1 = \frac{-Q \pm \sqrt{Q^2 - 4P.R}}{2P},$$

where $P = (n-1), Q = 2(a_2 + a_3 + a_4 + \ldots + a_n - a_1)$ and $R = a_2^2 + a_3^2 + a_4^2 + \ldots a_n^2 - a_1^2$. Values of $a_1, a_2, a_3, \ldots a_n$ can be assumed arbitrarily such that part under square root is rational and that determines x_1. Once x_1 is determined, $a_1, a_2, a_3, \ldots, a_n$ are known, $x_2, x_3, \ldots, x_n$ and y can be calculated. By assuming different set of $a_1, a_2, a_3, \ldots, a_n$, different set of $x_1, x_2, x_3, \ldots, x_n$ can be found out, hence n-tuples are generated.

2.3 Second Method: To Generate Pythagorean's Numbers

2.3a. Pythagorean's Triples

Pythagorean's Triples are given by the equation $a^2 + b^2 = c^2$. This can also be written as $a^2 = c^2 - b^2$ or $a^2 = (c-b)(c+b)$or $(\frac{c}{a} - \frac{b}{a})(\frac{c}{a} + \frac{b}{a}) = 1$. Let $x = (\frac{c}{a} + \frac{b}{a})$then from above equation, $\frac{1}{x} = (\frac{c}{a} - \frac{b}{a})$. On adding and subtracting,

(2.11) $x + \dfrac{1}{x} = 2\dfrac{c}{a}$

(2.12) $x - \dfrac{1}{x} = 2\dfrac{b}{a}.$

Since $c^2/a^2 - b^2/a^2 = 1$, therefore using Equations (2.11) and (2.12),

(2.13) $\dfrac{1}{4}(x + \dfrac{1}{x})^2 - \dfrac{1}{4}(x - \dfrac{1}{x})^2 = 1,$

(2.14) $(x - \dfrac{1}{x})^2 + 2^2 = (x + \dfrac{1}{x})^2,$

identity (2.14) generates Pythagorean's Triples $(x - 1/x)$, 2 and $(x + 1/x)$ satisfying $a^2 + b^2 = c^2$.[8].Let $x = \frac{3}{2}$then $(\frac{3}{2} - \frac{2}{3})$, 2 and $(\frac{3}{2} + \frac{2}{3})$ on normalisation are Pythagorean's triple as 5, 12 and 13. Procedure analogous to this has also been adopted by some mathematicians and it also makes mention in Wikipedia.[7], [8]. Some Pythagorean's triples using Equation (2.14) are given in ***Table* 2.6**.

***Table* 2.6:** Pythagorean's Triples

x	$(x - 1/x)$	2	$(x + 1/x)$	Normalised a	Normalised b	Normalised c
2	3/2	2	5/2	3	4	5
3/2	5/6	2	13/6	5	12	13
4/3	7/12	2	25/12	7	24	25
5/4	9/20	2	41/20	9	40	41
4	15/4	2	17/4	15	8	17
6	35/6	2	37/6	35	12	37
8/3	55/24	2	73/24	55	48	73
8/5	65/40	2	89/40	65	80	89
8/7	15/56	2	113/56	15	112	113
8	63/8	2	65	63	16	65

2.3b. Pythagorean's Quadruples of the Form $a^2+b^2+c^2 = d^2$ where $a^2+b^2 = e^2$and $e^2+c^2 = d^2$

Let $a^2 + b^2 = e^2$ then as proved in paragraph **2.3a**, $x - \frac{1}{x} = 2\frac{b}{a}$ and $x + \frac{1}{x} = 2\frac{e}{a}$. That makes $a^2 + b^2 = e^2$ as $\frac{1}{4}(x - \frac{1}{x})^2 + 1 = \frac{1}{4}(x + \frac{1}{x})^2$. On dividing left hand side and right hand side by $\frac{1}{4}(x + \frac{1}{x})^2$

(2.15) $(x - \dfrac{1}{x})^2(x + \dfrac{1}{x})^{-2} + 4(x + \dfrac{1}{x})^{-2} = 1.$

Since $a^2+b^2+c^2 = d^2$ and it is assumed $a^2+b^2 = e^2$, therefore,$e^2+c^2 = d^2$. That makes $y - \frac{1}{y} = 2\frac{c}{e}$ and $y + \frac{1}{y} = 2\frac{d}{e}$. Therefore,

(2.16) $1 + \dfrac{1}{4}(y - \dfrac{1}{y})^2 = \dfrac{1}{4}(y + \dfrac{1}{y})^2.$

On putting value of 1 from equation (2.15) in Equation (2.16),

(2.17) $(x - \dfrac{1}{x})^2(x + \dfrac{1}{x})^{-2} + 4(x + \dfrac{1}{x})^{-2} + \dfrac{1}{4}(y - \dfrac{1}{y})^2 = \dfrac{1}{4}(y + \dfrac{1}{y})^2.$

Equation (2.17) is an equation of Pythagorean's Quadruples for all real and rational values of$x$and y except $x = y = 1$. An example where $x = 2, y = 3$ is taken up below,

$(x-\frac{1}{x})^2(x+\frac{1}{x})^{-2} = (\frac{9}{4})(\frac{4}{25}) = \frac{9}{25}$,

$4(x+\frac{1}{x})^{-2} = 4(\frac{4}{25}) = \frac{16}{25}$,

$4(x+\frac{1}{x})^{-2} = 4(\frac{4}{25}) = \frac{16}{25}$,

$\frac{1}{4}(y-\frac{1}{y})^2 = \frac{1}{4}(\frac{64}{9}) = \frac{16}{9}$,

$\frac{1}{4}(y+\frac{1}{y})^2 = \frac{1}{4}(\frac{100}{9}) = \frac{25}{9}$.

That makes $\frac{9}{25}+\frac{16}{25}+\frac{16}{9} = \frac{25}{9}$ or $(\frac{3}{5})^2+(\frac{4}{5})^2+(\frac{4}{3})^2 = (\frac{5}{3})^2$and after normalisation,

$9^2+12^2+20^2 = 25^2$.

In this way, using equation (2.17), Pythagorean's Quadruples are generated and are given in ***Table* 2.7**.

***Table* 2.7:** Pythagorean's Quadruples of the Form $a^2+b^2+c^2 = d^2$ where $a^2+b^2 = e^2$ and $e^2+c^2 = d^2$

x	y	$\frac{(x-\frac{1}{x})}{(x+\frac{1}{x})}$	$2(x+\frac{1}{x})$	$\frac{1}{2}(y-\frac{1}{y})$	$\frac{1}{2}(y+\frac{1}{y})$	a	b	e	c	d
2	3	3/5	4/5	4/3	5/3	9	12	15	20	25
3	4	4/5	3/5	15/8	17/8	32	24	40	75	80
2	5	3/5	4/5	12/5	13/5	3	4	5	12	13
2	7	3/5	4/5	24/7	25/7	21	28	35	120	125
3	2	4/5	3/5	3/4	5/4	16	12	20	15	25
3	8	4/5	3/5	63/16	65/16	64	48	80	315	325
2	9	3/5	4/5	40/9	41/9	27	36	45	200	205
3	11	4/5	3/5	60/11	61/11	44	33	55	300	305
4	5	15/17	8/17	12/5	13/5	75	40	85	204	221
3	3/2	4/5	3/5	5/12	13/12	48	36	60	25	65

2.3c. Pythagorean's Quintuples of the Form $a^2+b^2+c^2+d^2 = g^2$ where $a^2+b^2 = e^2$, $e^2+c^2 = f^2$ and $f^2+d^2 = g^2$.

Pythagorean's Quadruples, generating equation has already been derived at (2.17), we shall proceed further from this equation which can also be written as

$$4(x-\frac{1}{x})^2(x+\frac{1}{x})^{-2}(y+\frac{1}{y})^{-2}+4^2(x+\frac{1}{x})^{-2}(y+\frac{1}{y})^{-2}+(y-\frac{1}{y})^2(y+\frac{1}{y})^{-2} = 1$$

or

$$(2.18)\quad \left\{2(x-\frac{1}{x})(x+\frac{1}{x})^{-1}(y+\frac{1}{y})^{-1}\right\}^2+\left\{4(x+\frac{1}{x})^{-1}(y+\frac{1}{y})^{-1}\right\}^2+\left\{(y-\frac{1}{y})(y+\frac{1}{y})^{-1}\right\}^2 = 1,$$

since $a^2+b^2 = e^2$, $e^2+c^2 = f^2$ and $f^2+d^2 = g^2$.

From discussion made in the paragraph related to generation of Quadruples, f, d and g can be given by relations, $\frac{f}{d} = \frac{1}{2}(z-\frac{1}{z})$ and $\frac{g}{d} = \frac{1}{2}(z+\frac{1}{z})$ where z is real rational number. Equation $f^2+d^2 = g^2$ can be written as $\frac{1}{4}(z-\frac{1}{z})^2+1 = \frac{1}{4}(z+\frac{1}{z})^2$and substituting 1 as given by equation (2.18), this equation takes the form

$$(2.19)\quad \left\{2(x-\frac{1}{x})(x+\frac{1}{x})^{-1}(y+\frac{1}{y})^{-1}\right\}^2+\left\{4(x+\frac{1}{x})^{-1}(y+\frac{1}{y})^{-1}\right\}^2+\left\{(y-\frac{1}{y})(y+\frac{1}{y})^{-1}\right\}^2+\left\{\frac{1}{2}(z-\frac{1}{z})\right\}^2 = \left\{\frac{1}{2}(z+\frac{1}{z})\right\}^2.$$

Equation (2.19) after normalisation generates Pythagorean's Quintuples for all real rational values of$x$, y and z. Let $A = \left\{2(x-\frac{1}{x})(x+\frac{1}{x})^{-1}(y+\frac{1}{y})^{-1}\right\}$, $B = \left\{4(x+\frac{1}{x})^{-1}(y+\frac{1}{y})^{-1}\right\}$, $C = \left\{(y-\frac{1}{y}).(y+\frac{1}{y})^{-1}\right\}$, $D = \left\{\frac{1}{2}(z-\frac{1}{z})\right\}$ and $G = \left\{\frac{1}{2}.(z+\frac{1}{z})\right\}$. Un-normaised Pythagorean's Quintuples are generated by equation $A^2+B^2+C^2+D^2 = G^2$ and after normalisation, we get $a^2+b^2+c^2+d^2 = g^2$.

Let us take an example where $x = 2, y = 3$ and $z = 4$. Then $A^2 = \left\{2(\frac{3}{2})(\frac{2}{5})(\frac{3}{10})\right\}^2 = (\frac{9}{25})^2, B^2 = \left\{4(\frac{2}{5})(\frac{3}{10})\right\}^2 = (\frac{12}{25})^2, C^2 = \left\{(\frac{8}{3})(\frac{3}{10})\right\}^2 = (\frac{4}{5})^2, D^2 = \left\{\frac{1}{2}(\frac{15}{4})\right\}^2 = (\frac{15}{8})^2$ and $G^2 = \left\{\frac{1}{2}(\frac{17}{4})\right\}^2 = (\frac{17}{8})^2$. Since $A^2+B^2+C^2+D^2 = G^2$, therefore,$(\frac{9}{25})^2+(\frac{12}{25})^2+(\frac{4}{5})^2+(\frac{15}{8})^2 = (\frac{17}{8})^2$.After normalisation, $a^2+b^2+c^2+d^2 = g^2$or $72^2+96^2+160^2+375^2 = 425^2$. In this way, using equation (2.19), Pythagorean's Quintuples are generated and are given in the ***Table* 2.8.**

Table **2.8:** Pythagorean's Quintuples Of The Form $a^2 + b^2 + c^2 + d^2 = g^2$

x	y	z	A	B	C	D	G	a	b	e	c	f	d	g
2	3	4	9/25	12/25	4/5	15/8	17/8	72	96	120	160	200	375	425
2	3	5	9/25	12/25	4/5	12/5	13/5	9	12	15	20	25	60	65
2	3	2	9/25	12/25	4/5	3/4	5/4	36	48	60	80	100	75	125
3	2	4	16/25	12/25	3/5	15/8	17/8	128	96	160	120	200	375	425
2	2	2	12/25	16/25	3/5	3/4	5/4	48	64	80	60	100	75	125
2	2	3	12/25	16/25	3/5	4/3	5/3	36	48	60	45	75	100	125
2	2	3/2	12/25	16/25	3/5	5/12	13/12	144	192	240	180	300	125	325
2	3	3	9/25	12/25	4/5	4/3	5/3	27	36	45	60	75	100	125
3	3	3/2	12/25	9/25	4/5	5/12	13/12	144	108	180	240	300	125	325
2	2	5	12/25	16/25	3/5	12/5	13/5	12	16	20	15	25	60	65
3	3	5/2	12/25	9/25	4/5	21/20	29/20	48	36	60	80	100	105	145

2.3d. Pythagorean's Sextuples of the Form $a^2 + b^2 + c^2 + d^2 + g^2 = h^2$ where $a^2 + b^2 = e^2$, $e^2 + c^2 = f^2$, $f^2 + d^2 = k^2$ and $k^2 + g^2 = h^2$

Equation (2.19) can also be written as

$$(2.20)\quad \left\{2^3(x+\frac{1}{x})^{-1}(y+\frac{1}{y})^{-1}(z+\frac{1}{z})^{-1}\right\}^2 + \left\{2^2(x-\frac{1}{x})(x+\frac{1}{x})^{-1}(y+\frac{1}{y})^{-1}(z+\frac{1}{z})^{-1}\right\}^2$$
$$+\left\{2(y-\frac{1}{y})(y+\frac{1}{y})^{-1}(z+\frac{1}{z})^{-1}\right\}^2 + \left\{2^0(z-\frac{1}{z})(z+\frac{1}{z})^{-1}\right\}^2 = 1.$$

Also $k^2 + g^2 = h^2$ can be written as $\left\{\frac{1}{2}(w-\frac{1}{w})\right\}^2 + 1 = \left\{\frac{1}{2}(w+\frac{1}{w})\right\}^2$ where $w + \frac{1}{w} = 2\frac{h}{k}$ and $w - \frac{1}{w} = 2\frac{g}{k}$ from equations (2.11) and (2.12). On substituting 1 as given in equation (2.20) in the above equation, we get

$$(2.21)\quad \left\{2^3(x+\frac{1}{x})^{-1}(y+\frac{1}{y})^{-1}(z+\frac{1}{z})^{-1}\right\}^2 + \left\{2^2(x-\frac{1}{x})(x+\frac{1}{x})^{-1}(y+\frac{1}{y})^{-1}(z+\frac{1}{z})^{-1}\right\}^2$$
$$+\left\{2(y-\frac{1}{y})(y+\frac{1}{y})^{-1}(z+\frac{1}{z})^{-1}\right\}^2 + \left\{2^0(z-\frac{1}{z})(z+\frac{1}{z})^{-1}\right\}^2 + \left\{2^{-1}(w-\frac{1}{w})\right\}^2 = \left\{2^{-1}(w+\frac{1}{w})\right\}^2.$$

Equation (2.21) after normalisation generates Pythagorean's Sextuples for all real rational values of x, y, z and w. Let

$$A^2 = \left\{2^3(x+\tfrac{1}{x})^{-1}(y+\tfrac{1}{y})^{-1}(z+\tfrac{1}{z})^{-1}\right\}^2$$
$$B^2 = \left\{2^2(x-\tfrac{1}{x})(x+\tfrac{1}{x})^{-1}(y+\tfrac{1}{y})^{-1}(z-\tfrac{1}{z})^{-1}\right\}^2$$
$$C^2 = \left\{2(y-\tfrac{1}{y})(y+\tfrac{1}{y})^{-1}(z+\tfrac{1}{z})^{-1}\right\}^2$$
$$D^2 = \left\{(z-\tfrac{1}{z})(z+\tfrac{1}{z})^{-1}\right\}^2$$
$$G^2 = \left\{\tfrac{1}{2}(w-\tfrac{1}{w})\right\}^2$$
$$H^2 = \left\{\tfrac{1}{2}(w+\tfrac{1}{w})\right\}^2,$$

then unnormalised Pythagorean's Sextuples are generated byA, B, C, D, G and H by the equation (2.21) where

$A^2 + B^2 + C^2 + D^2 + G^2 = H^2$.

After normalisation, we get

$a^2 + b^2 + c^2 + d^2 + g^2 = h^2$.

Let us take an example where$x = 2, y = 2, z = 2$ and $w = 3$ then

$$A^2 = \left\{2^3(\tfrac{2}{5})(\tfrac{2}{5})(\tfrac{2}{5})\right\}^2 = (\tfrac{64}{125})^2,$$
$$B^2 = \left\{2^2(\tfrac{3}{2})(\tfrac{2}{5})(\tfrac{2}{5})(\tfrac{2}{5})\right\}^2 = (\tfrac{48}{125})^2,$$

$C^2 = \left\{2(\frac{3}{2})(\frac{2}{5})(\frac{2}{5})\right\}^2 = (\frac{12}{25})^2,$
$D^2 = \left\{(\frac{3}{2})(\frac{2}{5})\right\}^2 = (\frac{3}{5})^2,$
$G^2 = \left\{(\frac{1}{2})(\frac{8}{3})\right\}^2 = (\frac{4}{3})^2,$
$H^2 = \left\{(\frac{1}{2})(\frac{10}{3})\right\}^2 = (\frac{5}{3})^2.$
Therefore,
$(\frac{64}{125})^2 + (\frac{48}{125})^2 + (\frac{12}{25})^2 + (\frac{3}{5})^2 + (\frac{4}{3})^2 = (\frac{5}{3})^2.$
After normalisation
$192^2 + 144^2 + 180^2 + 225^2 + 500^2 = 625^2.$
In this way, using Equation (2.21), Pythagorean's Sextuples are generated and unnormalised are given in the ***Table* 2.9 A** and normalised in ***Table* 2.9 B.**

***Table* 2.9 A:** Unnormalised Pythagorean's Sextuples

x	y	z	w	A	B	C	D	G	H
2	2	2	3	64/125	48/125	12/25	3/5	4/3	5/3
2	2	2	2	64/125	48/125	12/25	3/5	3/4	5/4
2	2	2	4	64/125	48/125	12/25	3/5	15/8	17/8
2	2	2	3/2	64/125	48/125	12/25	3/5	5/12	13/12
2	2	2	5	64/125	48/125	12/25	3/5	12/5	13/5
2	2	2	10	64/125	48/125	12/25	3/5	99/20	101/20
2	2	2	25	64/125	48/125	12/25	3/5	312/25	313/25
2	2	2	5/3	64/125	48/125	12/25	3/5	8/15	17/15

***Table* 2.9 B:** Pythagorean's Sextuples of the Form $a^2 + b^2 + c^2 + d^2 + g^2 = h^2$

x	y	z	w	a	b	c	d	g	h
2	2	2	3	192	144	180	225	500	625
2	2	2	2	256	192	240	300	375	625
2	2	2	4	512	384	480	600	1875	2125
2	2	2	3/2	768	576	720	900	625	1625
2	2	2	5	64	48	60	75	300	325
2	2	2	10	256	192	240	300	2475	2525
2	2	2	25	64	48	60	75	1560	1565
2	2	2	5/3	192	144	180	225	200	425

2.3e. Pythagorean's *N*-Tuples of the Form $x_1^2 + x_2^2 + x_3^2 + \ldots + x_n^2 = y^2$

Equations 2.17), (2.19) and (2.20) on changing $xtox_1, ytox_2, ztox_3$ so on can be written in the following way for number of Pythagorean's terms $n = 4$,

$$\left\{2(x_1 + \frac{1}{x_1})^{-1}\right\}^2 + \left\{2^0(x_1 - \frac{1}{x_1})(x_1 + \frac{1}{x_1})^{-1}\right\}^2 + \left\{2^{-1}(x_2 - \frac{1}{x_2})\right\}^2 = \left\{2^{-1}(x_2 + \frac{1}{x_2})\right\}^2.$$

For number of Pythagorean's terms $n = 5$,

$$\left\{2^2(x_1 + \frac{1}{x_1})^{-1}(x_2 + \frac{1}{x_2})^{-1}\right\}^2 + \left\{2^1(x_1 - \frac{1}{x_1})(x_1 + \frac{1}{x_1})^{-1}(x_2 + \frac{1}{x_2})^{-1}\right\}^2$$
$$+\left\{2^0(x_2 - \frac{1}{x_2})(x_2 + \frac{1}{x_2})^{-1}\right\}^2 + \left\{2^{-1}(x_3 - \frac{1}{x_3})\right\}^2 = \left\{2^{-1}(x_3 + \frac{1}{x_3})\right\}^2.$$

For number of Pythagorean's terms $n = 6$,

$$\left\{2^3(x_1 + \frac{1}{x_1})^{-1}(x_2 + \frac{1}{x_2})^{-1}(x_3 + \frac{1}{x_3})^{-1}\right\}^2 + \left\{2^2(x_1 - \frac{1}{x_1})(x_1 + \frac{1}{x_1})^{-1}(x_2 + \frac{1}{x_2})^{-1}(x_3 + \frac{1}{x_3})^{-1}\right\}^2$$

$$+\left\{2^1(x_2-\frac{1}{x_2})(x_2+\frac{1}{x_2})^{-1}(x_3+\frac{1}{x_3})^{-1}\right\}^2+\left\{2^0(x_3-\frac{1}{x_3}).(x_3+\frac{1}{x_3})^{-1}\right\}^2+\left\{2^{-1}(x_4-\frac{1}{x_4})\right\}^2=\left\{2^{-1}(x_4+\frac{1}{x_4})\right\}^2.$$

By mathematical induction for Pythagorean's terms n,

(2.22) $$\left\{2^{-1}(x_{n-2}+\frac{1}{x_{n-2}})\right\}^2$$
$$=\left\{2^{n-3}(x_1+\frac{1}{x_1})^{-1}(x_2+\frac{1}{x_2})^{-1}.(x_3+\frac{1}{x_3})^{-1}\ldots(x_{n-3}+\frac{1}{x_{n-3}})^{-1}\right\}^2$$
$$+\left\{2^{n-4}(x_1-\frac{1}{x_1})(x_1+\frac{1}{x_1})^{-1}(x_2+\frac{1}{x_2})^{-1}\ldots.(x_{n-3}+\frac{1}{x_{n-3}})^{-1}\right\}^2$$
$$+\left\{2^{n-5}(x_2-\frac{1}{x_2})(x_2+\frac{1}{x_2})^{-1}(x_3+\frac{1}{x_3})^{-1}\ldots(x_{n-3}+\frac{1}{x_{n-3}})^{-1}\right\}^2$$
$$+\left\{2^{n-6}(x_3-\frac{1}{x_3})(x_3+\frac{1}{x_3})^{-1}(x_4+\frac{1}{x_4})^{-1}\ldots(x_{n-3}+\frac{1}{x_{n-3}})^{-1}\right\}^2$$
$$+\ldots+\left\{2^0(x_{n-3}-\frac{1}{x_{n-3}}).(x_{n-3}+\frac{1}{x_{n-3}})^{-1}\right\}^2+\left\{2^{-1}(x_{n-2}-\frac{1}{x_{n-2}})\right\}^2.$$

This can also be written in mathematical notation,

$$\left\{2^{2(n-3)}\prod_{k=1}^{n-3}(x_k+\tfrac{1}{x_k})^{-2}\right\}+\left\{2^{2(n-4)}(x_1-\tfrac{1}{x_1})^2\prod_{k=1}^{n-3}(x_k+\tfrac{1}{x_k})^{-2}\right\}+\left\{2^{2(n-5)}(x_2-\tfrac{1}{x_2})^2\prod_{k=2}^{n-3}(x_k+\tfrac{1}{x_k})^{-2}\right\}$$
$$+\left\{2^{2(n-6)}(x_3-\frac{1}{x_3})^2\prod_{k=3}^{n-3}(x_k+\tfrac{1}{x_k})^{-2}\right\}+\ldots+\left\{2^2(x_{n-4}-\tfrac{1}{x_{n-4}})^2\prod_{k=n-4}^{n-3}(x_k+\tfrac{1}{x_k})^{-2}\right\}$$
$$+\left\{2^0(x_{n-3}-\frac{1}{x_{n-3}})^2(x_{n-3}+\frac{1}{x_{n-3}})^{-2}\right\}+\left\{2^{-2}(x_{n-2}-\frac{1}{x_{n-2}})^2\right\}=\left\{2^{-2}(x_{n-2}+\frac{1}{x_{n-2}})^2\right\}.$$

Symbol $\prod_{k=1}^{n-3}(x_k+\frac{1}{x_k})^{-2}$ denotes product of terms $(x_k+\frac{1}{x_k})^{-2}$ when k varies from 1 to $n-3$. Let us take an example of generation of Pythagorean's numbers when its terms n equals to 9 and $x_1=2, x_2=2, x_3=2, x_4=2, x_5=3, x_6=2$ and $x_7=3$. Identity (2.22) for $n=9$, transforms to

$$2^{2(6)}\prod_{k=1}^{6}(x_k+\tfrac{1}{x_k})^{-2}+2^{2(5)}(x_1-\tfrac{1}{x_1})^2\prod_{k=1}^{6}(x_k+\tfrac{1}{x_k})^{-2}+2^{2(4)}(x_2-\tfrac{1}{x_2})^2\prod_{k=2}^{6}(x_k+\tfrac{1}{x_k})^{-2}$$
$$+2^{2(3)}(x_3-\frac{1}{x_3})^2\prod_{k=3}^{6}(x_k+\tfrac{1}{x_k})^{-2}+\ldots+2^2(x_5-\tfrac{1}{x_5})^2\prod_{k=5}^{6}(x_k+\tfrac{1}{x_k})^{-2}$$
$$+2^0(x_6-\frac{1}{x_6})^2(x_6+\frac{1}{x_6})^{-2}+2^{-2}(x_7-\frac{1}{x_7})^2=2^{-2}(x_7+\frac{1}{x_7})^2.$$

Let us denote above terms as $A^2, B^2, C^2, D^2, E^2, F^2, G^2, H^2 and I^2$ respectively so that $A^2+B^2+C^2+D^2+E^2+F^2+G^2+H^2=I^2$. Values of $A^2, B^2, C^2, D^2, E^2, F^2, G^2, H^2$ and I^2 are calculated by putting values of say $x_1=2, x_2=2, x_3=2, x_4=2, x_5=3, x_6=2$ and $x_7=3$ in above identity as solved below.

$$A^2=\left\{2^6(\frac{2}{5})^6\right\}^2=(\frac{2^{12}}{5^6})^2, B^2=\left\{2^5(\frac{3}{2})(\frac{2}{5})^6\right\}^2=\left\{3(\frac{2^{10}}{5^6})\right\}^2, C^2=\left\{2^4(\frac{3}{2})(\frac{2}{5})^5\right\}^2=\left\{3(\frac{2^8}{5^5})\right\}^2,$$
$$D^2=\left\{2^3(\frac{3}{2})(\frac{2}{5})^4\right\}^2=\left\{3(\frac{2^6}{5^4})\right\}^2, E^2=\left\{2^2(\frac{3}{2})(\frac{2}{5})^3\right\}^2=\left\{3(\frac{2^4}{5^3})\right\}^2, F^2=\left\{2(\frac{3}{2})(\frac{2}{5})^2\right\}^2=\left\{3(\frac{2}{5})^2\right\}^2,$$
$$G^2=\left\{(\frac{3}{2})(\frac{2}{5})\right\}^2=(\frac{3}{5})^2, H^2=\left\{(\frac{1}{2})(\frac{8}{3})\right\}^2=(\frac{4}{3})^2, I^2=\left\{(\frac{1}{2})(\frac{10}{3})\right\}^2=(\frac{5}{3})^2.$$

Therefore, above identity can be written as

$$\left\{\frac{2^{12}}{5^6}\right\}^2+\left\{3(\frac{2^{10}}{5^6})\right\}^2+\left\{3(\frac{2^8}{5^5})\right\}^2+\left\{3(\frac{2^6}{5^4})\right\}^2+\left\{3(\frac{2^4}{5^3})\right\}^2+\left\{3(\frac{2}{5})^2\right\}^2+\left\{\frac{3}{5}\right\}^2+\left\{\frac{4}{3}\right\}^2=\left\{\frac{5}{3}\right\}^2.$$

On normalisation,

$$\left\{3(2^{12})\right\}^2+\left\{3^2(2^{10})\right\}^2+\left\{5(3^2)(2^8)\right\}^2+\left\{(5^2)(3^2)(2^6)\right\}^2+\left\{(5^3)(3^2)(2^4)\right\}^2+\left\{(5^4)(3^2)(2^2)\right\}^2+\left\{(5^5)(3^2)\right\}^2+\left\{(5^6)(2^2)\right\}^2$$
$$=\left\{5^7\right\}^2, \text{ or } 12288^2+9216^2+11520^2+14400^2+18000^2+22500^2+28125^2+62500^2=78125^2.$$

3 Results And Conclusions

Pythagorean's triples $x, y$and z satisfying equation $x^2 + y^2 = z^2$ can be given by equation $(x)^2 + (x+a)^2 = (x+b)^2$ or $x^2 - 2x(b-a) + (b^2 - a^2) = 0$ where $y = (x+a)$ and $z = (x+b)$. Above quadratic equation has two roots as $x = (b-a) \pm \left\{(b-a)^2 + (b^2 - a^2)\right\}^{\frac{1}{2}}$ wherex, a and b are real and rational quantities. Values of a and b are so assumed that roots are real and rationals. If the roots found are real but fractions or a or b or both a and b are real and rationals but fractions, then quantities $x, (x+a)$ and $(x+b)$ are normalised by multiplying with LCM to make these integers. Since different values of a and b can be chosen, therefore, a number of values of x can be generated, hence a number of Pythagorean's triples.

Pythagorean's quadruples x, y, z and w satisfying equation $x^2 + y^2 + w^2 = z^2$ can be found by equation $(x)^2 + (x+a)^2 + (x+b)^2 = (x+c)^2$ where $y = (x+a), z = (x+b)$ and $w = (x+c)$. This quadratic has roots as $x = \frac{1}{2}(c-b-a) \pm \frac{1}{2}\left\{(c-b-a)^2 + 2(c^2 - b^2 - a^2)\right\}^{\frac{1}{2}}$ where x, a, b and c are real and rational quantities. In assuming values of $a, b$and c care should be taken that roots are real and rationals. If the roots found are real but fractions or a or b or c or all are real and rationals but fractions, then quantities $x, (x+a), (x+b)$ and $(x+c)$are normalised by multiplying these with LCM to make these integers. Since different values of $a, b$and c can be chosen, therefore, a number of values of x can be generated, hence a number of Pythagorean's quadruples.

Pythagorean quadruples x, y, z and w satisfying equation $x^2 + y^2 = z^2 + w^2$ can be written as $(x)^2 + (x+a)^2 = (x+b)^2 + (x+c)^2$ where $y = (x+a), z = (x+b)$ and $w = (x+c)$. This equation on simplification, reduces to

$$x = \frac{b^2 + c^2 - a^2}{2(a-b-c)}.$$

Assigning different real and rational values of $a, b$and c will yield different values of x and hence different quadruples after normalisation.

Pythagorean Quintuples of form $x^2 + y^2 + z^2 + w^2 = v^2$can also be reduced to linear equation $(x)^2 + (x+a)^2 + (x+b)^2 + (x+c)^2 = (2.x+d)^2$ where $y = (x+a), z = (x+b), w = (x+c)$ and $v = (2.x+d)$. The above equation on simplification reduces to

$$x = \frac{d^2 - a^2 - b^2 - c^2}{2(a+b+c-2d)}.$$

Different real rational values are assigned to a, b, c and d so that real rational values of x are obtained. After normalisation, Pythagorean's Quintuples are generated.

Pythagorean n-tuples of form $x_1^2 + x_2^2 + x_3^2 + \ldots x_n^2 = y^2$can be reduced to quadratic equation $(n-1)x_1^2 + 2x_1(a_2 + a_3 + a_4 + \ldots + a_n - a_1) + a_2^2 + a_3^2 + a_4^2 + \ldots a_n^2 - a_1^2 = 0$ where $x_2 = (x_1 + a_2), x_3 = (x_1 + a_3), x_4 = (x_1 + a_4) \ldots x_n = (x_1 + a_n)$ and $y = (x_1 + a_1)$. Different real rational values of $a_1, a_2, a_3, \ldots a_n$are assumed so that real rational values of $x_1$are obtained as

$$x_1 = \frac{-Q \pm \sqrt{Q^2 - 4P.R}}{2P},$$

where $P = (n-1), Q = 2(a_2 + a_3 + a_4 + \ldots + a_n - a_1)$ and $R = a_2^2 + a_3^2 + a_4^2 + \ldots a_n^2 - a_1^2$.

Apart from this, there is a second method to generate Pythagorean's n-tuples. In this method, an identity is derived satisfying Pythagorean's n-tuples and then assigning different real rational values to variables in the identity, Pythagorean's numbers are generated.

Pythagorean's triples a, b and c given by the equation $a^2 + b^2 = c^2$ can be generated from identity

$$\left\{\frac{1}{2}(x - \frac{1}{x})\right\}^2 + 1^2 = \left\{\frac{1}{2}(x + \frac{1}{x})\right\}^2,$$

where $\frac{1}{2}(x - \frac{1}{x}) = \frac{b}{a}$, and $\frac{1}{2}(x + \frac{1}{x}) = \frac{c}{a}$ and $x$is real rational quantity. It may be integer or fraction of the form p/q where p and q are integers.

Pythagorean's Quadruples of the form $a^2 + b^2 + c^2 = d^2$ can be generated after normalisation, from the identity

$$(x - \frac{1}{x})^2(x + \frac{1}{x})^{-2} + 4(x + \frac{1}{x})^{-2} + \frac{1}{4}(y - \frac{1}{y})^2 = \frac{1}{4}(y + \frac{1}{y})^2,$$

where x and y are real rational quantities and these may be integers or fractions of kind p/q.

Pythagorean's Quintuples can be generated by the identity

$$\left\{2(x - \frac{1}{x})(x + \frac{1}{x})^{-1}(y + \frac{1}{y})^{-1}\right\}^2 + \left\{2^2(x + \frac{1}{x})^{-1}(y + \frac{1}{y})^{-1}\right\}^2 + \left\{2^0(y - \frac{1}{y})(y + \frac{1}{y})^{-1}\right\}^2 + \left\{2^{-1}(z - \frac{1}{z})\right\}^2 = \left\{2^{-1}(z + \frac{1}{z})\right\}^2,$$

after normalisation where x, y and z are real rational quantities and these may be integers or fractions of kind p/q .

Pythagorean's Sextuples can be generated by the identity

$$\left\{2^3(x+\frac{1}{x})^{-1}(y+\frac{1}{y})^{-1}(z+\frac{1}{z})^{-1}\right\}^2+\left\{2^2(x-\frac{1}{x})(x+\frac{1}{x})^{-1}(y+\frac{1}{y})^{-1}(z+\frac{1}{z})^{-1}\right\}^2$$
$$+\left\{2(y-\frac{1}{y})(y+\frac{1}{y})^{-1}(z+\frac{1}{z})^{-1}\right\}^2+\left\{2^0(z-\frac{1}{z})(z+\frac{1}{z})^{-1}\right\}^2+\left\{2^{-1}(w-\frac{1}{w})\right\}^2=\left\{2^{-1}(w+\frac{1}{w})\right\}^2,$$

after normalisation where $x, y, z$and w are real rational quantities and these may be integers or fractions of kind p/q.

Pythagorean's n-tuples can be generated by the identity

$$\left\{2^{-1}(x_{n-2}+\frac{1}{x_{n-2}})\right\}^2$$
$$=\left\{2^{n-3}(x_1+\frac{1}{x_1})^{-1}(x_2+\frac{1}{x_2})^{-1}(x_3+\frac{1}{x_3})^{-1}\dots(x_{n-3}+\frac{1}{x_{n-3}})^{-1}\right\}^2$$
$$+\left\{2^{n-4}(x_1-\frac{1}{x_1})(x_1+\frac{1}{x_1})^{-1}(x_2+\frac{1}{x_2})^{-1}\dots.(x_{n-3}+\frac{1}{x_{n-3}})^{-1}\right\}^2$$
$$+\left\{2^{n-5}(x_2-\frac{1}{x_2})(x_2+\frac{1}{x_2})^{-1}(x_3+\frac{1}{x_3})^{-1}\dots(x_{n-3}+\frac{1}{x_{n-3}})^{-1}\right\}^2$$
$$+\left\{2^{n-6}(x_3-\frac{1}{x_3})(x_3+\frac{1}{x_3})^{-1}(x_4+\frac{1}{x_4})^{-1}\dots(x_{n-3}+\frac{1}{x_{n-3}})^{-1}\right\}^2$$
$$+\dots+\left\{2^0(x_{n-3}-\frac{1}{x_{n-3}})(x_{n-3}+\frac{1}{x_{n-3}})^{-1}\right\}^2+\left\{2^{-1}(x_{n-2}-\frac{1}{x_{n-2}})\right\}^2$$

after normalisation where $x_1, x_2, x_3, \dots x_{n-2}$ are real rational quantities and these may be integers or fractions of kind p/q.

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