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(Dedicated to Honor Dr. R. C. Singh Chandel on His 75^{th} Birth Anniversary Celebrations)

ON BURR (4P) DISTRIBUTION: APPLICATION OF BREAKING STRESS DATA

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Abstract

In many problems of quality and reliability engineering processes and designs, fitting of a probability distribution to the tensile strength or breaking stress data may be helpful in predicting the probability or forecasting the frequency of occurrence of the breaking stress, and planning beforehand. In this paper, Burr (4P) distribution functions was fitted to such breaking stress data, and compared with Burr (3P), Dagum (3P) and Dagum (4P) distribution functions. Goodness of fit has been tested using the Kolmogorov-Smirnov, Anderson-Darling and Chi-Squared distribution tests. It is observed that the Burr (4P) distribution fits the best among these four distributions.
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1 Introduction

In order to deal with the random phenomena and data occurring in many applied problems in the fields of actuarial science, biological sciences, engineering, finance, hydrology, medical sciences, reliability, transportation, etc., probability distributions can be applied to make predictions and informed decisions under uncertainty. For example, according to Birnbaum and Saunders [3], "for the amount of fatigue data which can usually be obtained almost any two-dimensional parametric family of distributions can be made to fit reasonably well. In fact, in the region of central tendency the lognormal, the Weibull, the Gamma etc., can all be fitted by parametric estimation and because of the relatively small sample sizes hardly any can be rejected by, say a Chi-square Goodness of Fit test. However, when it becomes a question of predicting the "safe life", say, the one thousandth percentile, there is a wide discrepancy between these models. For this reason, a family of distributions which is obtained from considerations of the basic characteristics of the fatigue process should be more persuasive in its implications than any ad hoc family chosen for extraneous." As pointed out by Lio and Park [15] "the two-parameter Birnbaum–Saunders distributions have been shown to provide a better fit to the strength or breaking stress data such as carbon fiber or composite tensile strengths than the more commonly used Weibull distributions by Durham and Padgett [9], in addition to cycles to failure data, as was investigated by Birnbaum and Saunders [3, 4]."

The observed frequency distributions of the brittle materials, such as 6061-T6 aluminum, are the results of many complex parameters such as their tensile strength, among others, and it may not be possible to predict them exactly. Therefore, the statistical treatment of such data is an important aspect of their analysis and interpretation. As stated above, different probability models such as Birnbaum–Saunders and Weibull distributions have been applied to characterize the strength or breaking stress data such as carbon fiber or composite tensile strengths in the past, see Birnbaum and Saunders [2, 3], and Durham and Padgett [9], among others. However, it appears from the literature that no such studies have performed for the breaking stress of 6061-T6 aluminum, except the bootstrap control chart for Birnbaum-Saunders percentiles by Lio and Park [15]. As most of the distributions of the tensile strength or breaking stress of 6061-T6 aluminum data are continuous and skewed in nature, some of the various continuous skewed probability distributions, developed recently, seem to be better choices for such studies. Thus, a better selection of the best fitting probability distribution to the tensile strength or breaking stress of 6061-T6 aluminum data may help us in extrapolating the observed values to those which are more significant from the point of view of quality and reliability engineering standards. Motivated by the importance of such studies described above, we have considered the fitting of Burr (4P) distribution functions was fitted to to the breaking stress of 6061-T6 aluminum data set, which has been taken from Lio and Park [15] consisting of 200 observations on the breaking stress of 6061-T6 aluminum. Then, Burr (4P) was compared with Burr (3P), Dagum (3P) and Dagum (4P) distribution functions.

The organization of this paper is as follows: In ***Section*** **2**, we briefly provide a review of the Burr and Dagum type distributions and some of their basic distributional properties. ***Section*** **3** contains the data description, parameters estimation and fitting the distributions to breaking stress data. Some concluding remarks are given in ***Section*** **4**.

2 Review of the models and distributional properties

In what follows, we shall briefly provide a review of the Burr (4P) and Burr (3P) distribution models and some of their essential basic distributional properties. For details on these distributions, see, for example, Blischke and Murthy [1], Burr [5], Dey et al. [8], Johnson et al. [12], Kleiber and Kotz [13], Kleiber [14], and Tadikamalla [18], among others.

2.1 Burr (4P) Distribution:

A continuous non-negative random variable, X, is said to have a Burr (4P) distribution if its probability density function (pdf), cumulative distribution function (cdf) and hazard function (hf) are respectively given by

(2.1) $$f(x) = \frac{\alpha k \left(\frac{x-\gamma}{\beta}\right)^{\alpha-1}}{\beta\left(1+\left(\frac{x-\gamma}{\beta}\right)^{\alpha}\right)^{k+1}},$$

(2.2) $$F(x) = 1 - \left(1+\left(\frac{x-\gamma}{\beta}\right)^{\alpha}\right)^{-k},$$

and

(2.3) $$h(x) = \frac{\alpha k \left(\frac{x-\gamma}{\beta}\right)^{\alpha-1}}{\beta\left(1+\left(\frac{x-\gamma}{\beta}\right)^{\alpha}\right)},$$

where k (>0): shape parameter ; α (>0): shape parameter ; β (>0): scale parameter ; $-\infty < \gamma < +\infty$: location parameter ; and domain: $\gamma \leq x < \infty$. The possible shapes of the pdf (1) and cdf (2) are given for some selected values of the parameters in ***Figures*** **2.1(a)** and **2.1(b)** respectively.

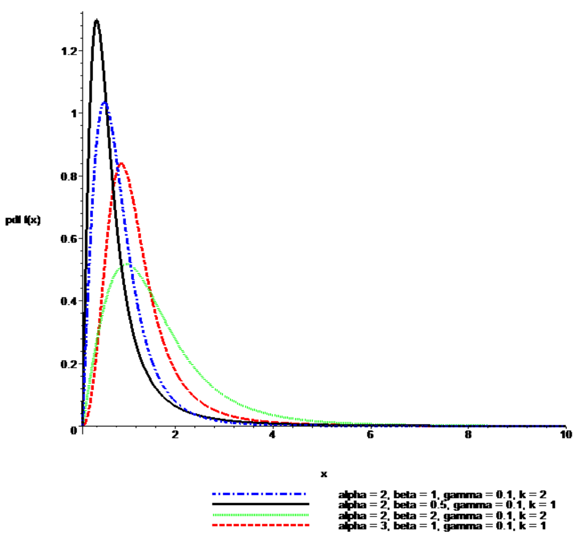

Figure **2.1(a):** Plots of the Burr (4P) pdf 1

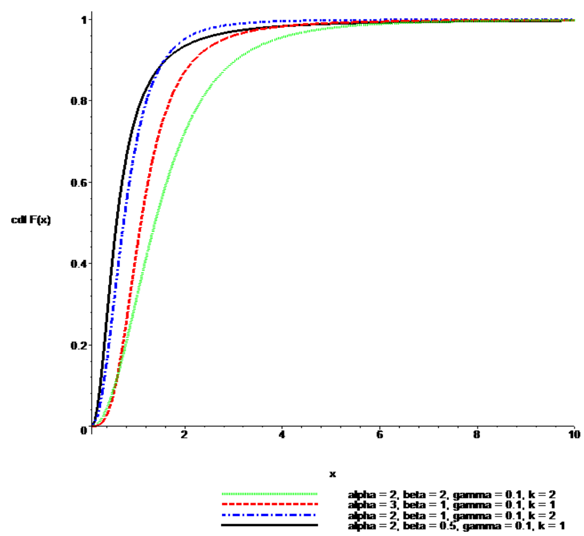

Figure 2.1(b): Plots of the Burr (4P) cdf (2)

The effects of the parameters can easily be seen from these graphs. For example, it is clear from these plots that the Burr (4P) distribution is positively right skewed with longer and heavier right tails for the selected values of the parameters.

2.2 Moments:

It is interesting to note that, after thorough search of literature, we did not find any expression for the j^{th} moment of the Burr (4P) distribution. Therefore, in what follows, we shall first derive the j^{th} moment of the Burr (4P) distribution independently. Then, various moments of the Burr (4P) distribution will be derived.

***j^{th}* Moment of the Burr (4P) Distribution:** For a positive integer j, the j^{th} moment of the random variable X of the Burr (4P) distribution is given by

(2.4) $$E\left(X^j\right) = \int_{\gamma}^{\infty} x^j \frac{\alpha k \left(\frac{x-\gamma}{\beta}\right)^{\alpha-1}}{\beta\left(1+\left(\frac{x-\gamma}{\beta}\right)^{\alpha}\right)^{k+1}} \, dx.$$

Letting $\frac{x-\gamma}{\beta} = u$ in equation (2.4), we have

(2.5) $E\left(X^j\right) = \left(\alpha k \beta^j\right) \int_0^\infty u^{\alpha-1} \left(u + \frac{\gamma}{\beta}\right)^j (1 + u^\alpha)^{-(k+1)} \, du.$

Now, using the binomial expansion for $\left(u + \frac{\gamma}{\beta}\right)^j$ in equation (2.5) and simplifying, we obtain

(2.6) $E\left(X^j\right) = \left(\alpha k \beta^j\right) \sum_{m=0}^{j} \binom{j}{m} \left(\frac{\gamma}{\beta}\right)^{j-m} \int_0^\infty u^{\alpha+m-1} (1 + u^\alpha)^{-(k+1)} \, du.$

Thus, using Gradshteyn and Ryzhik [10, page 295] Eq. 3.251.11, of the j^{th} moment of the Burr (4P) distribution is easily given by

(2.7) $E\left(X^j\right) = \left(k \beta^j\right) \sum_{m=0}^{j} \binom{j}{m} \left(\frac{\gamma}{\beta}\right)^{j-m} B\left(1 + \frac{m}{\alpha}, \, k - \frac{m}{\alpha}\right),$

where $0 \le m \le j < k\alpha$, $j > 0$ (a positive integer), $k > 0, \alpha > 0, \beta > 0, -\infty < \gamma < +\infty$, and $B()$ denotes the complete beta function.

First Moment (or Mean) of the Burr (4P) Distribution: Taking $j = 1$ in (2.7) and simplifying, the first moment (or the mean) of the Burr (4P) distribution is easily given by

(2.8) $E(X) = \gamma + (k\beta) \, B\left(1 + \frac{1}{\alpha}, \, k - \frac{1}{\alpha}\right), \; 0 < \frac{1}{\alpha} < k,$

where $k > 0, \alpha > 0, \beta > 0, -\infty < \gamma < +\infty$.

***j*^th^ (Central) Moment**: The j^{th} (central) moment of the Burr (4P) distribution can easily be derived as follows:

(2.9) $E\,[X - E\,(X)]^j = \int_0^\infty [x - E\,(X)]^j \, f_X(x) \, dx$

$$= \sum_{m=0}^{j} (-1)^m \binom{j}{m} (E(X))^m \, E\left(X^{j-m}\right),$$

where $E\left(X^{j-m}\right)$ and $(E(X))^m$ can be obtained from the equations (2.7) and (2.8) respectively. From the equation (2.9), one can easily obtain the second, third, and higher central moments.

Variance: Taking $j = 2$ in equation (2.9), the variance (or the second central moment) is given by

(2.10) Variance $= E\,[X - E\,(X)]^2 = \int_0^\infty [x - E\,(X)]^2 \, f_X(x) \, dx = E\left[X^2\right] - (E\,[X])^2.$

Coefficients of Skewness and Kurtosis: By taking $j = 3$ and $j = 4$ in the equation (2.9), the third and fourth central moments are respectively given by

(2.11) $E\,[X - E\,(X)]^3 = \sum_{m=0}^{3} (-1)^m \binom{3}{m} (E(X))^m \, E\left(X^{3-m}\right),$

and

(2.12) $E\,[X - E\,(X)]^4 = \sum_{m=0}^{4} (-1)^m \binom{4}{m} (E(X))^m \, E\left(X^{4-m}\right).$

Thus, using equations (2.11) and (2.12), the measure of skewness and kurtosis are respectively given by

(2.13) $\text{Skewness} = \dfrac{\sum_{m=0}^{3} (-1)^m \binom{3}{m} (E(X))^m \, E\left(X^{3-m}\right)}{\left(E\,[X^2] - E\,[X]^2\right)^{\frac{3}{2}}},$

and

(2.14) $\text{Kurtosis} = \dfrac{\sum_{m=0}^{4} (-1)^m \binom{4}{m} (E(X))^m \, E\left(X^{4-m}\right)}{\left(E\,[X^2] - E\,[X]^2\right)^{2}},$

where $E\left(X^{j-m}\right)$ and $(E(X))^m$ can be obtained from the equations (2.7) and (2.8) respectively.

2.3 Burr (3P) Distribution:

It is special case of Burr (4P) distribution, and can be obtained by taking the location parameter $\gamma = 0$ in Burr (4P) distribution, with its probability density function (pdf) and cumulative distribution function (cdf), respectively, given by

$$f(x) = \frac{\alpha k \left(\frac{x}{\beta}\right)^{\alpha-1}}{\beta \left(1 + \left(\frac{x}{\beta}\right)^\alpha\right)^{k+1}}, \text{ and } F(x) = 1 - \left(1 + \left(\frac{x}{\beta}\right)^\alpha\right)^{-k},$$

where k (> 0): shape parameter ; α (> 0): shape parameter ; β (> 0): scale parameter ; and domain: $0 \le x < \infty$.

Remark* 2.1** *Proceeding in the same manner as in* ***sub-sections **2.1** *and* **2.2** *above, we can draw the graphs of the pdf and cdf of Burr* (3P) *distribution for some selected values of the parameters, and also we can derive various moments of the Burr* (3P) *distribution.*

3 Data Analysis and Fitting Distributions

To illustrate the performance of the Burr (3P), Burr (4P), Dagum (3P) and Dagum (4P) distributions, we have considered the breaking stress of 6061-T6 aluminum data, consisting of 200 observations, as reported in Lio and Park [15], and determine their best fit.

3.1 Data Description:

In what follows, the breaking stress of 6061-T6 aluminum data is provided in ***Table* 3.1** below. The descriptive statistics of the data are computed in ***Table* 3.2.** Using the software statdisk (https://www.triolastats.com/statdisk), the histogram, boxplot and the probability plot of the data are drawn in ***Figure* 3.1,** followed by the testing of the normality of the data by Ryan-Joiner Test (Similar to Shapiro-Wilk Test), which is given in ***Table* 3.3**, along with a description of the methods of parameter estimates and goodness of fit tests, which are given subsequently.

***Table* 3.1:** Breaking Stress of 6061-T6 Aluminum

0.2187, 0.2802, 0.3026, 0.3693, 0.4136, 0.4155, 0.4348, 0.4357, 0.4436, 0.5193, 0.5239, 0.5319, 0.5322, 0.5355, 0.5365, 0.5459, 0.5653, 0.5654, 0.5680, 0.5950, 0.6235, 0.6270, 0.7027, 0.7174, 0.7382, 0.7527, 0.7666, 0.7666, 0.7707, 0.7716, 0.7735, 0.7842, 0.8068, 0.8069, 0.8156, 0.8168, 0.8243, 0.8541, 0.8861, 0.8899, 0.9028, 0.9263, 0.9522, 0.9569, 0.9583, 0.9660, 0.9880, 0.9917, 1.0150, 1.0380, 1.0440, 1.0470, 1.0760, 1.0800, 1.0830, 1.1040, 1.1140, 1.1150, 1.1170, 1.1210, 1.1260, 1.1400, 1.1420, 1.1450, 1.1450, 1.1520, 1.1550, 1.1570, 1.1590, 1.1670, 1.1730, 1.1860, 1.1880, 1.1900, 1.1980, 1.2000, 1.2090, 1.2160, 1.2210, 1.2210, 1.2260, 1.2470, 1.2470, 1.2650, 1.2900, 1.2970, 1.3000, 1.3080, 1.3080, 1.3160, 1.3200, 1.3360, 1.3370, 1.3380, 1.3410, 1.3430, 1.3440, 1.3520, 1.3580, 1.3690, 1.3840, 1.3870, 1.3870, 1.3870, 1.3910, 1.3990, 1.4090, 1.4170, 1.4300, 1.4320, 1.4360, 1.4360, 1.4630, 1.4730, 1.4890, 1.4900, 1.4940, 1.5040, 1.5060, 1.5110, 1.5130, 1.5140, 1.5180, 1.5260, 1.5310, 1.5510, 1.5560, 1.5750, 1.5860, 1.5870, 1.5920, 1.5920, 1.5950, 1.6030, 1.6030, 1.6100, 1.6150, 1.6310, 1.6390, 1.6460, 1.6490, 1.6500, 1.6510, 1.6690, 1.6800, 1.6830, 1.6880, 1.6890, 1.6890, 1.7080, 1.7260, 1.7360, 1.7540, 1.7670, 1.7790, 1.8260, 1.8450, 1.8570, 1.8760, 1.8850, 1.9000, 1.9070, 1.9460, 1.9810, 2.0220, 2.0280, 2.0400, 2.0510, 2.0660, 2.1080, 2.1130, 2.1900, 2.2010, 2.3440, 2.3460, 2.3460, 2.4700, 2.5640, 2.6140, 2.6250, 2.9400, 3.0470, 3.0870, 3.1030, 3.1470, 3.5950, 3.6320, 3.8360, 3.9910, 4.2170, 4.4480, 4.4740, 4.5500, 4.5620, 5.0100, 5.2110, 5.8540, 5.9990, 7.2170, 8.5320

***Table* 3.2:** Descriptive Statistics of the Breaking Stress data

Statistic	**Value**	**Statistic**	**Value**	**Percentile**	**Value**
Sample Size	200	Kurtosis	13.051	Min	0.2187
Range	8.3133	Mode	1.3870	5%	0.5216
Mean	1.6234	Midrange	4.37535	10%	0.60925
Variance	1.3699			25% (Q1)	1.041
Std. Deviation	1.1704			50% (Median)	1.3765
Coef. of Variation	0.72097			75% (Q3)	1.717
Std. Error	0.08276			90%	2.7825
Skewness	2.7615			95%	4.3325
Excess Kurtosis	10.005			Max	8.532

***Table* 3.3:** Ryan-Joiner Test of Normality Assessment

Ryan-Joiner Test
Test statistic, Rp: 0.849
Critical value for 0.05 significance level: 0.993
Critical value for 0.01 significance level: 0.99
Reject normality with a 0.05 significance level.
Reject normality with a 0.01 significance level.
Number of data values below Q1 by more than 1.5 IQR: 0
Number of data values above Q3 by more than 1.5 IQR: 20

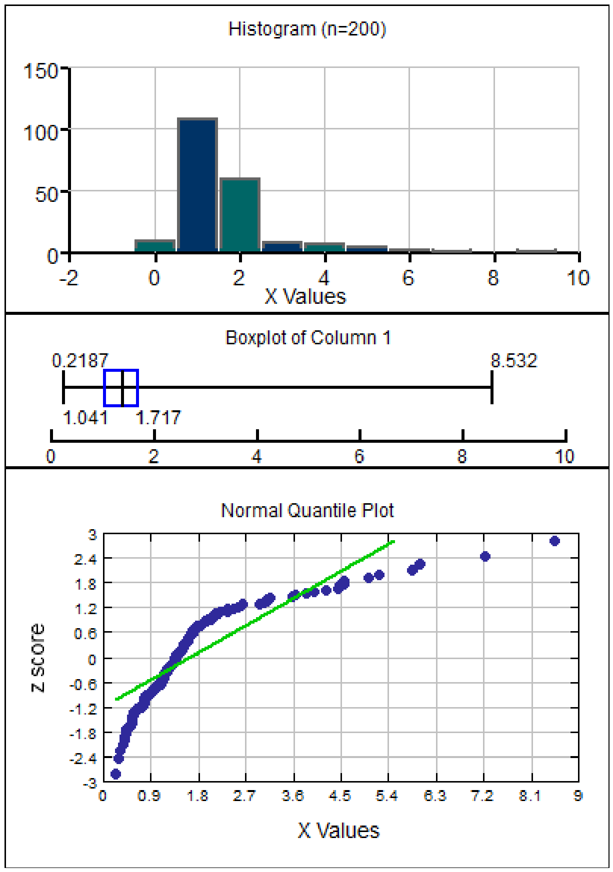

***Figure* 3.1:** Histogram, Boxplot and the Probability Plot of the Breaking Stress Data

From ***Table* 3.3** of Ryan-Joiner Test of Normality Assessment and ***Figure* 3.1** (for the histogram, boxplot and the probability plot), it is obvious that the shape of the breaking stress of 6061-T6 aluminum data is not normally distributed, instead of skewed to the right. This is also confirmed from the skewness (2.7615) and kurtosis (13.051) of the breaking stress of 6061-T6 aluminum data as computed in ***Table* 3.2.**

3.2 Estimation of parameters:

In what follows, we provide the estimation of the parameters of the Burr (3P), Burr (4P), Dagum (3P) and Dagum (4P) distributions by the method of moment and the method of maximum likelihood.

3.2.1 The Method of Moments:

If $\{X_i\}_{i=1}^{n}$ be an *iid* sample from a distribution with a m-dimensional parameter vector φ, then, according to the method of moment (MOM), the estimator $\tilde{\varphi}$ is the solution of the following system of equations:

$$(3.1)\quad E_{\tilde{\varphi}}\left(X^j\right) = \frac{\sum_{i=1}^{n} X_i^j}{n}, \quad j = 1, 2, 3, \ldots, m.$$

Thus, using the above-mentioned definition of MOM, we can obtain the respective moments from the equation (3.1) of the j[th] moment, $E\left(X^j\right)$, of the Burr (3P), Burr (4P), Dagum (3P) and Dagum (4P) distributions by taking the respective values of j, and evaluating the respective expressions of the respective moments numerically. Then, the moment estimations of the respective parameters of the Burr (3P), Burr (4P), Dagum (3P) and Dagum (4P) distributions can be determined by solving the system of respective equations thus obtained by Newton-Raphson's iteration method, and using some computer packages such as Maple, or Mathematica, or R, or MathCAD, or other software.

3.2.2 The Method of Maximum Likelihood:

Given a sample $\{x_i\}$, $i = 1, 2, 3, \ldots, n$, the likelihood functions of the respective pdf's of the Burr (3P), Burr (4P), Dagum (3P) and Dagum (4P) distributions are given by $L = \prod_{i=1}^{n} f(x_i)$. The objective of the likelihood function approach is to determine those values of the parameters that maximize the function L. Suppose $R = \ln(L) = \sum_{i=1}^{n} \ln[f(x_i)]$..Then, upon differentiation, the maximum likelihood estimates *(MLE)* of the respective parameters of the Burr (3P), Burr (4P), Dagum (3P) and Dagum (4P) distributions can be obtained by solving the respective maximum likelihood system of equations, applying the Newton-Raphson's iteration method and using some computer packages such as Maple, or Mathematica, or R, or MathCAD14, or other software.

3.3 Goodness-of-fit tests:

Since fitting of a probability distribution to of the breaking stress data may be helpful in predicting the probability or forecasting the frequency of occurrence of the breaking stress, this suggests that the breaking stress could possibly be modeled by some skewed distributions. As such we have tested the fitting of the Burr (3P), Burr (4P), Dagum (3P) and Dagum (4P) distributions based on their goodness of fit to the breaking stress of 6061-T6 aluminum data (***Table*** **3.1**). For this, we have used the Easyfit software for estimating the parameters of these distributions (http://www.mathwave.com/easyfit-distribution-fitting.html), which are provided in the ***Table*** **3.4.** The goodness of fit (GOF) tests, namely,

- Chi-Squared test
- Kolmogorov-Smirnov
- Anderson-Darling

are provided in the ***Tables*** **3.5, 3.6** and **3.7** respectively. For GOF tests, see, for example, Hogg and Tanis (2006), among others

Table **3.4:** Estimation of the Parameters

#	Distributions	Parameters
1	Burr (4P), with pdf as in Eq. (1).	k = 0.7002, α = 4.3802, β = 1.4014, γ = - 0.22955
2	Burr (3P), with pdf as given in sub-section 2.2.	k = 0.88229, α = 3.3412, β = 1.2771
3	Dagum (3P), with pdf as $f(x) = \frac{\alpha k \left(\frac{x}{\beta}\right)^{\alpha k - 1}}{\beta \left(1 + \left(\frac{x}{\beta}\right)^{\alpha}\right)^{k+1}}$, see Dagum [6, 7].	k = 0.94777, α = 3.2545, β = 1.3825
4	Dagum (4P), with pdf as $f(x) = \frac{\alpha k \left(\frac{x-\gamma}{\beta}\right)^{\alpha k - 1}}{\beta \left(1 + \left(\frac{x-\gamma}{\beta}\right)^{\alpha}\right)^{k+1}}$, see Dagum [6, 7].	k = 0.77762, α = 3.1754, β = 1.3882, γ = 0.11121

***Table* 3.5:** Comparison Criteria / Ranking of Fitted Distributions (P-Values and Test Statistics Analysis) (Based on the Chi-Square Test for Goodness-of-Fit at the Level of Significance = 0.05)

	Burr (4P) (Rank 1)	**Dagum (3P) (Rank 2)**	**Burr (3P) (Rank 3)**	**Dagum (4P) (Rank 4)**
Test Statistic	7.8209	9.609	10.277	10.926
Critical Value	14.067	14.067	14.067	14.067
P-Value	0.34865	0.21184	0.17341	0.14188

***Table* 3.6:** Comparison Criteria / Ranking of Fitted Distributions (P-Values and Test Statistics Analysis) (Based on the Kolmogorov Smirnov Test at the Level of Significance = 0.05)

		Kolmogorov Smirnov			
#	**Distribution**	**Test Statistic**	**Critical Value**	**P-Value**	**Rank**
1	Burr (4P)	0.07091	0.09603	0.25475	1
2	Burr (3P)	0.07244	0.09603	0.23316	2
3	Dagum (3P)	0.07307	0.09603	0.22475	3
4	Dagum (4P)	0.0735	0.09603	0.21913	4

***Table* 3.7:** Comparison Criteria / Ranking of Fitted Distributions (Test Statistics Analysis) (Based on the Anderson-Darling Goodness-of-Fit Test at the Level of Significance = 0.05)

		Anderson-Darling		
#	**Distribution**	**Test Statistic**	**Critical Value**	**Rank**
1	Burr (4P)	1.2487	2.5018	1
2	Dagum (3P)	1.3759	2.5018	2
3	Burr (3P)	1.3852	2.5018	3
4	Dagum (4P)	1.4157	2.5018	4

For the parameters estimated in ***Table* 3.4**, the probability density functions (pdf's) of the Burr (3P), Burr (4P), Dagum (3P) and Dagum (4P) distributions respectively have been superimposed on the histogram of the breaking stress of 6061-T6 aluminum data, which is provided in ***Figure* 3.2** below. For these distributions, we have also plotted the cumulative distribution function (cdf's), and probability difference, in ***Figures* 3.3** and **3.4** respectively.

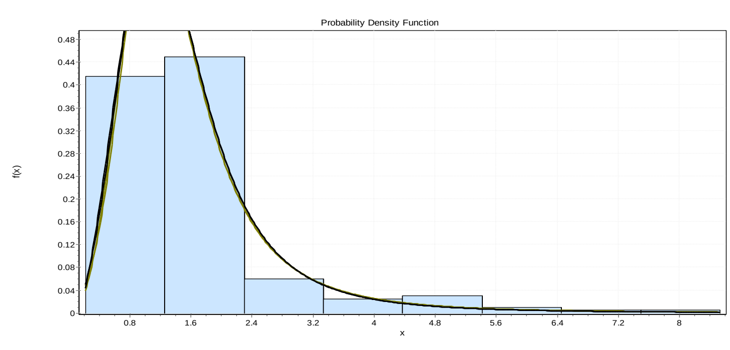

***Figure* 3.2:** Plots of the pdf's of the Fitted Burr (3P), Burr (4P), Dagum (3P) and Dagum (4P) Distributions to the Breaking Stress of 6061-T6 Aluminum Data

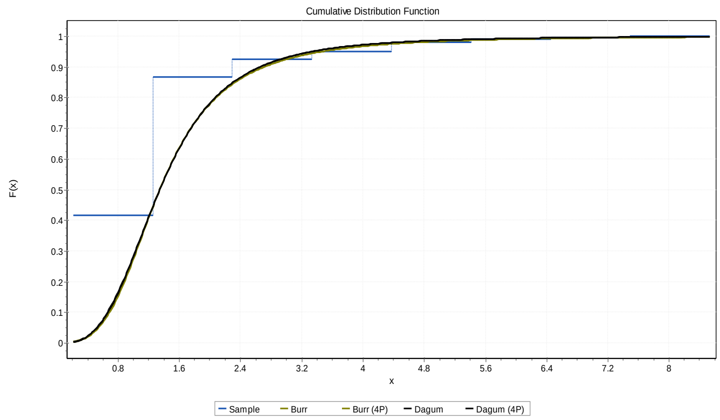

***Figure* 3.3:** Plots of the cdf's of the Fitted Burr (3P), Burr (4P), Dagum (3P) and Dagum (4P) Distributions to the Breaking Stress of 6061-T6 Aluminum Data

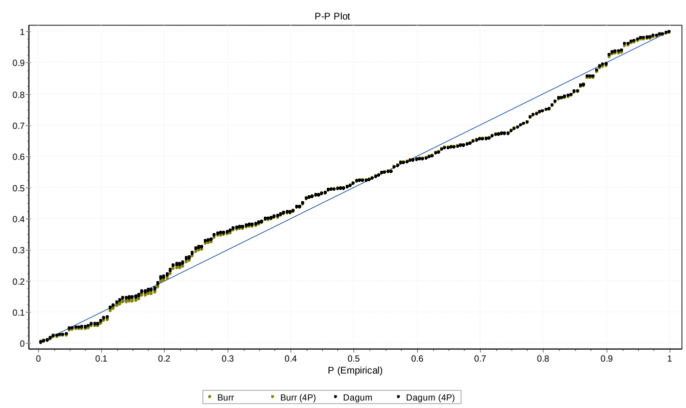

***Figure* 3.4:** P-P Plots of the Fitted Burr (3P), Burr (4P), Dagum (3P) and Dagum (4P) Distributions to the Breaking Stress of 6061-T6 Aluminum Data

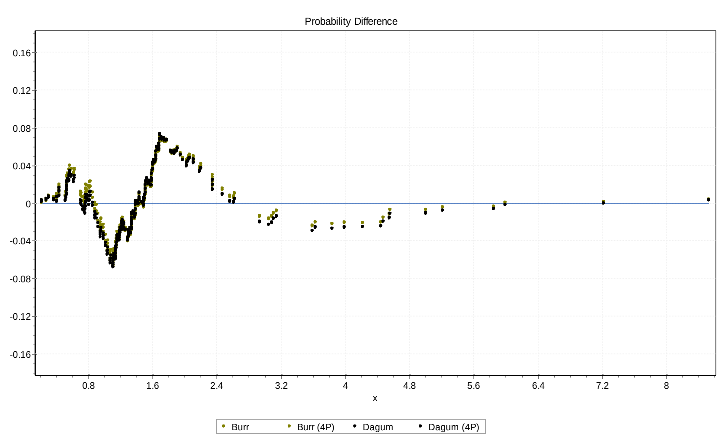

***Figure* 3.5:** Probability Difference of the Fitted Burr (3P), Burr (4P), Dagum (3P) and Dagum (4P) Distributions to the Breaking Stress of 6061-T6 Aluminum Data

3.4 Results Discussions

- From ***Table*** **3.3** of Ryan-Joiner Test of Normality Assessment and ***Figure*** **3.1** (for the histogram, boxplot and the probability plot of the data respectively), it is obvious that the shape of the breaking stress of 6061-T6 aluminum data is skewed to the right. This is also confirmed from the skewness (2.7615) and kurtosis (13.051) as computed in ***Table*** **3.2.**
- Based on the Chi-Squared test for goodness-of-fit, using the P-values and test statistics analysis, as provided in ***Table*** **3.5**, Burr (4P) distribution was found to be the best fit (Rank 1) for the breaking stress data, followed by the Dagum (3P) (Rank 2), Burr (4P) (Rank 3) and Dagum (4P) (Rank 4) distributions.
- From the Kolmogorov-Smirnov and Anderson-Darling GOF tests as provided in ***Tables*** **3.6** and **3.7** respectively, we observed that the Burr (4P) is the best fit amongst the four continuous probability distributions to the breaking stress data, since it has the lowest test statistic.
- The effects of the parameters can also be easily seen from the plots of the pdf's of the fitted Burr (3P), Burr (4P), Dagum (3P) and Dagum (4P) distributions to the breaking stress data in ***Figure*** **3.2**. For example, it is clear from these plots that the above-said distributions are positively right skewed with longer and heavier right tails for the estimated values of the parameters.
- The ***Figure*** **3.4** displays the P-P plot of the empirical cdf values plotted against the theoretical (fitted) cdf values. It is observed that the graph points fall approximately along on the diagonal line implying that the Burr (3P), Burr (4P), Dagum (3P) and Dagum (4P) distributions fit reasonably well to the observed data.
- The ***Figure*** **3.5** displays the probability difference graph, which is defined as a plot of the difference between the empirical cumulative distribution function and the fitted cdf. It is well-known that if the value of the maximum absolute difference is less than 0.05 (or 5%), we may consider the fit to be good. If the maximum absolute difference value is less than 0.01 (or 1%), then the fitting of the distributions are considered to be very good. These fact also confirmed from the probability difference plots of the fitted Burr (3P), Burr (4P), Dagum (3P) and Dagum (4P) distributions to the breaking stress data in ***Figure*** **3.5.**

4 Some Concluding Remarks

As we pointed out above, the strength or breaking stress data such as 6061-T6 aluminum or carbon fiber or composite tensile strengths are fundamental issues in many problems of quality and reliability engineering processes and designs. The statistical treatment of such data is an important aspect of their analysis and interpretation, and is therefore very crucial, and can play an important role in many studies quality and reliability engineering processes and designs. Fitting of a probability distribution to the breaking stress of 6061-T6 aluminum data may be helpful in predicting the probability or forecasting the frequency of occurrence of the breaking stress of 6061-T6 aluminum, and planning beforehand.

Motivated by the importance of such studies, in this paper, we have investigated the goodness of fit of the Burr (3P), Burr (4P), Dagum (3P) and Dagum (4P) distributions to a random sample of 200 observations of the breaking stress of 6061-T6 aluminum data set to determine their applicability and best fit to these data based on the Kolmogorov-Smirnov, Anderson-Darling, and Chi-Squared Goodness-of-Fit Tests. Based on the Chi-Squared test for goodness-of-fit, using the P-values and test statistics analysis, Burr (4P) distribution was found to be the best fit (Rank 1) to the breaking stress of 6061-T6 aluminum data. Moreover, Burr (4P) distribution is the best fit amongst the four continuous probability distributions to the breaking stress of 6061-T6 aluminum data based on the Kolmogorov-Smirnov and Anderson-Darling Goodness-of-Fit Tests, since it has the lowest test statistic.

It is hoped that this study will be helpful in many problems of quality and reliability engineering processes and designs. One can also consider of developing bootstrap control charts for the percentiles of the above-said distributions, which is an important area of studies in quality and reliability engineering.

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