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GROWTH ANALYSIS OF COMPOSITE *P*-ADIC ENTIRE FUNCTIONS FROM THE VIEW POINT OF RELATIVE (p, q)-TH ORDER AND RELATIVE (p, q)-TH TYPE

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Abstract

Suppose $\mathbb{K}$ be a complete ultrametric algebraically closed field and suppose $\mathcal{A}(\mathbb{K})$ be the $\mathbb{K}$-algebra of entire functions on $\mathbb{K}$. In this paper we study some growth properties of composite p-adic entire functions on the basis of their relative (p, q)-th order, relative (p, q)-th type and relative (p, q)-th weak type.
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1 Introduction and Definitions

Let us consider $\mathbb{K}$ be an algebraically closed field of characteristic 0, complete with respect to a p-adic absolute value $|\cdot|$ (example $\mathbb{C}_p$). For any $\alpha \in \mathbb{K}$ and $R \in]0, +\infty[$, the closed disk $\{x \in \mathbb{K} : |x-\alpha| \leq R\}$ and the open disk $\{x \in \mathbb{K} : |x-\alpha| < R\}$ are denoted by $d(\alpha, R)$ and $d(\alpha, R^-)$ respectively. Also $C(\alpha, r)$ denotes the circle $\{x \in \mathbb{K} : |x - a| = r\}$. Moreover $\mathcal{A}(\mathbb{K})$ represent the $\mathbb{K}$-algebra of analytic functions in $\mathbb{K}$ i.e. the set of power series with an infinite radius of convergence. For the most comprehensive study of analytic functions inside a disk or in the whole field $\mathbb{K}$, we refer the reader to the books [11, 12, 15, 17]. During the last several years the ideas of p-adic analysis have been studied from different aspects and many important results were gained (see [2] to [10], [13, 14]).

Let $f \in \mathcal{A}(\mathbb{K})$ and $r > 0$, then we denote by $|f|(r)$ the number $\sup\{|f(x)| : |x| = r\}$ where $|\cdot|(r)$ is a multiplicative norm on $\mathcal{A}(\mathbb{K})$. Moreover, if f is not a constant, the $|f|(r)$ is strictly increasing function of r and tends to $+\infty$ with r therefore there exists its inverse function $\widehat{|f|} : (|f(0)|, \infty) \to (0, \infty)$ with $\lim\limits_{s\to\infty} \widehat{|f|}(s) = \infty$.

For $x \in [0, \infty)$ and $k \in \mathbb{N}$, we define $\log^{[k]} x = \log(\log^{[k-1]} x)$ and $\exp^{[k]} x = \exp(\exp^{[k-1]} x)$ where $\mathbb{N}$ be the set of all positive integers. We also denote $\log^{[0]} x = x$ and $\exp^{[0]} x = x$. Throughout the paper, log denotes the Neperian logarithm. Further we assume that throughout the present paper p, q, m, n and l always denote positive integers. Now taking this into account the (p, q)-th order and (p, q)-th lower order of an entire function $f \in \mathcal{A}(\mathbb{K})$ are defined as follows:

***Definition* 1.1** [5] *Let $f \in \mathcal{A}(\mathbb{K})$ and $p, q \in \mathbb{N}$. Then the (p, q)-th order and (p, q)-th lower order of f are respectively defined as:*

$$\rho^{(p,q)}(f) = \limsup_{r\to\infty} \frac{\log^{[p]} |f|(r)}{\log^{[q]} r} \text{ and } \lambda^{(p,q)}(f) = \liminf_{r\to\infty} \frac{\log^{[p]} |f|(r)}{\log^{[q]} r}.$$

Definition 1.1 avoids the restriction $p \geq q$ of the original definition of (p, q)-th order (respectively (p, q)-th lower order) of entire functions introduced by Juneja et al. [16] *in complex context.*

When $q = 1$, we get the definitions of generalized order and generalized lower order of an entire function $f \in \mathcal{A}(\mathbb{K})$ which symbolize as $\rho^{(p)}(f)$ and $\lambda^{(p)}(f)$ respectively. If $p = 2$ and $q = 1$ then we write $\rho^{(2,1)}(f) = \rho(f)$ and $\lambda^{(2,1)}(f) = \lambda(f)$ where $\rho(f)$ and $\lambda(f)$ are respectively known as order and lower order of $f \in \mathcal{A}(\mathbb{K})$ introduced by Boussaf et al. [2].

In this connection we just introduce the following definition:

***Definition* 1.2** *An entire function $f \in \mathcal{A}(\mathbb{K})$ is said to have index-pair (p, q) if $b < \rho^{(p,q)}(f) < \infty$ and $\rho^{(p-1,q-1)}(f)$ is not a nonzero finite number, where $b = 1$ if $p = q$ and $b = 0$ for otherwise. Moreover if $0 < \rho^{(p,q)}(f) < \infty$, then*

$$\begin{cases} \rho^{(p-n,q)}(f) = \infty & \text{for } n < p, \\ \rho^{(p,q-n)}(f) = 0 & \text{for } n < q, \\ \rho^{(p+n,q+n)}(f) = 1 & \text{for } n = 1, 2, \cdots . \end{cases}.$$

Similarly for $0 < \lambda^{(p,q)}(f) < \infty$, one can easily verify that

$$\begin{cases} \lambda^{(p-n,q)}(f) = \infty & \text{for } n < p, \\ \lambda^{(p,q-n)}(f) = 0 & \text{for } n < q, \\ \lambda^{(p+n,q+n)}(f) = 1 & \text{for } n = 1, 2, \cdots . \end{cases}.$$

An entire function $f \in \mathcal{A}(\mathbb{K})$ of index-pair (p, q) is said to be of regular (p, q)-th growth if its (p, q)-th order coincides with its (p, q)-th lower order, otherwise f is said to be of irregular (p, q)-th growth.

Next, to compare the growth of entire functions on $\mathbb{K}$ having the same (p, q)-th order, we give the definitions of (p, q)-th type and (p, q)-th lower type in the following manner :

***Definition* 1.3** [5] *Let $f \in \mathcal{A}(\mathbb{K})$. The (p, q)-th type $\sigma^{(p,q)}(f)$ and the (p, q)-th lower type $\overline{\sigma}^{(p,q)}(f)$ of f having finite positive (p, q)-th order $\rho^{(p,q)}(f)$ $(0 < \rho^{(p,q)}(f) < \infty)$ are defined as:*

$$\sigma^{(p,q)}(f) = \limsup_{r\to\infty} \frac{\log^{[p-1]} |f|(r)}{(\log^{[q-1]} r)^{\rho^{(p,q)}(f)}} \text{ and } \overline{\sigma}^{(p,q)}(f) = \liminf_{r\to\infty} \frac{\log^{[p-1]} |f|(r)}{(\log^{[q-1]} r)^{\rho^{(p,q)}(f)}}.$$

***Remark* 1.1** *If $p = 2$ and $q = 1$ then we write $\sigma^{(p,q)}(f) = \sigma(f)$ where $\sigma(f)$ is known as type of $f \in \mathcal{A}(\mathbb{K})$ introduced by Boussaf et al.* [2].

Likewise, to compare the growth of entire functions on $\mathbb{K}$ having the same (p, q)-th lower order, one can also introduce the concepts of (p, q)-th weak type in the following manner :

***Definition* 1.4** [5] *Let $f \in \mathcal{A}(\mathbb{K})$. The (p, q)-th weak type $\tau^{(p,q)}(f)$ of f having finite positive (p, q)-th lower order $\lambda^{(p,q)}(f)$ $(0 < \lambda^{(p,q)}(f) < \infty)$ is defined as :*

$$\tau^{(p,q)}(f) = \liminf_{r\to\infty} \frac{\log^{[p-1]} |f|(r)}{(\log^{[q-1]} r)^{\lambda^{(p,q)}(f)}}.$$

Similarly one may define the growth indicator $\overline{\tau}^{(p,q)}(f)$ of an entire function $f \in \mathcal{A}(\mathbb{K})$ in the following way :

$$\overline{\tau}^{(p,q)}(f) = \limsup_{r\to\infty} \frac{\log^{[p-1]} |f|(r)}{(\log^{[q-1]} r)^{\lambda^{(p,q)}(f)}}, \ 0 < \lambda^{(p,q)}(f) < \infty.$$

The notion of relative order was first introduced by Bernal [1]. *In order to make some progress in the study of p-adic analysis, recently Biswas* [4] *introduce the definition of relative order $\rho_g(f)$ and relative lower order $\lambda_g(f)$ of entire function $f \in \mathcal{A}(\mathbb{K})$ with respect to another entire function $g \in \mathcal{A}(\mathbb{K})$ in the following way:*

$$\rho_g(f) = \limsup_{r\to\infty} \frac{\log \widehat{|g|}(|f|(r))}{\log r} \text{ and } \lambda_g(f) = \liminf_{r\to\infty} \frac{\log \widehat{|g|}(|f|(r))}{\log r}.$$

Further the function $f \in \mathcal{A}(\mathbb{K})$, for which relative order and relative lower order with respect to another function $g \in \mathcal{A}(\mathbb{K})$ are the same is called a function of regular relative growth with respect to g otherwise, f is said to be irregular relative growth.with respect to g.

In the case of relative order, it therefore seems reasonable to define suitably the relative (p, q)-th order of entire function belonging to $\mathcal{A}(\mathbb{K})$. With this in view one may introduce the definition of relative (p, q)-th order $\rho_g^{(p,q)}(f)$ and relative (p, q)-th lower order $\lambda_g^{(p,q)}(f)$ of an entire function $f \in \mathcal{A}(\mathbb{K})$ with respect to another entire function $g \in \mathcal{A}(\mathbb{K})$, in the light of index-pair which are as follows:

***Definition* 1.5** [5] *Let f, $g \in \mathcal{A}(\mathbb{K})$. Also let the index-pairs of f and g are (m, q) and (m, p), respectively. Then the relative (p, q)-th order $\rho_g^{(p,q)}(f)$ and relative (p, q)-th lower order $\lambda_g^{(p,q)}(f)$ of f with respect to g are defined as*

$$\rho_g^{(p,q)}(f) = \limsup_{r\to\infty} \frac{\log^{[p]} \widehat{|g|}(|f|(r))}{\log^{[q]} r} \text{ and } \lambda_g^{(p,q)}(f) = \liminf_{r\to\infty} \frac{\log^{[p]} \widehat{|g|}(|f|(r))}{\log^{[q]} r}.$$

In order to refine the above growth scale, now we introduce the definitions of an another growth indicator, called relative (p,q)-th type and relative (p,q)-th lower type respectively of entire function belonging to $\mathcal{A}(\mathbb{K})$ with respect to another entire function belonging to $\mathcal{A}(\mathbb{K})$ in the light of their index-pair which are as follows:

***Definition* 1.6** [5] *Let $f, g \in \mathcal{A}(\mathbb{K})$. Also let the index-pairs of f and g are (m,q) and (m,p), respectively. The relative (p,q)-th type and relative (p,q)-th lower type of f with respect to g having finite positive relative (p,q)-th order $\rho_g^{(p,q)}(f)$ $(0 < \rho_g^{(p,q)}(f) < \infty)$ are defined as:*

$$\sigma_g^{(p,q)}(f) = \limsup_{r\to\infty} \frac{\log^{[p-1]} \widehat{|g|}(|f|(r))}{(\log^{[q-1]} r)^{\rho_g^{(p,q)}(f)}} \text{ and } \overline{\sigma}_g^{(p,q)}(f) = \liminf_{r\to\infty} \frac{\log^{[p-1]} \widehat{|g|}(|f|(r))}{(\log^{[q-1]} r)^{\rho_g^{(p,q)}(f)}}.$$

Analogously, to determine the relative growth of two entire functions belonging to $\mathcal{A}(\mathbb{K})$ and having same non zero finite relative (p,q)-th lower order with respect to another entire function belonging to $\mathcal{A}(\mathbb{K})$, one can introduce the definition of relative (p,q)-th weak type of an entire function $f \in \mathcal{A}(\mathbb{K})$ with respect to another entire function $g \in \mathcal{A}(\mathbb{K})$ of finite positive relative (p,q)-th lower order $\lambda_g^{(p,q)}(f)$ in the following way:

***Definition* 1.7** [5] *Let $f, g \in \mathcal{A}(\mathbb{K})$. Also let the index-pair of f and g are (m,q) and (m,p), respectively. The relative (p,q)-th weak type $\tau_g^{(p,q)}(f)$ and the growth indicator $\overline{\tau}_g^{(p,q)}(f)$ of f with respect to g having finite positive relative (p,q)-th lower order $\lambda_g^{(p,q)}(f)$ $(0 < \lambda_g^{(p,q)}(f) < \infty)$ are defined as:*

$$\overline{\tau}_g^{(p,q)}(f) = \limsup_{r\to\infty} \frac{\log^{[p-1]} \widehat{|g|}(|f|(r))}{(\log^{[q-1]} r)^{\lambda_g^{(p,q)}(f)}} \text{ and } \tau_g^{(p,q)}(f) = \liminf_{r\to\infty} \frac{\log^{[p-1]} \widehat{|g|}(|f|(r))}{(\log^{[q-1]} r)^{\lambda_g^{(p,q)}(f)}}.$$

The main purpose of this paper is to ascertain some results associated to the growth properties of composite p-adic entire functions on the basis of relative (p,q)-th order, relative (p,q)-th type and relative (p,q)-th weak type.

2 Lemma

In this section we present the following lemma which can be found in [2] or [3] and will be needed in the sequel.

Lemma 2.1. Let $f, g \in \mathcal{A}(\mathbb{K})$. Then for all sufficiently large positive numbers of r the following equality holds

$$|f \circ g|(r) = |f|(|g|(r)).$$

3 Main Results

In this section we present the main results of the paper.

***Theorem* 3.1** *Let f, g, $h \in \mathcal{A}(\mathbb{K})$. Also let $0 < \lambda_h^{(p,q)}(f) \le \rho_h^{(p,q)}(f) < \infty$, $\lambda^{(m,n)}(g) > 0$ and γ be a positive continuous function defined on $[0,+\infty)$ increasing to ∞ as $r \to \infty$. Then for any number $\alpha \ge 0$,*

$$\lim_{r\to\infty} \frac{\log^{[p]} \widehat{|h|}(|f \circ g|(\exp^{[n-1]} r))}{\{\log^{[p]} \widehat{|h|}(|f|(\exp^{[q]} \gamma(r)))\}^{1+\alpha}} = \infty \text{ when } q < m \text{ and } \lim_{r\to+\infty} \frac{\log \gamma(r)}{\log r} = 0,$$

and

$$\lim_{r\to\infty} \frac{\log^{[p]} \widehat{|h|}(|f \circ g|(\exp^{[n-1]} r))}{\{\log^{[p]} \widehat{|h|}(|f|(\exp^{[q]} \gamma(r)))\}^{1+\alpha}} = \infty \text{ when } q \geqslant m \text{ and } \lim_{r\to+\infty} \frac{\log \gamma(r)}{\log^{[q-m+1]} r} = 0.$$

Proof. From the definition of $\rho_h^{(p,q)}(f)$, it follows for all sufficiently large positive numbers of r that

$$(3.1)\quad \log^{[p]} \widehat{|h|}(|f|(\exp^{[q]} \gamma(r))) \le (\rho_h^{(p,q)}(f) + \varepsilon)\gamma(r).$$

Since $\widehat{|h|}(r)$ is an increasing function of r, it follows from ***Lemma* 2.1** and for all sufficiently large positive numbers of r that

$$(3.2)\quad \log^{[p]} \widehat{|h|}(|f \circ g|(\exp^{[n-1]} r)) \ge (\lambda_h^{(p,q)}(f) - \varepsilon) \log^{[q]} |g|(\exp^{[n-1]} r).$$

***Case* I.** Let $q < m$. Then from (3.2) it follows for all sufficiently large positive numbers of r that

$$(3.3)\quad \log^{[p]} \widehat{|h|}(|f \circ g|(\exp^{[n-1]} r)) \ge (\lambda_h^{(p,q)}(f) - \varepsilon) \exp^{[m-q-1]} \log^{[m-1]} |g|(\exp^{[n-1]} r)$$

i.e.,

$$(3.4)\quad \log^{[p]} \widehat{|h|}(|f \circ g|(\exp^{[n-1]} r)) \ge (\lambda_h^{(p,q)}(f) - \varepsilon) \exp^{[m-q-1]} r^{(\lambda^{(m,n)}(g) - \varepsilon)}.$$

***Case* II.** Let $q \geqslant m$. Then from (3.2) we get for all sufficiently large positive numbers of r that

$$(3.5)\quad \log^{[p]} \widehat{|h|}(|f \circ g|(\exp^{[n-1]} r)) \ge (\lambda_h^{(p,q)}(f) - \varepsilon) \log^{[q-m]} \log^{[m]} |g|(\exp^{[n-1]} r)$$

i.e.,

$$\log^{[p]} \widehat{|h|}(|f \circ g|(\exp^{[n-1]} r)) \geq (\lambda_h^{(p,q)}(f) - \varepsilon) \log^{[q-m+1]} r + O(1)$$

i.e.,

$$\log^{[p-1]} \widehat{|h|}(|f \circ g|(\exp^{[n-1]} r)) \geqslant (\log^{[q-m]} r)^{(\lambda_h^{(p,q)}(f)-\varepsilon)} + O(1). \tag{3.6}$$

Now combining (3.1) and (3.4) of ***Case* I** it follows for all sufficiently large positive numbers of r that

$$\frac{\log^{[p]} \widehat{|h|}(|f \circ g|(\exp^{[n-1]} r))}{\{\log^{[p]} \widehat{|h|}(|f|(\exp^{[q]} \gamma(r)))\}^{1+\alpha}} \geq \frac{(\lambda_h^{(p,q)}(f) - \varepsilon) \exp^{[m-q-1]} r^{(\lambda^{(m,n)}(g)-\varepsilon)}}{(\rho_h^{(p,q)}(f) + \varepsilon)^{1+\alpha}\{\gamma(r)\}^{1+\alpha}}.$$

Since $\lim_{r\to+\infty} \frac{\log \gamma(r)}{\log r} = 0$, therefore $\frac{\exp^{[m-q-1]} r^{(\lambda^{(m,n)}(g)-\varepsilon)}}{\{\gamma(r)\}^{1+\alpha}} \to +\infty$ as $r \to +\infty$, then from above it follows that

$$\lim_{r\to\infty} \frac{\log^{[p]} \widehat{|h|}(|f \circ g|(\exp^{[n-1]} r))}{\{\log^{[p]} \widehat{|h|}(|f|(\exp^{[q]} \gamma(r)))\}^{1+\alpha}} = \infty,$$

from which the first part of ***Theorem* 3.1** follows.

Again combining (3.1) and (3.6) of ***Case* II** it follows for all sufficiently large positive numbers of r that

$$\frac{\log^{[p-1]} \widehat{|h|}(|f \circ g|(\exp^{[n-1]} r))}{\{\log^{[p]} \widehat{|h|}(|f|(\exp^{[q]} \gamma(r)))\}^{1+\alpha}} \geq \frac{(\log^{[q-m]} r)^{(\lambda_h^{(p,q)}(f)-\varepsilon)} + O(1)}{(\rho_h^{(p,q)}(f) + \varepsilon)^{1+\alpha}\{\gamma(r)\}^{1+\alpha}}.$$

As $\lim_{r\to+\infty} \frac{\log \gamma(r)}{\log^{[q-m+1]} r} = 0$, so $\frac{(\log^{[q-m]} r)^{(\lambda_h^{(p,q)}(f)-\varepsilon)}}{\{\gamma(r)\}^{1+\alpha}} \to +\infty$ as $r \to +\infty$. Thus it follows from above that

$$\lim_{r\to\infty} \frac{\log^{[p-1]} \widehat{|h|}(|f \circ g|(\exp^{[n-1]} r))}{\{\log^{[p]} \widehat{|h|}(|f|(\exp^{[q]} \gamma(r)))\}^{1+\alpha}} = \infty.$$

This proves the second part of ***Theorem* 3.1**.

Thus ***Theorem* 3.1** follows.

***Remark* 3.1** ***Theorem* 3.1** *is still valid with "superior limit" instead of "limit" if we replace the condition "* $0 < \lambda_h^{(p,q)}(f) \leq \rho_h^{(p,q)}(f) < \infty$*" by "* $0 < \lambda_h^{(p,q)}(f) < \infty$*".*

In the line of ***Theorem* 3.1** one may state the following theorem without proof:

***Theorem* 3.2** *Let* $f, g, h, k \in \mathcal{A}(\mathbb{K})$*. Also let* g *is of finite* (m,n)*-th lower order,* $\lambda_h^{(p,q)}(f) > 0$, $\rho_k^{(l,n)}(g) < \infty$ *and* γ *be a positive continuous function defined on* $[0, +\infty)$ *increasing to* ∞ *as* $r \to \infty$*. Then for any number* $\alpha \geq 0$,

$$\lim_{r\to\infty} \frac{\log^{[p]} \widehat{|h|}(|f \circ g|(\exp^{[n-1]} r))}{\{\log^{[l]} \widehat{|k|}(|g|(\exp^{[n]} \gamma(r)))\}^{1+\alpha}} = \infty \text{ when } q < m \text{ and } \lim_{r\to+\infty} \frac{\log \gamma(r)}{\log r} = 0,$$

and

$$\lim_{r\to\infty} \frac{\log^{[p-1]} \widehat{|h|}(|f \circ g|(\exp^{[n-1]} r))}{\{\log^{[l]} \widehat{|k|}(|g|(\exp^{[n]} \gamma(r)))\}^{1+\alpha}} = \infty \text{ when } q \geqslant m \text{ and } \lim_{r\to+\infty} \frac{\log \gamma(r)}{\log^{[q-m+1]} r} = 0.$$

***Remark* 3.2** *In* ***Theorem* 3.2** *if we take the condition* $\lambda_k^{(l,n)}(g) < \infty$ *instead of* $\rho_k^{(l,n)}(g) < \infty$*, then also* ***Theorem* 3.2** *remains true with "superior limit" in place of " limit ".*

***Theorem* 3.3** *Let* $f, g, h \in \mathcal{A}(\mathbb{K})$*. Also let* $0 < \lambda_h^{(p,q)}(f) \leq \rho_h^{(p,q)}(f) < \infty$, $\rho^{(m,n)}(g) < \infty$ *and* γ *be a positive continuous function defined on* $[0, +\infty)$ *increasing to* ∞ *as* $r \to \infty$*. Then for any number* $\alpha \geq 0$,

$$\lim_{r\to\infty} \frac{\{\log^{[p]} \widehat{|h|}(|f \circ g|(r))\}^{1+\alpha}}{\log^{[p]} \widehat{|h|}(|f|(\exp^{[q]}\{\gamma(r)\}))} = 0 \text{ if } q \geqslant m$$

and

$$\lim_{r\to+\infty} \frac{\{\log^{[p+m-q-1]} \widehat{|h|}(|f \circ g|(r))\}^{1+\alpha}}{\log^{[p]} \widehat{|h|}(|f|(\exp^{[q]} \gamma(r)))} = 0 \text{ if } q < m,$$

where

$$\lim_{r\to\infty} \frac{\log \gamma(r)}{\log r} = \infty.$$

Proof. Since $\widehat{|h|}(r)$ is an increasing function of r, it follows from ***Lemma* 2.1** and for all sufficiently large positive numbers of r that

(3.7) $\log^{[p]} \widehat{|h|}(|f \circ g|(r)) \leqslant (\rho_h^{(p,q)}(f) + \varepsilon) \log^{[q]} |g|(r)$.

Now the following cases may arise :

***Case* I.** Let $q \geqslant m$. Then we get from (3.7) for all sufficiently large positive numbers of r that

(3.8) $\log^{[p]} \widehat{|h|}(|f \circ g|(r)) \leqslant (\rho_h^{(p,q)}(f) + \varepsilon) \log^{[m-1]} |g|(r)$.

Now from the definition of (m, n)-th order of g, for arbitrary positive ε and for all sufficiently large positive numbers of r, we have

$$\log^{[m]} |g|(r) \leqslant (\rho^{(m,n)}(g) + \varepsilon) \log^{[n]} r$$

i.e.,

(3.9) $\log^{[m]} |g|(r) \leq (\rho^{(m,n)}(g) + \varepsilon) \log r$

and

(3.10) $\log^{[m-1]} |g|(r) \leq r^{(\rho^{(m,n)}(g)+\varepsilon)}$.

So from (3.8) and (3.10) it follows for all sufficiently large positive numbers of r that

(3.11) $\log^{[p]} \widehat{|h|}(|f \circ g|(r)) \leqslant (\rho_h^{(p,q)}(f) + \varepsilon) r^{(\rho^{(m,n)}(g)+\varepsilon)}$.

***Case* II**. Let $q < m$. Then we obtain from (3.7) for all sufficiently large positive numbers of r that

(3.12) $\log^{[p]} \widehat{|h|}(|f \circ g|(r)) \leqslant (\rho_h^{(p,q)}(f) + \varepsilon) \exp^{[m-q]} \log^{[m]} |g|(r)$.

Also we get from (3.9) for all sufficiently large positive numbers of r that

$$\exp^{[m-q]} \log^{[m]} |g|(r) \leqslant \exp^{[m-q]} \log r^{(\rho^{(m,n)}(g)+\varepsilon)}$$

i.e.,

(3.13) $\exp^{[m-q]} \log^{[m]} |g|(r) \leqslant \exp^{[m-q-1]} r^{(\rho^{(m,n)}(g)+\varepsilon)}$.

Now from (3.12) and (3.13) we have for all sufficiently large positive numbers of r that

$$\log^{[p]} \widehat{|h|}(|f \circ g|(r)) \leqslant (\rho_h^{(p,q)}(f) + \varepsilon) \exp^{[m-q-1]} r^{(\rho^{(m,n)}(g)+\varepsilon)}$$

i.e.,

(3.14) $\log^{[p+m-q-1]} \widehat{|h|}(|f \circ g|(r)) \leqslant r^{(\rho^{(m,n)}(g)+\varepsilon)} + O(1)$.

Again for all sufficiently large positive numbers of r that

(3.15) $\log^{[p]} \widehat{|h|}(|f|(\exp^{[q]} \gamma(r))) \geq (\lambda_h^{(p,q)}(f) - \varepsilon)\gamma(r)$.

Now if $q \geqslant m$, we obtain from (3.11) and (3.15) for all sufficiently large positive numbers of r that

$$\frac{\{\log^{[p]} \widehat{|h|}(|f \circ g|(r))\}^{1+\alpha}}{\log^{[p]} \widehat{|h|}(|f|(\exp^{[q]} \gamma(r)))} \leqslant \frac{(\rho_h^{(p,q)}(f) + \varepsilon)^{1+\alpha} r^{(\rho^{(m,n)}(g)+\varepsilon)(1+\alpha)}}{(\lambda_h^{(p,q)}(f) - \varepsilon)\gamma(r)}.$$

Since $\lim\limits_{r\to\infty} \frac{\log \gamma(r)}{\log r} = \infty$, therefore $\frac{r^{(\rho^{(m,n)}(g)+\varepsilon)(1+\alpha)}}{\gamma(r)} \to 0$ as $r \to \infty$, then the first part of ***Theorem* 3.3** follows from above.

Further when $q < m$, we get from (3.14) and (3.15) for all sufficiently large positive numbers of r that

$$\frac{\{\log^{[p+m-q-1]} \widehat{|h|}(|f \circ g|(r))\}^{1+\alpha}}{\log^{[p]} \widehat{|h|}(|f|(\exp^{[q]} \gamma(r)))} \leqslant \frac{r^{(\rho^{(m,n)}(g)+\varepsilon)(1+\alpha)}(1 + \frac{O(1)}{r^{(\rho^{(m,n)}(g)+\varepsilon)}})^{1+\alpha}}{(\lambda_h^{(p,q)}(f) - \varepsilon)\gamma(r)}$$

i.e.,

$$\lim_{r\to\infty} \frac{\{\log^{[p+m-q-1]} \widehat{|h|}(|f \circ g|(r))\}^{1+\alpha}}{\log^{[p]} \widehat{|h|}(|f|(\exp^{[q]} \gamma(r)))} = 0,$$

This proves the second part of ***Theorem* 3.3**.

Remark* 3.3** *In* ***Theorem 3.3 *if we take the condition* $\rho_h^{(p,q)}(f) > 0$ *instead of* $\lambda_h^{(p,q)}(f) > 0$*, the theorem remains true with "inferior limit" in place of "limit ".*

***Theorem* 3.4** *Let $f, g, h, k \in \mathcal{A}(\mathbb{K})$. Also let g is of finite (m,n)-th order, $\rho_h^{(p,q)}(f) < +\infty$, $\lambda_k^{(l,n)}(g) > 0$ and γ be a positive continuous function defined on $[0,+\infty)$ increasing to ∞ as $r \to \infty$. Then for any number $\alpha \geq 0$,*

$$\lim_{r\to\infty} \frac{\{\log^{[p]} \widehat{|h|}(|f \circ g|(r))\}^{1+\alpha}}{\log^{[l]} \widehat{|k|}(|g|(\exp^{[n]} \gamma(r)))} = 0 \text{ if } q \geqslant m$$

and

$$\lim_{r\to\infty} \frac{\{\log^{[p+m-q-1]} \widehat{|h|}(|f \circ g|(r))\}^{1+\alpha}}{\log^{[l]} \widehat{|k|}(|g|(\exp^{[n]} \gamma(r)))} = 0 \text{ if } q < m,$$

where

$$\lim_{r\to+\infty} \frac{\log \gamma(r)}{\log r} = \infty.$$

The proof of ***Theorem* 3.4** *would run parallel to that of* ***Theorem* 3.3**. *We omit the details.*

***Remark* 3.4** *In* ***Theorem* 3.4**, *if we take the condition $\rho_k^{(l,n)}(g) > 0$ instead of $\lambda_k^{(l,n)}(g) > 0$,* ***Theorem* 3.4** *remains true with "limit replaced by limit inferior".*

***Theorem* 3.5** *Let $f, g, h \in \mathcal{A}(\mathbb{K})$. Also let $\rho^{(m,n)}(g) < \lambda_h^{(p,q)}(f) \leq \rho_h^{(p,q)}(f) < \infty$. Then*

$$(i) \lim_{r\to\infty} \frac{\log^{[p]} \widehat{|h|}(|f \circ g|(r))}{\log^{[p-1]} \widehat{|h|}(|f|(\exp^{[q-1]} r^A))} = 0 \text{ if } q \geqslant m$$

and

$$(ii) \lim_{r\to\infty} \frac{\log^{[p+m-q-1]} \widehat{|h|}(|f \circ g|(r))}{\log^{[p-1]} \widehat{|h|}(|f|(\exp^{[q-1]} r^A))} = 0 \text{ if } q < m$$

where $A > 0$.

Proof. From the definition of relative (p,q)-th lower order, we get for all sufficiently large positive numbers of r that

$$(3.16)\quad \log^{[p-1]} \widehat{|h|}(|f|(\exp^{[q-1]} r^A)) \geqslant r^{A(\lambda_h^{(p,q)}(f)-\varepsilon)}.$$

As $\rho^{(m,n)}(g) < \lambda_h^{(p,q)}(f)$, we can choose $\varepsilon(>0)$ in such a way that

$$(3.17)\quad \rho^{(m,n)}(g) + \varepsilon < A(\lambda_h^{(p,q)}(f) - \varepsilon).$$

Now if $q \geqslant m$, combining (3.11), (3.16) and in view of (3.17) we obtain for all sufficiently large positive numbers of r that

$$\frac{\log^{[p]} \widehat{|h|}(|f \circ g|(r))}{\log^{[p-1]} \widehat{|h|}(|f|(\exp^{[q-1]} r^A))} \leqslant \frac{(\rho_h^{(p,q)}(f) + \varepsilon) r^{(\rho^{(m,n)}(g)+\varepsilon)}}{r^{A(\lambda_h^{(p,q)}(f)-\varepsilon)}}$$

i.e.,

$$\lim_{r\to\infty} \frac{\log^{[p]} \widehat{|h|}(|f \circ g|(r))}{\log^{[p-1]} \widehat{|h|}(|f|(\exp^{[q-1]} r^A))} = 0.$$

This proves the first part of ***Theorem* 3.5**.

When $q < m$, combining (3.14) and (3.16) it follows for all sufficiently large positive numbers of r that

$$\frac{\log^{[p+m-q-1]} \widehat{|h|}(|f \circ g|(r))}{\log^{[p-1]} \widehat{|h|}(|f|(\exp^{[q-1]} r^A))} \leqslant \frac{r^{(\rho^{(m,n)}(g)+\varepsilon)}\left(1 + \frac{O(1)}{r^{(\rho^{(m,n)}(g)+\varepsilon)}}\right)}{r^{A(\lambda_h^{(p,q)}(f)-\varepsilon)}}.$$

Since $\rho^{(m,n)}(g) < \lambda_h^{(p,q)}(f)$ and $\varepsilon(>0)$ is arbitrary, we get from above

$$\lim_{r\to\infty} \frac{\log^{[p+m-q-1]} \widehat{|h|}(|f \circ g|(r))}{\log^{[p-1]} \widehat{|h|}(|f|(\exp^{[q-1]} r^A))} = 0,$$

which is the second part of ***Theorem* 3.5**.

***Remark* 3.5** *In* ***Theorem* 3.5**, *if we take the condition $\lambda^{(m,n)}(g) < \lambda_h^{(p,q)}(f) \leq \rho_h^{(p,q)}(f) < \infty$ instead of $\rho^{(m,n)}(g) < \lambda_h^{(p,q)}(f) \leq \rho_h^{(p,q)}(f) < \infty$,* ***Theorem* 3.5** *remains true with "inferior limit" in place of "limit ".*

***Theorem* 3.6** *Let $f,\ g,\ h \in \mathcal{A}(\mathbb{K})$. Also let $0 < \lambda_h^{(p,q)}(f) \le \rho_h^{(p,q)}(f) < \infty$ and $\rho^{(m,q)}(g) < \infty$. Then*

$$\limsup_{r\to\infty} \frac{\log^{[p+m-q]} \widehat{|h|}(|f\circ g|(r))}{\log^{[p]} \widehat{|h|}(|f|(r))} \le \frac{\rho^{(m,q)}(g)}{\lambda_h^{(p,q)}(f)},$$

where $m > q$.

Proof. Since $q < m$, we have from (3.7) for all sufficiently large positive numbers of r that

$$\log^{[p+m-q]} \widehat{|h|}(|f\circ g|(r)) \leqslant \log^{[m]} |g|(r) + O(1)$$

i.e.,

$$\frac{\log^{[p+m-q]} \widehat{|h|}(|f\circ g|(r))}{\log^{[p]} \widehat{|h|}(|f|(r))} \le \frac{\log^{[m]} |g|(r) + O(1)}{\log^{[q]} r} \cdot \frac{\log^{[q]} r}{\log^{[p]} \widehat{|h|}(|f|(r))}$$

i.e.,

$$\limsup_{r\to\infty} \frac{\log^{[p+m-q]} \widehat{|h|}(|f\circ g|(r))}{\log^{[p]} \widehat{|h|}(|f|(r))} \le \frac{\rho^{(m,q)}(g)}{\lambda_h^{(p,q)}(f)}.$$

This proves ***Theorem* 3.6**.

In the line of ***Theorem* 3.6** we may state the following theorem without proof.

***Theorem* 3.7** *Let $f,\ g,\ h, k \in \mathcal{A}(\mathbb{K})$. Also let $\rho_h^{(p,q)}(f) < \infty$, $\lambda_k^{(l,n)}(g) > 0$ and $\rho^{(m,n)}(g) < \infty$. Then*

$$\limsup_{r\to\infty} \frac{\log^{[p+m-q]} \widehat{|h|}(|f\circ g|(r))}{\log^{[l]} \widehat{|k|}(|g|(r))} \le \frac{\rho^{(m,n)}(g)}{\lambda_k^{(l,n)}(g)},$$

where $m > n$.

***Theorem* 3.8** *Let $f,\ g,\ h \in \mathcal{A}(\mathbb{K})$. Also let $0 < \lambda_h^{(p,q)}(f) \le \rho_h^{(p,q)}(f) < \infty$ and $\sigma^{(m,n)}(g) < \infty$ where $q = m-1$. Then*

$$\limsup_{r\to\infty} \frac{\log^{[p]} \widehat{|h|}(|f\circ g|(r))}{\log^{[p]} \widehat{|h|}(|f|(\exp^{[q]}(\log^{[n-1]} r)^{\rho^{(m,n)}(g)}))} \le \frac{\sigma^{(m,n)}(g)\cdot \rho_h^{(p,q)}(f)}{\lambda_h^{(p,q)}(f)}.$$

Proof. Since $q = m-1$, we obtain from (3.7) for all sufficiently large positive numbers of r that

$$\log^{[p]} \widehat{|h|}(|f\circ g|(r)) \leqslant (\rho_h^{(p,q)}(f) + \varepsilon) \log^{[m-1]} |g|(r)$$

i.e.,

(3.18) $\log^{[p]} \widehat{|h|}(|f\circ g|(r)) \le (\rho_h^{(p,q)}(f) + \varepsilon)(\sigma^{(m,n)}(g) + \varepsilon)(\log^{[n-1]}(r))^{\rho^{(m,n)}(g)}$.

Now from the definition of $\lambda_h^{(p,q)}(f)$, we get for all sufficiently large positive numbers of r that

(3.19) $\log^{[p]} \widehat{|h|}(|f|(\exp^{[q]}(\log^{[n-1]} r)^{\rho^{(m,n)}(g)})) \ge (\lambda_h^{(p,q)}(f) - \varepsilon)(\log^{[n-1]} r)^{\rho^{(m,n)}(g)}$.

Therefore from (3.18) and (3.19), it follows for all sufficiently large positive numbers of r that

$$\frac{\log^{[p]} \widehat{|h|}(|f\circ g|(r))}{\log^{[p]} \widehat{|h|}(|f|(\exp^{[q]}(\log^{[n-1]} r)^{\rho^{(m,n)}(g)}))} \le \frac{(\rho_h^{(p,q)}(f) + \varepsilon)(\sigma^{(m,n)}(g) + \varepsilon)(\log^{[n-1]}(r))^{\rho^{(m,n)}(g)}}{(\lambda_h^{(p,q)}(f) - \varepsilon)(\log^{[n-1]} r)^{\rho^{(m,n)}(g)}}.$$

Since $\varepsilon(>0)$ is arbitrary, we get from above that

$$\limsup_{r\to\infty} \frac{\log^{[p]} \widehat{|h|}(|f\circ g|(r))}{\log^{[p]} \widehat{|h|}(|f|(\exp^{[q]}(\log^{[n-1]} r)^{\rho^{(m,n)}(g)}))} \le \frac{\sigma^{(m,n)}(g)\cdot \rho_h^{(p,q)}(f)}{\lambda_h^{(p,q)}(f)}.$$

Thus ***Theorem* 3.8** is established.

***Remark* 3.6** *In **Theorem 3.8**, if we will replace "$\sigma^{(m,n)}(g)$" by "$\overline{\sigma}^{(m,n)}(g)$", then **Theorem 3.8** remains valid with "inferior limit" replacing "superior limit".*

*Now we state the following theorem without its proof as it can easily be carried out in the line of **Theorem 3.8**.*

***Theorem* 3.9** *Let $f, g, h, k \in \mathcal{A}(\mathbb{K})$. Also let $\lambda_k^{(l,n)}(g) > 0$, $\rho_h^{(p,q)}(f) < \infty$ and $\sigma^{(m,n)}(g) < \infty$ where $q = m-1$. Then*

$$\limsup_{r\to\infty} \frac{\log^{[p]} \widehat{|h|}(|f\circ g|(r))}{\log^{[l]} \widehat{|k|}(|g|(\exp^{[n]}(\log^{[n-1]} r)^{\rho^{(m,n)}(g)}))} \le \frac{\sigma^{(m,n)}(g)\cdot \rho_h^{(p,q)}(f)}{\lambda_k^{(l,n)}(g)}.$$

Remark* 3.7** *In* ***Theorem* 3.9, if we will replace "$\sigma^{(m,n)}(g)$" by "$\overline{\sigma}^{(m,n)}(g)$", then* ***Theorem* 3.9** *remains valid with "inferior limit" in place of "superior limit".*

Remark* 3.8** *We remark that in* ***Theorem* 3.9, if we will replace the condition "* $\rho_h^{(p,q)}(f) < \infty$*" by "* $\lambda_h^{(p,q)}(f) < \infty$*" , then*

$$(3.20)\quad \liminf_{r\to\infty} \frac{\log^{[p]} |\widehat{h}|(|f\circ g|(r))}{\log^{[l]} |\widehat{k}|(|g|(\exp^{[n]}(\log^{[n-1]} r)^{\rho^{(m,n)}(g)}))} \le \frac{\sigma^{(m,n)}(g)\cdot \lambda_h^{(p,q)}(f)}{\lambda_k^{(l,n)}(g)}.$$

Remark* 3.9** *In* ***Remark* 3.8, if we will replace the conditions "* $\lambda_k^{(l,n)}(g) > 0$ *and* $\lambda_h^{(p,q)}(f) < \infty$*" by "*$\rho_k^{(l,n)}(g) > 0$ *and* $\rho_h^{(p,q)}(f) < \infty$*" respectively, then is need to go the same replacement in right part of* (3.20)*.*

Using the concept of the growth indicator $\overline{\tau}^{(m,n)}(g)$ *of a p-adic entire function g, we may state the subsequent two theorems without their proofs since those can be carried out in the line of* ***Theorem* 3.8** *and* ***Theorem* 3.9** *respectively.*

***Theorem* 3.10** *Let* $f, g, h \in \mathcal{A}(\mathbb{K})$*. Also let* $0 < \lambda_h^{(p,q)}(f) \le \rho_h^{(p,q)}(f) < \infty$ *and* $\overline{\tau}^{(m,n)}(g) < \infty$ *where* $q = m-1$*. Then*

$$\limsup_{r\to\infty} \frac{\log^{[p]} |\widehat{h}|(|f\circ g|(r))}{\log^{[p]} |\widehat{h}|(|f|(\exp^{[q]}(\log^{[n-1]} r)^{\lambda^{(m,n)}(g)}))} \le \frac{\overline{\tau}^{(m,n)}(g)\cdot \rho_h^{(p,q)}(f)}{\lambda_h^{(p,q)}(f)}.$$

Remark* 3.10** *We remark that in* ***Theorem* 3.10, if we will replace the condition "* $0 < \lambda_h^{(p,q)}(f) \le \rho_h^{(p,q)}(f) < \infty$ *and* $\overline{\tau}^{(m,n)}(g) < \infty$*" by "* $0 < \lambda_h^{(p,q)}(f) < \infty$ *or* $0 < \rho_h^{(p,q)}(f) < \infty$ *and* $\sigma^{(m,n)}(g) < \infty$*" , then*

$$\liminf_{r\to\infty} \frac{\log^{[p]} |\widehat{h}|(|f\circ g|(r))}{\log^{[p]} |\widehat{h}|(|f|(\exp^{[q]}(\log^{[n-1]} r)^{\rho^{(m,n)}(g)}))} \le \sigma^{(m,n)}(g).$$

***Theorem* 3.11** *Let* $f, g, h, k \in \mathcal{A}(\mathbb{K})$*. Also let* $\lambda_k^{(l,n)}(g) > 0$, $\rho_h^{(p,q)}(f) < \infty$ *and* $\overline{\tau}^{(m,n)}(g) < \infty$ *where* $q = m-1$*. Then*

$$\limsup_{r\to\infty} \frac{\log^{[p]} |\widehat{h}|(|f\circ g|(r))}{\log^{[l]} |\widehat{k}|(|g|(\exp^{[n]}(\log^{[n-1]} r)^{\lambda^{(m,n)}(g)}))} \le \frac{\overline{\tau}^{(m,n)}(g)\cdot \rho_h^{(p,q)}(f)}{\lambda_k^{(l,n)}(g)}.$$

Further using the notion of (p,q)*-th weak type we may also state the following two theorems without proof because it can be carried out in the line of* ***Theorem* 3.10** *and* ***Theorem* 3.11** *respectively.*

***Theorem* 3.12** *Let* $f, g, h \in \mathcal{A}(\mathbb{K})$*. Also let* $0 < \lambda_h^{(p,q)}(f) \le \rho_h^{(p,q)}(f) < \infty$ *and* $\tau^{(m,n)}(g) < \infty$ *where* $q = m-1$*. Then*

$$\liminf_{r\to\infty} \frac{\log^{[p]} |\widehat{h}|(|f\circ g|(r))}{\log^{[p]} |\widehat{h}|(|f|(\exp^{[q]}(\log^{[n-1]} r)^{\lambda^{(m,n)}(g)}))} \le \frac{\tau^{(m,n)}(g)\cdot \rho_h^{(p,q)}(f)}{\lambda_h^{(p,q)}(f)}.$$

Remark* 3.11** *We remark that in* ***Theorem* 3.12, if we will replace the condition "* $0 < \lambda_h^{(p,q)}(f) \le \rho_h^{(p,q)}(f) < \infty$ *and* $\tau^{(m,n)}(g) < \infty$*" by "* $0 < \lambda_h^{(p,q)}(f) < \infty$ *or* $0 < \rho_h^{(p,q)}(f) < \infty$ *and* $\overline{\tau}^{(m,n)}(g) < \infty$*" , then*

$$\liminf_{r\to\infty} \frac{\log^{[p]} |\widehat{h}|(|f\circ g|(r))}{\log^{[p]} |\widehat{h}|(|f|(\exp^{[q]}(\log^{[n-1]} r)^{\lambda^{(m,n)}(g)}))} \le \overline{\tau}^{(m,n)}(g).$$

***Theorem* 3.13** *Let* $f, g, h, k \in \mathcal{A}(\mathbb{K})$*. Also let* $\lambda_k^{(l,n)}(g) > 0$, $\rho_h^{(p,q)}(f) < \infty$ *and* $\tau^{(m,n)}(g) < \infty$ *where* $q = m-1$*. Then*

$$\liminf_{r\to\infty} \frac{\log^{[p]} |\widehat{h}|(|f\circ g|(r))}{\log^{[l]} |\widehat{k}|(|g|(\exp^{[n]}(\log^{[n-1]} r)^{\lambda^{(m,n)}(g)}))} \le \frac{\tau^{(m,n)}(g)\cdot \rho_h^{(p,q)}(f)}{\lambda_k^{(l,n)}(g)}.$$

Remark* 3.12** *We remark that in* ***Theorem* 3.13, if we will replace the condition "* $\rho_h^{(p,q)}(f) < \infty$ *and* $\tau^{(m,n)}(g) < \infty$*" by "* $\lambda_h^{(p,q)}(f) < \infty$ *and* $\overline{\tau}^{(m,n)}(g) < \infty$*" , then*

$$(3.21)\quad \liminf_{r\to\infty} \frac{\log^{[p]} |\widehat{h}|(|f\circ g|(r))}{\log^{[l]} |\widehat{k}|(|g|(\exp^{[n]}(\log^{[n-1]} r)^{\lambda^{(m,n)}(g)}))} \le \frac{\overline{\tau}^{(m,n)}(g)\cdot \lambda_h^{(p,q)}(f)}{\lambda_k^{(l,n)}(g)}.$$

Remark* 3.13** *In* ***Remark* 3.12, if we will replace the conditions "* $\lambda_k^{(l,n)}(g) > 0$ *and* $\lambda_h^{(p,q)}(f) < \infty$*" by "*$\rho_k^{(l,n)}(g) > 0$ *and* $\rho_h^{(p,q)}(f) < \infty$*" respectively, then is need to go the same replacement in right part of* (3.21)*.*

***Theorem* 3.14** *Let f, g, $h \in \mathcal{A}(\mathbb{K})$. Also let $0 < \rho_h^{(p,q)}(f) < \infty$, $\rho_h^{(p,q)}(f) = \rho^{(m,n)}(g)$, $\sigma^{(m,n)}(g) < \infty$ and $0 < \sigma_h^{(p,q)}(f) < \infty$ where $q = n = m - 1$. Then*

$$(3.22)\quad \liminf_{r\to\infty} \frac{\log^{[p]} \widehat{|h|}(|f \circ g|(r))}{\log^{[p-1]} \widehat{|h|}(|f|(r))} \leq \frac{\rho_h^{(p,q)}(f) \cdot \sigma^{(m,n)}(g)}{\sigma_h^{(p,q)}(f)}.$$

Proof. Since $\rho_h^{(p,q)}(f) = \rho^{(m,n)}(g)$ and $q = m-1$, we have from (3.7) for all sufficiently large positive numbers of r that

$$(3.23)\quad \log^{[p]} \widehat{|h|}(|f \circ g|(r)) \leq (\rho_h^{(p,q)}(f) + \varepsilon)(\sigma^{(m,n)}(g) + \varepsilon)(\log^{[n-1]}(r))^{\rho_h^{(p,q)}(f)}.$$

As $q = n$, then we obtain in view of the definition of $\sigma_h^{(p,q)}(f)$ for a sequence of positive numbers of r tending to infinity that

$$(3.24)\quad \log^{[p-1]} \widehat{|h|}(|f|(r)) \geq (\sigma_h^{(p,q)}(f) - \varepsilon)(\log^{[n-1]}(r))^{\rho_h^{(p,q)}(f)}.$$

Now from (3.23) and (3.24), it follows for a sequence of positive numbers of r tending to infinity that

$$\frac{\log^{[p]} \widehat{|h|}(|f \circ g|(r))}{\log^{[p-1]} \widehat{|h|}(|f|(r))} \leq \frac{(\rho_h^{(p,q)}(f) + \varepsilon)(\sigma^{(m,n)}(g) + \varepsilon)(\log^{[n-1]}(r))^{\rho_h^{(p,q)}(f)}}{(\sigma_h^{(p,q)}(f) - \varepsilon)(\log^{[n-1]}(r))^{\rho_h^{(p,q)}(f)}}.$$

Since $\varepsilon(> 0)$ is arbitrary, it follows from above that

$$\liminf_{r\to\infty} \frac{\log^{[p]} \widehat{|h|}(|f \circ g|(r))}{\log^{[p-1]} \widehat{|h|}(|f|(r))} \leq \frac{\rho_h^{(p,q)}(f) \cdot \sigma^{(m,n)}(g)}{\sigma_h^{(p,q)}(f)}.$$

***Remark* 3.14** *In **Theorem 3.14**, if we will replace the conditions " $\sigma^{(m,n)}(g) < \infty$" and "$0 < \sigma_h^{(p,q)}(f) < \infty$" by "$\overline{\sigma}^{(m,n)}(g) < \infty$" and "$0 < \overline{\sigma}_h^{(p,q)}(f) < \infty$", then is need to go the same replacement in right part of* (3.22). *Also if we replace the conditions $0 < \rho_h^{(p,q)}(f) < \infty$ and $0 < \sigma_h^{(p,q)}(f) < \infty$ of **Theorem 3.14** by $0 < \lambda_h^{(p,q)}(f) \leq \rho_h^{(p,q)}(f) < \infty$ and $0 < \overline{\sigma}_h^{(p,q)}(f) < \infty$ respectively, then*

$$\liminf_{r\to\infty} \frac{\log^{[p]} \widehat{|h|}(|f \circ g|(r))}{\log^{[p-1]} \widehat{|h|}(|f|(r))} \leq \frac{\lambda_h^{(p,q)}(f) \cdot \sigma^{(m,n)}(g)}{\overline{\sigma}_h^{(p,q)}(f)}.$$

*Further if, in **Theorem 3.14**, we replace $\sigma_h^{(p,q)}(f)$ by $\overline{\sigma}_h^{(p,q)}(f)$, then **Theorem 3.14** remains valid with "superior limit" in place of "inferior limit".*

*Now we state the following three theorems without their proofs as those can easily be carried out in the line of **Theorem 3.14**.*

***Theorem* 3.15** *Let f, g, $h \in \mathcal{A}(\mathbb{K})$. Also let $0 < \lambda_h^{(p,q)}(f) \leq \rho_h^{(p,q)}(f) < \infty$, $\lambda_h^{(p,q)}(f) = \lambda^{(m,n)}(g)$, $\overline{\tau}^{(m,n)}(g) < \infty$ and $0 < \overline{\tau}_h^{(p,q)}(f) < \infty$ where $q = n = m - 1$. Then*

$$(3.25)\quad \liminf_{r\to\infty} \frac{\log^{[p]} \widehat{|h|}(|f \circ g|(r))}{\log^{[p-1]} \widehat{|h|}(|f|(r))} \leq \frac{\rho_h^{(p,q)}(f) \cdot \overline{\tau}^{(m,n)}(g)}{\overline{\tau}_h^{(p,q)}(f)}.$$

***Remark* 3.15** *In **Theorem 3.15**, if we will replace the conditions " $\overline{\tau}^{(m,n)}(g) < \infty$" and "$0 < \overline{\tau}_h^{(p,q)}(f) < \infty$" by "$\tau^{(m,n)}(g) < \infty$" and "$0 < \tau_h^{(p,q)}(f) < \infty$", then is need to go the same replacement in right part of* (3.25). *Also if we replace the conditions $0 < \lambda_h^{(p,q)}(f) \leq \rho_h^{(p,q)}(f) < \infty$ and $0 < \overline{\tau}_h^{(p,q)}(f) < \infty$ of **Theorem 3.15** by $0 < \lambda_h^{(p,q)}(f) < \infty$ and $0 < \tau_h^{(p,q)}(f) < \infty$ respectively, then*

$$\liminf_{r\to\infty} \frac{\log^{[p]} \widehat{|h|}(|f \circ g|(r))}{\log^{[p-1]} \widehat{|h|}(|f|(r))} \leq \frac{\lambda_h^{(p,q)}(f) \cdot \overline{\tau}^{(m,n)}(g)}{\tau_h^{(p,q)}(f)}.$$

*Further, in **Theorem 3.15**, if we replace $\overline{\tau}_h^{(p,q)}(f)$ by $\tau_h^{(p,q)}(f)$, then **Theorem 3.15** remains valid with "superior limit" instead of "inferior limit".*

***Theorem* 3.16** *Let f, g, $h \in \mathcal{A}(\mathbb{K})$. Also let $0 < \lambda_h^{(p,q)}(f) \leq \rho_h^{(p,q)}(f) < \infty$, $\lambda_h^{(p,q)}(f) = \rho^{(m,n)}(g)$, $\sigma^{(m,n)}(g) < \infty$ and $0 < \overline{\tau}_h^{(p,q)}(f) < \infty$ where $q = n = m - 1$. Then*

$$(3.26)\quad \liminf_{r\to\infty} \frac{\log^{[p]} \widehat{|h|}(|f \circ g|(r))}{\log^{[p-1]} \widehat{|h|}(|f|(r))} \leq \frac{\rho_h^{(p,q)}(f) \cdot \sigma^{(m,n)}(g)}{\overline{\tau}_h^{(p,q)}(f)}.$$

***Remark* 3.16** *In **Theorem** 3.16, if we will replace the conditions "* $\sigma^{(m,n)}(g) < \infty$*" and "*$0 < \overline{\tau}_h^{(p,q)}(f) < \infty$*" by "*$\overline{\sigma}^{(m,n)}(g) < \infty$*" and "*$0 < \tau_h^{(p,q)}(f) < \infty$*", then is need to go the same replacement in right part of* (3.26). *Also if we replace the conditions* $0 < \lambda_h^{(p,q)}(f) \le \rho_h^{(p,q)}(f) < \infty$ *and* $0 < \overline{\tau}_h^{(p,q)}(f) < \infty$ *of **Theorem** 3.16 by* $0 < \lambda_h^{(p,q)}(f) < \infty$ *and* $0 < \tau_h^{(p,q)}(f) < \infty$ *respectively, then*

$$\liminf_{r\to\infty} \frac{\log^{[p]} |\widehat{h}|(|f \circ g|(r))}{\log^{[p-1]} |\widehat{h}|(|f|(r))} \le \frac{\lambda_h^{(p,q)}(f) \cdot \sigma^{(m,n)}(g)}{\tau_h^{(p,q)}(f)}.$$

*Further if, in **Theorem** 3.16, we replace* $\overline{\tau}_h^{(p,q)}(f)$ *by* $\tau_h^{(p,q)}(f)$*, then **Theorem** 3.16 remains valid with "superior limit" replacing "inferior limit".*

***Theorem* 3.17 *Theorem* 3.17.** *Let* $f, g, h \in \mathcal{A}(\mathbb{K})$*. Also let* $0 < \rho_h^{(p,q)}(f) < \infty$, $\rho_h^{(p,q)}(f) = \lambda^{(m,n)}(g)$, $\overline{\tau}^{(m,n)}(g) < \infty$ *and* $0 < \sigma_h^{(p,q)}(f) < \infty$ *where* $q = n = m - 1$*. Then*

$$\liminf_{r\to\infty} \frac{\log^{[p]} |\widehat{h}|(|f \circ g|(r))}{\log^{[p-1]} |\widehat{h}|(|f|(r))} \le \frac{\rho_h^{(p,q)}(f) \cdot \overline{\tau}^{(m,n)}(g)}{\sigma_h^{(p,q)}(f)}. \tag{3.27}$$

***Remark* 3.17** *In **Theorem** 3.17, if we will replace the conditions "* $\overline{\tau}^{(m,n)}(g) < \infty$*" and "*$0 < \sigma_h^{(p,q)}(f) < \infty$*" by "*$\tau^{(m,n)}(g) < \infty$*" and "*$0 < \overline{\sigma}_h^{(p,q)}(f) < \infty$*", then is need to go the same replacement in right part of* (3.27). *Also if we replace the conditions* $0 < \rho_h^{(p,q)}(f) < \infty$ *and* $0 < \sigma_h^{(p,q)}(f) < \infty$ *of **Theorem** 3.17 by* $0 < \lambda_h^{(p,q)}(f) \le \rho_h^{(p,q)}(f) < \infty$ *and* $0 < \overline{\sigma}_h^{(p,q)}(f) < \infty$ *respectively, then*

$$\liminf_{r\to\infty} \frac{\log^{[p]} |\widehat{h}|(|f \circ g|(r))}{\log^{[p-1]} |\widehat{h}|(|f|(r))} \le \frac{\lambda_h^{(p,q)}(f) \cdot \overline{\tau}^{(m,n)}(g)}{\overline{\sigma}_h^{(p,q)}(f)}.$$

*Further if, in **Theorem** 3.17, we replace* $\sigma_h^{(p,q)}(f)$ *by* $\overline{\sigma}_h^{(p,q)}(f)$*, then **Theorem** 3.17 remains valid with "superior limit" in place of "inferior limit".*

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