***Jñānābha*, Vol. 50(2) (2020), 160-166**
(Dedicated to Honor Dr. R. C. Singh Chandel on His 75th Birth Anniversary Celebrations)

UNIQUENESS OF ALGEBROID FUNCTIONS IN CONNECTION TO NEVANLINNA'S FIVE-VALUE THEOREM

By
Ashok Rathod
Department of Mathematics
KLE Society's G I Bagewadi Arts Science and Commerce College, Nippani - 591237, India
S. H. Naveenkumar
Department of Mathematics
GITAM University, Bangalore-562163, India
Email:ashokmrmaths@gmail.com and naveenkumarsh.220@gmail.com
(*Received : July* 03, 2019 *; Revised: December* 19, 2020)

Abstract

In this paper, we prove a uniqueness theorem for derivatives of algebroid functions which improve and generalize the Navenlinna's five-value theorem for algebroid functions.
2010 Mathematics Subject Classifications: 30D35.
Keywords and phrases: Value Distribution Theory; Nevanlinna theory; algebroid functions, uniqueness.

1 Introduction

The value distribution theory of meromorphic functions was extended to the corresponding theory of algebroid functions by Ullarich [27] and Valiron [28] around 1930, and important results on uniqueness for algebroid functions have been obtained. It is well known that Valiron obtained a famous $4\nu + 1$-valued theorem. The uniqueness theory of algebroid functions is an interesting problem in the value distribution theory. Many researchers like Valiron [28], Baganas [1], He et al.[11, 12] and others have done a lot of work in this area (see [1], [4]-[6], [9]-[26], [30], [31]). In this paper, we discuss a result of Indrajit Lahiri and Rupa Pal [13] on the Nevanlinna's value distribution theory of meromorphic functions for Nevanlinna's five values theorem to algebroid functions

Let $A_\nu(z), A_{\nu-1}(z), ..., A_0(z)$ be analytic functions with no common zeros in the complex plane, then the following equation

(1.1) $A_\nu(z)W^\nu + A_{\nu-1}(z)W^{\nu-1} + ... + A_1(z)W + A_0(z) = 0.$

Then equation (1.1) defines a ν-valued algebroid function $W(z)$ [29].

It is well known from [12] that on the complex plane with a cutting the projection of the critical points of the function W, the Nevanlinna characteristic $T(r, W)$ is defined as

$$T(r, W) = m(r, W) + N(r, W),$$

where

$$m(r, W) = \frac{1}{2\pi\nu} \sum_{j=1}^{\nu} \int_0^{2\pi} \log^+ |w_j(re^{i\theta})|d\theta,$$

$$N(r, W) = \frac{1}{\nu} \int_0^r \frac{n(t,W)-n(0,W)}{t}dt + \frac{n(0,W)}{\nu} \log r,$$

where $w_j(z)(j = 1, 2, 3, ..., \nu)$ is one valued branch $W(z)$ and $n(t, W)$ is the counting function of poles of the function of $W(z)$ in the whole of the complex plane. Let $w_i(z)$ and $m_j(z)$ be one valued branches of two algebroid (μ-valued and ν-valued)functions. It follows from Prokopovich [16] that we consider their quotient in the domain of the complex plane with cutting through the projection of the critical points of both functions. The one-valued branches of the function W/M $(W.M)$ will be defined by w_i/m_j $(w_i.m_j)$, where $1 \le i \le m,\ 1 \le j \le n$. The Nevanlinna characteristic $T(r, W/M)$ or $T(r, W.M)$ is defined as follows

$$m(r, W.M) = \frac{1}{\mu\nu} \sum_{1\le i\le\mu;1\le j\le\nu} m(r, w_i(z).m_j(z))$$

$$= \frac{1}{\mu\nu} \sum_{1\le i\le\mu;1\le j\le\nu} \frac{1}{2\pi} \log^+ |w_i(z).m_j(z)|d\theta$$

$$= \frac{1}{\mu\nu} \left(\nu \sum_{i=1}^{\mu} \frac{1}{2\pi} \log^+ |w_i(z)|d\theta + \mu \sum_{j=1}^{\nu} \frac{1}{2\pi} \log^+ |m_j(z)|d\theta\right)$$

$$= \frac{1}{\mu} \sum_{i=1}^{\mu} \frac{1}{2\pi} \log^+ |w_i(z)| d\theta + \frac{1}{\nu} \sum_{j=1}^{\nu} \frac{1}{2\pi} \log^+ |m_j(z)| d\theta$$
$$= m(r, W) + m(r, M),$$

and

$$N(r, W.M) = \frac{1}{\mu\nu} \int_0^r \frac{n(t, W.M)}{t} dt$$
$$\leq \frac{1}{\mu\nu} \left(\nu \int_0^r \frac{n(t, W.M)}{t} dt + \mu \int_0^r \frac{n(t, W.M)}{t} dt \right)$$
$$= \frac{1}{\mu} \int_0^r \frac{n(t, W.M)}{t} dt + \frac{1}{\nu} \int_0^r \frac{n(t, W.M)}{t} dt$$
$$= N(r, W) + N(r, M).$$

Therefore $T(r, W.M) \leq T(r, W) + T(r, M)$.
Similarly $T(r, W/M) \leq T(r, W) + T(r, M)$.

Let $W(z)$ be a ν-valued algebroid function and $a \in \overline{\mathbb{C}}$ be any complex number. $\overline{E}_{k)}(W = a)$ denotes the set of zeros of $W(z) - a$, whose multiplicities are not greater than k. $\overline{n}_{k)}(W = a)$ denotes the number of distinct zeros of $W(z) - a$ in $|z| \leq r$, whose multiplicities are not greater than k and are counted only once. Similarly, we define the functions $\overline{n}_{(k+1}(r, W = a)$, $\overline{N}_{k)}(r, W = a)$ and $\overline{N}_{(k+1}(r, W = a)$.

***Lemma* 1.1** [9] *Let $W(z)$ be a ν-valued algebroid function and $\{a_j\}_{j=1}^q \subset \overline{\mathbb{C}}$ be q distinct complex numbers and let $\{k_j\}_{j=1}^q \subset \mathbb{N}$ be q positive integers. Then*

$$(q - 2\nu)T(r, W) \leq \sum_{k=1}^{q} \frac{k_j}{k_j+1} \overline{N}_{k_j)} \left(r, W = a_j\right) + \sum_{k=1}^{q} \frac{1}{k_j+1} N(r, W = a_j) + S(r, W),$$
$$\left(q - 2\nu - \sum_{k=1}^{q} \frac{1}{k_j+1}\right) T(r, W) \leq \sum_{k=1}^{q} \frac{k_j}{k_j+1} \overline{N}_{k_j)} \left(r, W = a_j\right) + N(r, W = a_j) + S(r, W).$$

In 2006 *Zu-Xing Xuan and Zong-Sheng Gao* [29] *improved the Nevanlinna Five Value Theorem for algebroid functions in the following manner.*

***Theorem* 1.1** *Let $W(z)$ and $M(z)$ be two ν-valued, non-constant algebroid functions, let a_j ($j = 1, 2, ..., 4\nu + 1$) be $4\nu + 1$ distinct complex numbers in $\overline{\mathbb{C}}$. If*

$$\overline{E}_{2\nu+1)}(a_j, W) = \overline{E}_{2\nu+1)}(a_j, M) \quad (j = 1, 2, ..., 2\nu + 1)$$

and

$$\overline{E}_{2\nu)}(a_j, W) = \overline{E}_{2\nu)}(a_j, M) \quad (j = 1, 2, ..., 4\nu + 1),$$

then $W(z) = M(z)$.

***Definition* 1.1** *For $B \subset \mathbb{A}$ and $a \in \overline{C}$, we denote by $\overline{N}_B(r, \frac{1}{f-a})$ the reduced counting function of those zeros of $f - a$ on $\mathbb{A}$, which belong to the set B.*

In 2018 *Rathod* [20] *proved the following theorem for algebroid functions*

***Theorem* 1.2** *Let $W_1(z)$ and $W_2(z)$ be two ν-valued, non-constant algebroid functions, let a_j ($j = 1, 2, ..., q$) be $q \geq 4\nu+1$ distinct complex numbers or ∞. Suppose that $k_1 \geq k_2 \geq ... \geq k_q, m$ are positive integers or ∞; $1 \leq m \leq q$ and $\delta_j(\geq 0)(j = 1, 2, ..., q)$ are such that*

$$\left(1 + \frac{1}{k_m}\right) \sum_{j=m}^{q} \frac{1}{1+k_j} + 3\nu + \sum_{j=1}^{q} \delta_j < (q - m - 1)\left(1 + \frac{1}{k_m}\right) + m.$$

Let $B_j = \overline{E}_{k_j}(a_j, f) \backslash \overline{E}_{k_j}(a_j, g)$ for $j = 1, 2, ..., q$. If

$$\overline{N}_{B_j}(r, \frac{1}{W_1 - a_j}) \leq \delta_j T(r, W_1)$$

and

$$\liminf_{r\to\infty} \frac{\sum_{j=1}^{q} \overline{N}_{k_j}(r, \frac{1}{W_1-a_j})}{\sum_{j=1}^{q} \overline{N}_{k_j}(r, \frac{1}{W_2-a_j})} > \frac{\nu k_m}{(1 + k_m) \sum_{j=1}^{q} \frac{k_j}{k_j+1} - 2\nu(1 + k_m) + (m - 2\nu - \sum_{j=1}^{q} \delta_j)k_m}$$

then $W_1(z) \equiv W_2(z)$.

2 Main Results

In the paper we wish to further investigate the problem on the Nevanlinna's five value theorem for algebroid functions. To state our main Theorem, we wish to introduce the following ***Lemma* 2.1**.

***Lemma* 2.1** *Let $W(z)$ be a ν-valued algebroid function and $a_1, a_2, ..., a_q$ be $q(\geq 2\nu+1)$ distinct complex numbers. If for a non-negative integer n, $E(0;W) \subseteq E(0, W^{(n)})$, then*

$$(q-2\nu+o(1))T(r,W) \leq \sum_{j=1}^{q} \overline{N}\left(r, \frac{1}{W^{(n)}-a_j}\right).$$

Proof. By Nevanlinna's first fundamental theorem for algebroid functions, we have

$$(2.1)\quad T(r,W) = T\left(r, \frac{1}{W}\right) + O(1)$$

$$\leq N\left(r, \frac{1}{W}\right) + m\left(r, \frac{W^{(n)}}{W}\right) + m\left(r, \frac{1}{W^{(n)}}\right) + O(1)$$

$$\leq N\left(r, \frac{1}{W}\right) + T(r, W^{(n)}) - N\left(r, \frac{1}{W^{(n)}}\right) + S(r,W).$$

By the Nevanlinna's second fundamental theorem for algebroid functions, we get

$$(q-1)\,T(r,W^{(n)}) \leq \overline{N}(r,W^{(n)}) + \sum_{j=1}^{q-1} \overline{N}\left(r, \frac{1}{W^{(n)}-a_j}\right) + \overline{N}\left(r, \frac{1}{W^{(n)}}\right) + S(r,W).$$

Without loss of generality, we may assume that $a_q = 0$. Otherwise a suitable linear transformation is done. Then the above inequality reduces to

$$(2.2)\quad (q-1)\,T(r,W^{(n)}) \leq \overline{N}(r,W^{(n)}) + \sum_{j=1}^{q} \overline{N}\left(r, \frac{1}{W^{(n)}-a_j}\right) + S(r,W).$$

Using (2.2) in (2.1), we obtain

$$(q-1)T(r,W) \leq (q-1)\,T\left(r, \frac{1}{W}\right) + \overline{N}(r,W^{(n)}) + \sum_{j=1}^{q} \overline{N}\left(r, \frac{1}{W^{(n)}-a_j}\right)$$

$$-\,(q-1)N\left(r, \frac{1}{W^{(n)}}\right) + S(r,W).$$

Thus

$$(2.3)\quad (q-1)T(r,W) \leq (q-1)\,T\left(r, \frac{1}{W}\right) + \overline{N}(r,W) + \sum_{j=1}^{q} \overline{N}\left(r, \frac{1}{W^{(n)}-a_j}\right)$$

$$-\,(q-1)N\left(r, \frac{1}{W^{(n)}}\right) + S(r,W).$$

Since $E(0,W) \subseteq E(0, W^{(n)})$, we have from (2.3)

$$(q-1)T(r,W) \leq \overline{N}(r,W) + \sum_{j=1}^{q} \overline{N}\left(r, \frac{1}{W^{(n)}-a_j}\right) + S(r,W).$$

Hence

$$(q-2\nu+o(1))T(r,W) \leq \sum_{j=1}^{q} \overline{N}\left(r, \frac{1}{W^{(n)}-a_j}\right).$$

This completes the proof of the ***Lemma* 2.1**.

In this paper we wish to obtain a generalization of ***Theorem* 1.2**. Now we state and prove our main result in the following way.

***Theorem* 2.1** *Let $W_1(z)$ and $W_2(z)$ be two ν-valued, non-constant algebroid functions, let $a_j(j=1,2,...,q)$ be $q \geq 4\nu+1$ distinct complex numbers or ∞. Suppose that $k_1 \geq k_2 \geq ... \geq k_q$ are positive integers or ∞ and $\delta_j(\geq 0)(j=1,2,...,q)$ are such that*

$$\frac{1}{k_1} + \left(1+\frac{1}{k_m}\right)\sum_{j=2\nu}^{q} \frac{1}{1+k_j} + 1 + \delta < \frac{q-2\nu}{n+1}\left(1+\frac{1}{k_1}\right).$$

for a non-negative integer n. Let $B_j = \overline{E}_{k_j}(a_j, W_1)\backslash \overline{E}_{k_j}(a_j, W_2)$ for $j = 1, 2\nu, ..., q$ and $E(0, W_i) \subseteq E(0, W_i^{(n)})$ for $i = 1, 2$. If

$$\overline{N}_{B_j}(r, \frac{1}{W_1^{(n)}-a_j}) \leq \delta_j T(r, W_1^{(n)})$$

and

$$\liminf_{r\to\infty} \frac{\sum_{j=1}^{q} \overline{N}_{k_{j)}}(r, \frac{1}{W_1^{(n)}-a_j})}{\sum_{j=1}^{q} \overline{N}_{k_{j)}}(r, \frac{1}{W_2^{(n)}-a_j})} > \frac{(n+1)k_1}{(p-2\nu)(1+k_1) - (n+1)(1+k_1)\sum_{j=2\nu}^{q} \frac{1}{1+k_j} - (n+1)\{(1+\delta)k_1+1\}},$$

then $W_1^{(n)}(z) \equiv W_2^{(n)}(z)$.

Proof. By ***Lemma* 2.1**, we have

(2.4) $(q-2\nu+o(1))T(r,W_1)\le \sum_{j=1}^{q}\overline{N}\left(r,\frac{1}{W_1^{(n)}-a_j}\right)$

and

(2.5) $(q-2\nu+o(1))T(r,W_2)\le \sum_{j=1}^{q}\overline{N}\left(r,\frac{1}{W_2^{(n)}-a_j}\right)$.

From (2.4), we have

$$\begin{aligned}(q-2\nu+o(1))T(r,W_1)&\le \sum_{j=1}^{q}\left\{\overline{N}_{k_j)}\left(r,\frac{1}{W_1^{(n)}-a_j}\right)+\overline{N}_{(k_j+1}\left(r,\frac{1}{W_1^{(n)}-a_j}\right)\right\}\\ &\le \sum_{j=1}^{q}\left\{\overline{N}_{k_j)}\left(r,\frac{1}{W_1^{(n)}-a_j}\right)+\frac{1}{1+k_j}N_{(k_q+1}\left(r,\frac{1}{W_1^{(n)}-a_j}\right)\right\}\\ &\le \sum_{j=1}^{q}\left\{\frac{k_j}{1+k_j}\overline{N}_{k_j)}\left(r,\frac{1}{W_1^{(n)}-a_j}\right)+\frac{1}{1+k_j}N\left(r,\frac{1}{W_1^{(n)}-a_j}\right)\right\}\\ &\le \sum_{j=1}^{q}\frac{k_j}{1+k_j}\overline{N}_{k_j)}\left(r,\frac{1}{W_1^{(n)}-a_j}\right)+\sum_{j=1}^{q}\frac{1}{1+k_j}T\left(r,W_1^{(n)}\right)\\ &\le \sum_{j=1}^{q}\frac{k_j}{1+k_j}\overline{N}_{k_j)}\left(r,\frac{1}{W_1^{(n)}-a_j}\right)+(n+1)\sum_{j=1}^{q}\frac{1}{1+k_j}T\left(r,W_1^{(n)}\right).\end{aligned}$$

Therefore

$$(q-2\nu-(n+1)\sum_{j=1}^{q}\frac{1}{1+k_j}+o(1))T(r,W_1)\le \sum_{j=1}^{q}\frac{k_j}{1+k_j}\overline{N}_{k_j)}\left(r,\frac{1}{W_1^{(n)}-a_j}\right).$$

Similarly from (2.5), we get

$$(q-2\nu-(n+1)\sum_{j=1}^{q}\frac{1}{1+k_j}+o(1))T(r,W_2)\le \sum_{j=1}^{j}\frac{k_j}{1+k_j}\overline{N}_{k_j)}\left(r,\frac{1}{W_2^{(n)}-a_j}\right).$$

Let $B_j=\overline{E}_{k_j}(a_j,W_1^{(n)})\backslash A_j$ for $j=1,2\nu,...,q$.

Now

$$\begin{aligned}\sum_{j=1}^{q}\overline{N}_{k_j)}\left(r,\frac{1}{W_1^{(n)}-a_j}\right)&=\sum_{j=1}^{q}\overline{N}_{A_j}\left(r,\frac{1}{W_1^{(n)}-a_j}\right)+\sum_{j=1}^{q}\overline{N}_{B_j}\left(r,\frac{1}{W_1^{(n)}-a_j}\right)\\ &\le\delta T(r,W_1^{(n)})+N\left(r,\frac{1}{W_1^{(n)}-W_2^{(n)}}\right)\\ &\le(1+\delta)(n+1)T(r,W_1)+(n+1)T(r,W_2).\end{aligned}$$

Hence

$$\begin{aligned}&\left(q-2\nu-(n+1)\sum_{j=1}^{q}\frac{1}{1+k_j}+o(1)\right)\sum_{j=1}^{q}\overline{N}_{k_j)}\left(r,\frac{1}{W_1^{(n)}-a_j}\right)\\ &\le(1+\delta)(n+1)\sum_{j=1}^{q}\frac{k_j}{1+k_j}\overline{N}_{k_j)}\left(r,\frac{1}{W_1^{(n)}-a_j}\right)+(n+1)\sum_{j=1}^{q}\frac{k_j}{1+k_j}\overline{N}_{k_j)}\left(r,\frac{1}{W_2^{(n)}-a_j}\right).\end{aligned}$$

Since $1\ge\dfrac{k_1}{k_1+1}\ge\dfrac{k_2}{k_2+1}\ge...\ge\dfrac{k_q}{k_q+1}\ge\dfrac{1}{2}$, we get from the above inequality

$$\begin{aligned}&\left(q-2\nu-(n+1)\sum_{j=1}^{q}\frac{1}{1+k_j}+o(1)\right)\sum_{j=1}^{q}\overline{N}_{k_j)}\left(r,\frac{1}{W_1^{(n)}-a_j}\right)\\ &\le(1+\delta)(n+1)\frac{k_1}{1+k_1}\sum_{j=1}^{q}\overline{N}_{k_j)}\left(r,\frac{1}{W_1^{(n)}-a_j}\right)+(n+1)\frac{k_1}{1+k_1}\sum_{j=1}^{q}\overline{N}_{k_j)}\left(r,\frac{1}{W_2^{(n)}-a_j}\right).\end{aligned}$$

Since that implies

$$\begin{aligned}&\left(q-2\nu-(n+1)\sum_{j=1}^{q}\frac{1}{1+k_j}-(1+\delta)(n+1)\frac{k_1}{1+k_1}+o(1)\right)\sum_{j=1}^{q}\overline{N}_{k_j)}\left(r,\frac{1}{W_1^{(n)}-a_j}\right)\\ &\le(n+1)\frac{k_1}{1+k_1}\sum_{j=1}^{q}\overline{N}_{k_j)}\left(r,\frac{1}{W_2^{(n)}-a_j}\right).\end{aligned}$$

Therefore

$$\begin{aligned}&\liminf_{r\to\infty}\frac{\sum_{j=1}^{q}\overline{N}_{k_j}(r,\frac{1}{W_1-a_j})}{\sum_{j=1}^{q}\overline{N}_{k_j}(r,\frac{1}{W_2-a_j})}\\ &\le\frac{(n+1)k_1}{(q-2\nu)(1+k_1)-(n+1)(1+k_1)\sum_{j=1}^{q}\frac{1}{1+k_j}-(n+1)\{(1+\delta)k_1}\end{aligned}$$

$$\leq \frac{(n+1)k_1}{(q-2\nu)(1+k_1)-(n+1)(1+k_1)\sum_{j=2\nu}^{q}\frac{1}{1+k_j}-(n+1)\{(1+\delta)k_1+1}.$$

which is a contradiction.

Thus, we have $W_1^{(n)}(z) \not\equiv W_2^{(n)}(z)$.

Therefore we complete the proof of ***Theorem* 2.1**.

From ***Theorem* 2.1**, we can get the following consequences.

***Corollary* 2.1** *Let $k_j = \infty$ for $j = 1, 2\nu, ..., q$ and*

$$\gamma = \liminf_{r\to\infty} \frac{\overline{N}_{k_j)}\left(r, \frac{1}{W_1^{(n)}-a_j}\right)}{\overline{N}_{k_j)}\left(r, \frac{1}{W_2^{(n)}-a_j}\right)} > \frac{n+1}{q-(n+2\nu+1)}.$$

If $\overline{N}_{A_j}(r, \frac{1}{W_1^{(n)}-a_j}) \leq \delta_j T(r, W_1^{(n)})$ where $\delta(\geq 0)$ satisfy $0 \leq \delta_j < \frac{q-(n+2\nu+1)}{n+1} - \frac{1}{\gamma}$.

If we assume $E_{\infty)}(a_j, W_1^{(n)}) \subseteq E_{\infty)}(a_j, W_2^{(n)})$, then $A_j = \phi$ for $j = 1, 2\nu, ..., q$ and so we can choose $\delta = 0$.

Therefore ***Theorem* 2.1** *is an improvement of following theorem.*

***Theorem* 2.2** *Let $W_1(z)$ and $W_2(z)$ be two ν-valued, non-constant algebroid functions, let a_j $(j = 1, 2, ..., q)$ be $q \geq 4\nu+1$ distinct complex numbers or ∞. and for a non-negative integer n, $E_{\infty)}(a_j, W_1^{(n)}) \subseteq E_{\infty)}(a_j, W_2^{(n)})$ for $1 \leq j \leq q$, $E_{\infty)}(0, W_1) \subseteq E_{\infty)}(0, W_1^{(n)})$, $E_{\infty)}(0, W_2) \subseteq E_{\infty)}(0, W_2^{(n)})$ and*

$$\liminf_{r\to\infty} \frac{\sum_{j=1}^{q} \overline{N}_{k_j)}(r, \frac{1}{W_1^{(n)}-a_j})}{\sum_{j=1}^{q} \overline{N}_{k_j)}(r, \frac{1}{W_2^{(n)}-a_j})} > \frac{(n+1)}{q-(n+2\nu+1)},$$

then $W_1^{(n)}(z) \equiv W_2^{(n)}(z)$.

***Corollary* 2.2** *Let $n = 0$, $k_j = \infty$ for $j = 1, 2\nu, ..., q$ and*

$$\gamma = \liminf_{r\to\infty} \frac{\overline{N}_{k_j)}\left(r, \frac{1}{W_1-a_j}\right)}{\overline{N}_{k_j)}\left(r, \frac{1}{W_2-a_j}\right)} > \frac{1}{q-2\nu+1}.$$

If $\overline{N}_{B_j}(r, \frac{1}{W_1-a_j}) \leq \delta_j T(r, W_1)$ where $\delta(\geq 0)$ satisfy $0 \leq \sum_{j=1}^{q} \delta_j < k - (2\nu+1) - \frac{1}{\gamma}$, then $W_1(z) \equiv W_2(z)$. If we take $q = 4\nu + 1$ and $\overline{E}(a_j, f) \subseteq \overline{E}(a_j, f)$, then $A_j = \phi$ for $j = 1, 2, ..., 4\nu + 1$. Therefore, if we choose $\delta_j = 0$ for $j = 1, 2, ..., 4\nu + 1$ and take any constant γ, such that $0 \leq 2\nu - \frac{1}{\gamma}$ in ***Corollary* 2.2***; we can get that $W_1(z) \equiv W_2(z)$. Especially, if $q = 4\nu + 1$ and $\overline{E}(a_j, W_1) = \overline{E}(a_j, W_2)$, then $\gamma = 1$ and $\delta_j = 0$ for $j = 1, 2, ..., 4\nu + 1$. We can obtain $W_1(z) \equiv W_2(z)$. Then* ***Corollary* 2.2** *is an improvement of* ***Theorem* 1.1**.

***Corollary* 2.3** *Let $W_1(z)$ and $W_2(z)$ be two ν-valued, non-constant algebroid functions, let a_j $(j = 1, 2, ..., q)$ be $q \geq 5$ distinct complex numbers or ∞. Suppose that $k_1, k_2, ..., k_q$ are positive integers or ∞; with $k_1 \geq k_2 \geq ... \geq k_q$ if $\overline{E}_{k_j)}(a_j, W_1) \subseteq \overline{E}_{k_j)}(a_j, W_2)$ and :*

$$\sum_{j=2\nu}^{q} \frac{k_j}{k_j+1} - \frac{k_1}{\gamma(k_1+1)} - 2\nu > 0,$$

where γ is stated as in Corollary 2.2; then $W_1(z) \equiv W_2(z)$.

***Corollary* 2.4** *Under the assumptions of* ***Corollary* 2.2**, *$\overline{E}_{k_j)}(a_j, W_1) = \overline{E}_{k_j)}(a_j, W_2)$ and :*

$$\sum_{j=2\nu}^{q} \frac{k_j}{k_j+1} - \frac{k_1}{(k_1+1)} - 2\nu > 0,$$

***Corollary* 2.5** *Let $W_1(z)$ and $W_2(z)$ be two ν-valued, non-constant algebroid functions, let a_j $(j = 1, 2, ..., q)$ be $q \geq 5$ distinct complex numbers or ∞. Suppose that $k_1, k_2, ..., k_q$ are positive integers or ∞; with $k_1 \geq k_2 \geq ... \geq k_q$ if $\overline{E}_{k_j)}(a_j, f) \subseteq \overline{E}_{k_j)}(a_j, g)$ and :*

$$\sum_{j=2\nu}^{q} \frac{k_j}{k_j+1} - 2\nu + \frac{(m-2\nu-\frac{1}{\gamma})k_m}{\gamma(k_m+1)} - 2\nu > 0,$$

where γ is stated as in Corollary 2.2; then $W_1(z) \equiv W_2(z)$.

In Corollary 2.1 if $n = 0$ and $q = 4\nu + 1$ then we get the following theorem.

***Theorem* 2.3** *Let $W_1(z)$ and $W_2(z)$ be two ν-valued, non-constant algebroid functions such that $E_{\infty)}(a_j, W_1) \subseteq E_{\infty)}(a_j, W_2)$ for $a_1, a_2, ..., a_5$ of $\mathbb{C} \cup \infty$. If*

$$\liminf_{r\to\infty} \frac{\sum_{j=1}^{4\nu+1} \overline{N}_{k_j)}(r, \frac{1}{W_1-a_j})}{\sum_{j=1}^{4\nu+1} \overline{N}_{k_j)}(r, \frac{1}{W_2-a_j})} > \frac{1}{2},$$

then $W_1(z) \equiv W_2(z)$.

3 Conclusion

In this paper, we discussed on the Nevanlinna's value distribution theory of meromorphic functions to Nevanlinna's five values theorem for algebroid functions and we further investigated the problems on the Nevanlinna's five value theorem for algebroid functions.

Acknowledgement. Authors are thankful to Editors and Reviewers for their suggestions to improve the paper in its present form.

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