***Jñānābha*, Vol. 50(2) (2020), 139-145**
(Dedicated to Honor Dr. R. C. Singh Chandel on His 75^{th} Birth Anniversary Celebrations)

HYPERGEOMETRIC FORMS OF CERTAIN COMPOSITE FUNCTIONS INVOLVING ARCSINE(x) USING MACLAURIN SERIES AND THEIR APPLICATIONS

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(*Received : July* 02, 2019 *; Revised: August* 18, 2020)

Abstract

In this article, we obtain hypergeometric forms of some composite functions containing arcsine(x) like: $\exp(a\sin^{-1}x)$, $\frac{\exp(a\sin^{-1}x)}{\sqrt{(1-x^2)}}$, $\cos(b\sin^{-1}x)$, $\sin(b\sin^{-1}x)$, $\cosh(d\sin^{-1}x)$, $\sinh(d\sin^{-1}x)$, $\frac{\cos(b\sin^{-1}x)}{\sqrt{(1-x^2)}}$, $\frac{\sin(b\sin^{-1}x)}{\sqrt{(1-x^2)}}$ and arcsinh (x) like: $\exp(b\sinh^{-1}x)$, $\frac{\exp(b\sinh^{-1}x)}{\sqrt{(1+x^2)}}$, $\cos(g\sinh^{-1}x)$, $\sin(g\sinh^{-1}x)$, $\cosh(b\sinh^{-1}x)$, $\sinh(b\sinh^{-1}x)$, by using Leibniz theorem for successive differentiation, Maclaurin's series expansion and Taylor's series expansion, as the proof of the hypergeometric forms of the above functions is not available in the literature.
2010 Mathematics Subject Classifications: 33C05, 34A35, 41A58, 33B10.
Keywords and phrases: Hypergeometric function; Maclaurin's series; Taylor's series; Leibniz theorem.

1 Introduction and Preliminaries

In this paper, we shall use the following standard notations:
$\mathbb{N} := \{1,2,3,\cdots\}$; $\mathbb{N}_0 := \mathbb{N}\bigcup\{0\}$; and $\mathbb{Z}_0^- := \mathbb{Z}^-\bigcup\{0\} = \{0,-1,-2,-3,\cdots\}$.

The symbols $\mathbb{C},\mathbb{R},\mathbb{N},\mathbb{Z},\mathbb{R}^+$ and $\mathbb{R}^-$ denote the sets of complex numbers, real numbers, natural numbers, integers, positive and negative real numbers respectively.

The Pochhammer symbol $(\alpha)_p(\alpha, p\in\mathbb{C})$ is defined by ([10, p.22 Eq.(1), p.32, Q.N.(8) and Q.N.(9)],see also [12, p.23, Eq.(22) and Eq.(23)]).

A natural generalization of the Gaussian hypergeometric series ${}_2F_1[\alpha,\beta;\gamma;z]$ is accomplished by introducing any arbitrary number of numerator and denominator parameters[12, p.42, Eq.(1)].

Relation between hyperbolic and trigonometric functions:

(1.1) $\cos(i\theta) = \cosh(\theta), \quad \sin(i\theta) = i\sinh(\theta).$

(1.2) $\sin^{-1}(x) = -i\sinh^{-1}(ix), \ \ \sinh^{-1}(x) = -i\sin^{-1}(ix).$

The Taylor's series of a real or complex-valued function $y(x)$ which is infinitely differentiable at a real or complex number a, is the power series:

(1.3) $$y(x) = (y)_{x=a} + (x-a)(y_1)_{x=a} + \frac{(x-a)^2}{2!}(y_2)_{x=a} + \frac{(x-a)^3}{3!}(y_3)_{x=a} + \frac{(x-a)^4}{4!}(y_4)_{x=a} + \cdots$$

(1.4) $$= \sum_{n=0}^{\infty}\frac{(x-a)^n}{n!}(y_n)_{x=a}$$

(1.5) $$= \sum_{n=0}^{\infty}\frac{(x-a)^{2n}}{(2n)!}(y_{2n})_{x=a} + \sum_{n=0}^{\infty}\frac{(x-a)^{2n+1}}{(2n+1)!}(y_{2n+1})_{x=a}.$$

The Maclaurin's series is a particular case of Taylor's series expansion of a function, about the origin i.e, when $a=0$ in equation (1.3), the Maclaurin series is given as:

$$y(x) = (y)_0 + x(y_1)_0 + \frac{x^2}{2!}(y_2)_0 + \frac{x^3}{3!}(y_3)_0 + \frac{x^4}{4!}(y_4)_0 + \frac{x^5}{5!}(y_5)_0 + \cdots$$

(1.6) $$= \sum_{n=0}^{\infty}\frac{x^n}{n!}(y_n)_0$$

(1.7) $$= \sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!}(y_{2n})_0 + \sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2n+1)!}(y_{2n+1})_0,$$

where, $(y_m)_0 = \left(\frac{d^m y}{dx^m}\right)_{x=0}$.

The general Leibniz rule, named after Gottfried Wilhelm Leibniz, generalizes the product rule (which is also known as "Leibniz's rule"), which states that if $U(x)$ and $T(x)$ are n-times differentiable functions, then the product $U(x).T(x)$ is also n-times differentiable and its nth derivative is given by:

$$D^n[U(x)\, T(x)] = ({}^nC_0)(D^n\, U)(D^0\, T) + ({}^nC_1)(D^{n-1}\, U)(D^1\, T) + ({}^nC_2)(D^{n-2}\, U)(D^2\, T) + \cdots +$$

(1.8) $+({}^nC_{n-1})(D^1\, U)(D^{n-1}\, T) + ({}^nC_n)(D^0\, U)(D^n\, T)$

(1.9) $= \sum_{r=0}^{n} {}^nC_r(D^rT)(D^{n-r}U)$

(1.10) $= \sum_{r=0}^{n} {}^nC_r(D^{n-r}T)(D^rU)$,

where $D = \frac{d}{dx}$.

Euler's linear transformation

(1.11) $${}_2F_1\left[\begin{array}{c} \beta,\ \lambda; \\ \mu; \end{array} z\right] = (1-z)^{\mu-\beta-\lambda}\ {}_2F_1\left[\begin{array}{c} \mu-\beta,\ \mu-\lambda; \\ \mu; \end{array} z\right],$$

where $\mu \in \mathbb{C}\backslash\mathbb{Z}_0^-$ and $|arg(1-z)| < \pi$.

The present article is organized as follows. In section 3 we have given the proof of the hypergeometric forms of presented functions, because their proofs are not available in the literature[1, 2, 3, 4, 5, 6, 7, 8, 9] see also [11, 13]. So we are interested to give the proof of hypergeometric forms of some composite functions containing arcsine(x), using Maclaurin series. In section 4 we have obtained hypergeometric forms of some more functions by using the relations between inverse trigonometric and inverse hyperbolic functions. In section 5 we have discussed some applications of hypergeometric forms (2.1),(2.2) and (2.3). In section 6 we discussed some applications of hypergeometric forms (5.4) and (5.5).

2 Main Hypergeometric Forms of Certain Composite Functions

When the values of numerator, denominator parameters and arguments leading to the results which do not make sense are tacitly excluded, then each of the following hypergeometric form holds true:

(2.1) $$\exp(a\sin^{-1}x) = {}_2F_1\left[\begin{array}{c} \frac{ia}{2},\ -\frac{ia}{2}; \\ \frac{1}{2}; \end{array} x^2\right] + ax\ {}_2F_1\left[\begin{array}{c} \frac{1+ia}{2},\ \frac{1-ia}{2}; \\ \frac{3}{2}; \end{array} x^2\right],$$

where $|x| < 1$.

(2.2) $$\exp(b\sin^{-1}x) = \exp\left(\frac{\pi b}{2}\right) {}_2F_1\left[\begin{array}{c} ib,\ -ib; \\ \frac{1}{2}; \end{array} \frac{1-x}{2}\right],$$

where $|\frac{1-x}{2}| < 1$.

(2.3) $$\frac{\exp(a\sin^{-1}x)}{\sqrt{(1-x^2)}} = {}_2F_1\left[\begin{array}{c} \frac{1+ia}{2},\ \frac{1-ia}{2}; \\ \frac{1}{2}; \end{array} x^2\right] + ax\ {}_2F_1\left[\begin{array}{c} \frac{2+ia}{2},\ \frac{2-ia}{2}; \\ \frac{3}{2}; \end{array} x^2\right],$$

where $|x| < 1$.

Note: In the above hypergeometric functions x, a and b can be purely real or purely imaginary or complex numbers.

3 Proof of Hypergeometric Forms

Proof of hypergeometric form (2.1)

Let

(3.1) $y = \exp(a\sin^{-1}x)$.

Put $x = 0$ in equation (3.1), we get

(3.2) $(y)_0 = 1$.

Differentiate equation (3.1) w.r.t. x and put $x = 0$, we get

(3.3) $\sqrt{1-x^2}\, y_1 = ay$,

(3.4) $(y_1)_0 = a$.

Differentiate equation (3.3) w.r.t. x and put $x = 0$, we get

(3.5) $(1 - x^2)y_2 - xy_1 - a^2 y = 0,$

(3.6) $(y_2)_0 = a^2.$

Differentiate equation (3.5) w.r.t. x and put $x = 0$, we get

(3.7) $(1 - x^2)y_3 - 3xy_2 - (1 + a^2)y_1 = 0,$

(3.8) $(y_3)_0 = (1 + a^2)a.$

Now differentiate equation (3.5) n-times w.r.t. x, and applying Leibnitz theorem we get

$$D^n\left\{(1 - x^2)y_2\right\} - D^n(xy_1) - D^n(a^2 y) = 0 \quad ; n \geq 2,$$

(3.9) $(1 - x^2)y_{n+2} - (2n + 1)xy_{n+1} - (n^2 + a^2)y_n = 0 \quad ; n \geq 2.$

Put $x = 0$ in equation(3.9) we get

(3.10) $(y_{n+2})_0 = (n^2 + a^2)(y_n)_0 \quad ; n \geq 2.$

Put $n = 2, 3, 4, 5, 6, 7, 8...$ in equations (3.10), we get

(3.11) $(y_4)_0 = (2^2 + a^2)a^2,$

(3.12) $(y_5)_0 = (3^2 + a^2)(1 + a^2)a,$

(3.13) $(y_6)_0 = (4^2 + a^2)(2^2 + a^2)a^2,$

(3.14) $(y_7)_0 = (5^2 + a^2)(3^2 + a^2)(1 + a^2)a,$

(3.15) $(y_8)_0 = (6^2 + a^2)(4^2 + a^2)(2^2 + a^2)a^2,$

(3.16) $(y_9)_0 = (7^2 + a^2)(5^2 + a^2)(3^2 + a^2)(1 + a^2)a,$

(3.17) $(y_{10})_0 = (8^2 + a^2)(6^2 + a^2)(4^2 + a^2)(2^2 + a^2)a^2,$

$\vdots$

Recurrence Relation

In case of odd

(3.18) $(y_{2n+1})_0 = a \prod_{j=1}^{n}\left\{(2j - 1)^2 + a^2\right\}.$

In case of even

(3.19) $(y_{2n})_0 = \prod_{j=1}^{n}\left\{(2j - 2)^2 + a^2\right\}.$

We know by Maclaurin series expansion

(3.20) $y = (y)_0 + x(y_1)_0 + \frac{x^2}{2!}(y_2)_0 + \frac{x^3}{3!}(y_3)_0 + \frac{x^4}{4!}(y_4)_0 + \frac{x^5}{5!}(y_5)_0 + ...$

Substitute the values of $(y)_0, (y_1)_0, (y_2)_0, (y_3)_0, (y_4)_0, (y_5)_0, ...$ in equation (3.20), we get

$$y = \sum_{n=0}^{\infty} \frac{x^{2n}}{2n!} \prod_{j=1}^{n}\left\{(2j - 2)^2 + a^2\right\} + a \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} \prod_{j=1}^{n}\left\{(2j - 1)^2 + a^2\right\},$$

$$y = \sum_{n=0}^{\infty} \frac{x^{2n}}{2n!}\left\{\prod_{j=1}^{n}[(2j - 2) + ia]\prod_{j=1}^{n}[(2j - 2) - ia]\right\} +$$

$$+ ax\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n+1)!}\left\{\prod_{j=1}^{n}[(2j - 1) + ia]\prod_{j=1}^{n}[(2j - 1) - ia]\right\},$$

$$y = \sum_{n=0}^{\infty} \frac{x^{2n}2^{2n}}{2n!}\left\{\prod_{j=1}^{n}\left[(j - 1) + \frac{ia}{2}\right]\prod_{j=1}^{n}\left[(j - 1) - \frac{ia}{2}\right]\right\} +$$

$$+ ax\sum_{n=0}^{\infty} \frac{x^{2n}2^{2n}}{(2n+1)!}\left\{\prod_{j=1}^{n}\left[(j - 1) + \frac{1}{2} + \frac{ia}{2}\right]\prod_{j=1}^{n}\left[(j - 1) + \frac{1}{2} - \frac{ia}{2}\right]\right\},$$

$$y = \sum_{n=0}^{\infty} \frac{x^{2n}\left(\frac{ia}{2}\right)_n\left(-\frac{ia}{2}\right)_n}{\left(\frac{1}{2}\right)_n n!} + ax\sum_{n=0}^{\infty} \frac{x^{2n}\left(\frac{1+ia}{2}\right)_n\left(\frac{1-ia}{2}\right)_n}{\left(\frac{3}{2}\right)_n n!}.$$

Using definition of generalized hypergeometric function of one variable, we get the required result (2.1).

Proof of hypergeometric forms (2.2) and (2.3)

The proof of hypergeometric form (2.2) can be given by following same approach and making use of Taylors series expansion. Similarly the proof of hypergeometric form (2.3) can be given by following same approach and making use of Maclaurin's series expansion. So we omit the details here.

4 Some Inverse Hyperbolic Sine Functions as Special Cases

Replacing x by ix in equation (2.1) and putting $a = -ib$, we get

$$(4.1)\quad \exp(b\sinh^{-1} x) = {}_2F_1\left[\begin{matrix} -\frac{b}{2}, & \frac{b}{2}; & \\ & & -x^2 \\ & \frac{1}{2}; & \end{matrix}\right] + bx\,{}_2F_1\left[\begin{matrix} \frac{1+b}{2}, & \frac{1-b}{2}; & \\ & & -x^2 \\ & \frac{3}{2}; & \end{matrix}\right].$$

Replacing x by ix in equation (2.2), and putting $b = -ia$, we get

$$(4.2)\quad \exp(a\sinh^{-1} x) = \exp\left(-\frac{i\pi a}{2}\right)\,{}_2F_1\left[\begin{matrix} a, & -a; & \\ & & \frac{1-ix}{2} \\ & \frac{1}{2}; & \end{matrix}\right].$$

Replacing x by ix in equation (2.3) and putting $a = -ib$, we get

$$(4.3)\quad \frac{\exp(b\sinh^{-1} x)}{\sqrt{(1+x^2)}} = {}_2F_1\left[\begin{matrix} \frac{1+b}{2}, & \frac{1-b}{2}; & \\ & & -x^2 \\ & \frac{1}{2}; & \end{matrix}\right] + bx\,{}_2F_1\left[\begin{matrix} \frac{2+b}{2}, & \frac{2-b}{2}; & \\ & & -x^2 \\ & \frac{3}{2}; & \end{matrix}\right].$$

5 Some Applications

5.1 Special Cases of Hypergeometric form (2.1)

Suppose $x \in \mathbb{R}$ and a is purely imaginary in equation (2.1), then put $a = ib$, where b is purely real, we get

$$(5.1)\quad \exp(ib\sin^{-1} x) = {}_2F_1\left[\begin{matrix} -\frac{b}{2}, & \frac{b}{2}; & \\ & & x^2 \\ & \frac{1}{2}; & \end{matrix}\right] + ibx\,{}_2F_1\left[\begin{matrix} \frac{1+b}{2}, & \frac{1-b}{2}; & \\ & & x^2 \\ & \frac{3}{2}; & \end{matrix}\right].$$

Applying Euler's formula on left hand side of equation (5.1), then on equating real and imaginary parts, we get

$$(5.2)\quad \cos(b\sin^{-1} x) = {}_2F_1\left[\begin{matrix} -\frac{b}{2}, & \frac{b}{2}; & \\ & & x^2 \\ & \frac{1}{2}; & \end{matrix}\right],$$

$$(5.3)\quad \sin(b\sin^{-1} x) = bx\,{}_2F_1\left[\begin{matrix} \frac{1+b}{2}, & \frac{1-b}{2}; & \\ & & x^2 \\ & \frac{3}{2}; & \end{matrix}\right].$$

Put $x = \sin(\theta)$ in equation (5.2) and (5.3), we get

$$(5.4)\quad \cos(b\theta) = {}_2F_1\left[\begin{matrix} -\frac{b}{2}, & \frac{b}{2}; & \\ & & \sin^2\theta \\ & \frac{1}{2}; & \end{matrix}\right],$$

$$(5.5)\quad \sin(b\theta) = b\sin(\theta){}_2F_1\left[\begin{matrix} \frac{1+b}{2}, & \frac{1-b}{2}; & \\ & & \sin^2\theta \\ & \frac{3}{2}; & \end{matrix}\right].$$

Using Euler's linear transformation (1.11) in the right hand side of equations (5.4) and (5.5), we get

$$(5.6)\quad \cos(b\theta) = \cos\theta{}_2F_1\left[\begin{matrix} \frac{1+b}{2}, & \frac{1-b}{2}; & \\ & & \sin^2\theta \\ & \frac{1}{2}; & \end{matrix}\right],$$

$$(5.7)\quad \sin(b\theta) = b(\sin\theta)(\cos\theta){}_2F_1\left[\begin{matrix} \frac{2-b}{2}, & \frac{2+b}{2}; & \\ & & \sin^2\theta \\ & \frac{3}{2}; & \end{matrix}\right].$$

Put $b = id$ in equation (5.2) and (5.3), where d is purely imaginary, we get

$$(5.8)\quad \cosh(d\sin^{-1} x) = {}_2F_1\left[\begin{matrix} -\frac{id}{2}, & \frac{id}{2}; & \\ & & x^2 \\ & \frac{1}{2}; & \end{matrix}\right],$$

(5.9) $$\sinh(d\sin^{-1}x) = dx\,{}_2F_1\left[\begin{matrix}\frac{1+id}{2},\ \frac{1-id}{2};\\ \frac{3}{2};\end{matrix}\ x^2\right].$$

Putting $x = iy$ in equation (5.2) and (5.3), where y is purely imaginary, we get

(5.10) $$\cosh(b\sinh^{-1}y) = {}_2F_1\left[\begin{matrix}-\frac{b}{2},\ \frac{b}{2};\\ \frac{1}{2};\end{matrix}\ -y^2\right],$$

(5.11) $$\sinh(b\sinh^{-1}y) = by\,{}_2F_1\left[\begin{matrix}\frac{1-b}{2},\ \frac{1+b}{2};\\ \frac{3}{2};\end{matrix}\ -y^2\right].$$

Putting $x = iy$ and $b = ig$ in equation (5.2) and (5.3), where y and g are purely imaginary, we get

(5.12) $$\cos(g\sinh^{-1}y) = {}_2F_1\left[\begin{matrix}-\frac{ig}{2},\ \frac{ig}{2};\\ \frac{1}{2};\end{matrix}\ -y^2\right],$$

(5.13) $$\sin(g\sinh^{-1}y) = gy\,{}_2F_1\left[\begin{matrix}\frac{1+ig}{2},\ \frac{1-ig}{2};\\ \frac{3}{2};\end{matrix}\ -y^2\right].$$

5.2 Special Cases of Hypergeometric form (2.2)

Suppose $x \in \mathbb{R}$ and b is purely imaginary in equation (2.2), then put $b = ia$, where a is purely real, we get

(5.14) $$\exp(ia\sin^{-1}x) = \exp\left(\frac{i\pi a}{2}\right)\,{}_2F_1\left[\begin{matrix}-a,\ a;\\ \frac{1}{2};\end{matrix}\ \frac{1-x}{2}\right].$$

Applying Euler's formula on left hand side of equation (5.14), then on equating real and imaginary parts, we get

(5.15) $$\cos(a\sin^{-1}x) = \cos\left(\frac{\pi a}{2}\right)\,{}_2F_1\left[\begin{matrix}-a,\ a;\\ \frac{1}{2};\end{matrix}\ \frac{1-x}{2}\right],$$

(5.16) $$\sin(a\sin^{-1}x) = \sin\left(\frac{\pi a}{2}\right)\,{}_2F_1\left[\begin{matrix}-a,\ a;\\ \frac{1}{2};\end{matrix}\ \frac{1-x}{2}\right].$$

Put $a = id$ in equation (5.15) and (5.16), where d is purely imaginary, we get

(5.17) $$\cosh(d\sin^{-1}x) = \cosh\left(\frac{\pi d}{2}\right)\,{}_2F_1\left[\begin{matrix}-id,\ id;\\ \frac{1}{2};\end{matrix}\ \frac{1-x}{2}\right],$$

(5.18) $$\sinh(d\sin^{-1}x) = \sinh\left(\frac{\pi d}{2}\right)\,{}_2F_1\left[\begin{matrix}-id,\ id;\\ \frac{1}{2};\end{matrix}\ \frac{1-x}{2}\right].$$

Putting $x = iy$ in equation (5.15) and (5.16), where y is purely imaginary, we get

(5.19) $$\cosh(a\sinh^{-1}y) = \cos\left(\frac{\pi a}{2}\right)\,{}_2F_1\left[\begin{matrix}-a,\ a;\\ \frac{1}{2};\end{matrix}\ \frac{1-iy}{2}\right],$$

(5.20) $$\sinh(a\sinh^{-1}y) = -i\sin\left(\frac{\pi a}{2}\right)\,{}_2F_1\left[\begin{matrix}-a,\ a;\\ \frac{1}{2};\end{matrix}\ \frac{1-iy}{2}\right].$$

Putting $x = iy$ and $a = ig$, in equation (5.15) and (5.16), where y and g are purely imaginary, we get

(5.21) $$\cos(g\sinh^{-1}y) = \cosh\left(\frac{\pi g}{2}\right)\,{}_2F_1\left[\begin{matrix}-ig,\ ig;\\ \frac{1}{2};\end{matrix}\ \frac{1-iy}{2}\right],$$

(5.22) $$\sin(g\sinh^{-1}y) = -i\sinh\left(\frac{\pi g}{2}\right)\,{}_2F_1\left[\begin{matrix}-ig,\ ig;\\ \frac{1}{2};\end{matrix}\ \frac{1-iy}{2}\right].$$

5.3 Special Cases of Hypergeometric form(2.3)

Suppose $x \in \mathbb{R}$ and a is purely imaginary in equation (2.3), then put $a = ib$, where b is purely real, we get

$$(5.23)\quad \frac{\exp(ib\sin^{-1}x)}{\sqrt{(1-x^2)}} = {}_2F_1\left[\begin{array}{c} \frac{1-b}{2},\ \frac{1+b}{2};\\ \quad\quad x^2\\ \frac{1}{2}; \end{array}\right] + ibx\,{}_2F_1\left[\begin{array}{c} \frac{2-b}{2},\ \frac{2+b}{2};\\ \quad\quad x^2\\ \frac{3}{2}; \end{array}\right].$$

Applying Euler's formula on left hand side of equation (5.23), then on equating real and imaginary parts, we get

$$(5.24)\quad \frac{\cos(b\sin^{-1}x)}{\sqrt{(1-x^2)}} = {}_2F_1\left[\begin{array}{c} \frac{1-b}{2},\ \frac{1+b}{2};\\ \quad\quad x^2\\ \frac{1}{2}; \end{array}\right],$$

$$(5.25)\quad \frac{\sin(b\sin^{-1}x)}{\sqrt{(1-x^2)}} = bx\,{}_2F_1\left[\begin{array}{c} \frac{2-b}{2},\ \frac{2+b}{2};\\ \quad\quad x^2\\ \frac{3}{2}; \end{array}\right].$$

Put $b = id$, in equation (5.24) and (5.25), where d is purely imaginary, we get

$$(5.26)\quad \frac{\cosh(d\sin^{-1}x)}{\sqrt{(1-x^2)}} = {}_2F_1\left[\begin{array}{c} \frac{1-id}{2},\ \frac{1+id}{2};\\ \quad\quad x^2\\ \frac{1}{2}; \end{array}\right],$$

$$(5.27)\quad \frac{\sinh(d\sin^{-1}x)}{\sqrt{(1-x^2)}} = dx\,{}_2F_1\left[\begin{array}{c} \frac{2-id}{2},\ \frac{2+id}{2};\\ \quad\quad x^2\\ \frac{3}{2}; \end{array}\right].$$

Putting $x = iy$ in equation (5.24) and (5.25), where y is purely imaginary, we get

$$(5.28)\quad \frac{\cosh(b\sinh^{-1}y)}{\sqrt{(1+y^2)}} = {}_2F_1\left[\begin{array}{c} \frac{1-b}{2},\ \frac{1+b}{2};\\ \quad\quad -y^2\\ \frac{1}{2}; \end{array}\right],$$

$$(5.29)\quad \frac{\sinh(b\sinh^{-1}y)}{\sqrt{(1+y^2)}} = by\,{}_2F_1\left[\begin{array}{c} \frac{2-b}{2},\ \frac{2+b}{2};\\ \quad\quad -y^2\\ \frac{3}{2}; \end{array}\right].$$

Putting $x = iy$ and $b = ig$ in equation (5.24) and (5.25) where y and g are purely imaginary, we get

$$(5.30)\quad \frac{\cos(g\sinh^{-1}y)}{\sqrt{(1+y^2)}} = {}_2F_1\left[\begin{array}{c} \frac{1-ig}{2},\ \frac{1+ig}{2};\\ \quad\quad -y^2\\ \frac{1}{2}; \end{array}\right],$$

$$(5.31)\quad \frac{\sin(g\sinh^{-1}y)}{\sqrt{(1+y^2)}} = gy\,{}_2F_1\left[\begin{array}{c} \frac{2-ig}{2},\ \frac{2+ig}{2};\\ \quad\quad -y^2\\ \frac{3}{2}; \end{array}\right].$$

6 Some Applications of Hypergeometric Forms (5.4) and (5.5)

Replacing θ by $\left(\frac{\pi}{2} - \theta\right)$ in equations (5.4) and (5.5), we get

$$(6.1)\quad \cos(b\theta) = \cos\left(\frac{b\pi}{2}\right){}_2F_1\left[\begin{array}{c} -\frac{b}{2},\ \frac{b}{2};\\ \quad\quad \cos^2\theta\\ \frac{1}{2}; \end{array}\right] + b\cos(\theta)\sin\left(\frac{b\pi}{2}\right){}_2F_1\left[\begin{array}{c} \frac{1+b}{2},\ \frac{1-b}{2};\\ \quad\quad \cos^2\theta\\ \frac{3}{2}; \end{array}\right].$$

$$(6.2)\quad \sin(b\theta) = \sin\left(\frac{b\pi}{2}\right){}_2F_1\left[\begin{array}{c} -\frac{b}{2},\ \frac{b}{2};\\ \quad\quad \cos^2\theta\\ \frac{1}{2}; \end{array}\right] - b\cos(\theta)\cos\left(\frac{b\pi}{2}\right){}_2F_1\left[\begin{array}{c} \frac{1+b}{2},\ \frac{1-b}{2};\\ \quad\quad \cos^2\theta\\ \frac{3}{2}; \end{array}\right].$$

7 Conclusion

In this paper, we have obtained hypergeometric forms of some composite functions involving arcsin(x) and arcsinh(x), by using Maclaurin's series expansion and Taylor's series expansion.We conclude our present investigation by observing that hypergeometric forms of some other functions can be derived in an analogous manner. More over the results derived are significant. These are expected to find some potential applications in the fields of Mathematics and Engineering Sciences.

Acknowledgement. We are very much thankful to the Editor and Reviewer for their kind suggestions to improve the paper in its present form.

References

[1] M. Abramowitz, and I. A. Stegun, *Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables*,Reprint of the 1972 Edition, Dover Publications, Inc., New York, 1992.

[2] G. E. Andrews, R. Askey and R. Roy, *Special Functions,* Cambridge University Press, Cambridge, UK, 1999.

[3] L. C. Andrews, *Special Functions for Engineers and Applied Mathematicians*, Macmillan Publishing Company, New York, 1985.

[4] L. C. Andrews, *Special Functions of Mathematics for Engineers*, Reprint of the 1992 Second Edition, SPIE Optical Engineering Press, Bellingham, W.A, Oxford University Press, Oxford, 1998.

[5] A. Erdélyi, W. Magnus, F. Oberhettinger and F. G. Tricomi, *"Higher Transcendental Functions"*, Vol. I, McGraw-Hill, New York, Toronto, London, 1953.

[6] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series and Products*, Fifth Edition, Academic Press, New York, 1994.

[7] W. Magnus, F. Oberhettinger and R. P. Soni, *Formulas and Theorems for the Special Functions of Mathematical Physics*, Third Enlarged Edition, Springer-Verlag, New York, 1966.

[8] A. P. Prudnikov, Yu. A. Brychknov and O. I. Marochev, *Integrals and Series* Vol.3; *(More Special Functions)*, Nauka Moscow, 1986, (Translated from the Russianby G.G.Gould), Gordon and Breach Science Publishers, New York, Philadelphia.London, Paris, Montreux, Tokyo, Melbourne, 1990.

[9] M. I. Qureshi, J. Majid and A. H. Bhat, Hypergeometric forms of some composite functions containing arccosine(x) using Maclaurin's expansion, *South East Asian Journal of Mathematics and Mathematical Sciences*, (Accepted).

[10] E. D. Rainville, *Special Functions*, The Macmillan Co. Inc., New York 1960, Reprinted by Chelsea publ. Co., Bronx, New York, 1971.

[11] H. M. Srivastava, and P. W. Karlsson, *"Multiple Gaussian Hypergeometric Series"*, Halsted Press (Ellis Horwood Limited, Chichester), John Wiley, New York, Chichester, Brisbane, Toronto, 1985.

[12] H. M. Srivastava and H. L. Manocha, *A Treatise on Generating Functions*,Halsted Press (Ellis Horwood Limited, Chichester), John Wiley and Sons, New York, Chichester Brisbane, Toronto, 1984.

[13] G. Szegö, *Orthogonal Polynomials*,Vol. XXIII Amer. Math. Soc., Providence, Rhode Island, Colloquium publ., New York, 1939.