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(Dedicated to Honor Dr. R. C. Singh Chandel on His 75^{th} *Birth Anniversary Celebrations)*

A QUADRUPLE FIXED POINT THEOREM FOR A MULTIMAP IN A HAUSDORFF FUZZY METRIC SPACE

By
M. S. Chauhan
Institute for Excellence in Higher Education, Bhopal-462016, Madhya Pradesh, India.
Email:dr.msc@rediffmail.com
Rajesh Shrivastava
Govt. Science and Commerce College Benazir, Bhopal-462008, Madhya Pradesh, India.
Email:rajeshraju0101@rediffmail.com
Rupali Verma
Institute for Excellence in Higher Education, Bhopal-462016, Madhya Pradesh, India.
Email:rupali.varma1989@gmail.com
Qazi Aftab Kabir
Saifia Science College, Bhopal-462001, Madhya Pradesh, India.
Email:aftabqazi168@gmail.com
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Abstract

Rao and Rao[16] obtained a triple fixed point theorem for a multimap in Hausdorff fuzzy metric space. Extending this idea we generalize the concept of triple fixed point, we define quadruple fixed point. In this paper we have established a result regarding it in Hausdorff fuzzy metric space.
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1 Introduction and Preliminaries

Zadeh [23] introduced the concept of fuzzy sets in 1965. Since then, it was developed extensively by many authors. Fuzzy metric spaces have been defined by several researchers in several ways (e.g.[6,7]). The concepts of fuzzy metric space introduced by Kramosil and Michlek [12] have been modified by George and Veeramani [7] and also induced a Hausdro topology on such fuzzy metric space. The contraction principle in the setting of fuzzy metric spaces introduced in [7] was later proved by Grabiec[9]. Some interesting references for fixed point theorems in fuzzy metric spaces are given in [3,4,5,21].

Nadler [14] initiated the study of fixed points for multivalued contraction mappings using the Hausdor metric. In 2004, Rodrguez-Lpez and Romaguera [17] introduced Hausdors fuzzy metric on the set of the nonempty compact subsets of a given fuzzy metric space. Later some fixed point theorems for multivalued maps in fuzzy metric spaces (e.g., [1,11,20,22]) were proved by several authors. Many authors studied the existence of fixed points for various multivalued contractive mappings under dierent conditions, refer to [12-14] and the references therein.

In 2006 coupled fixed point in partially ordered metric spaces was introduced by Gnana Bhaskar and Lakshmikantham [8] and some problems of the uniqueness of a coupled fixed point was discussed and the results were applied to periodic boundary value problems. In 2011, Samet and Vetro [18] proved the coupled fixed point theorems for a multivalued mapping. Berinde and Borcut [2] also introduced the concept of triple fixed points and obtained a triple fixed point theorem for a single valued map in partially ordered metric spaces.

In this paper, we obtain a quadruple fixed point theorem for a multimap in a Hausdor fuzzy metric space and using it, we obtain a common quadruple fixed point for a multi- and single valued maps.

For this we need the following.

***Definition* 1.1** [19] *A binary operation* $* : [0,1] \times [0,1] \to [0,1]$ *is a continuous t-norm if it satisfies the following conditions:*

1. $*$ *is associative and commutative,*
2. $*$ *is continuous,*
3. $a * 1 = a$ *for all* $a \epsilon [0,1]$*,*
4. $a * b \le c * d$ *whenever* $a \le c$ *and* $b \le d$*, for each* $a, b, c, d \in [0,1]$.

Two typical examples of continuous t-norm are $a * b = ab$ *and* $a * b = \min\{a, b\}$.

***Definition* 1.2** [7] *A 3-tuple* $(X, M, *)$ *is called a fuzzy metric space if X is an arbitrary (nonempty) set,* $*$ *is a continuous t-norm, and is a fuzzy set on* $X^2 \times (0, \infty)$*, satisfying the following conditions for each* $x, y, z \in X$ *and each t and* $s > 0$*,*
1. $M(x, y, t) > 0$*,*
2. $M(x, y, t) = 1$ *if and only if* $x = y$*,*
3. $M(x, y, t) = M(y, x, t)$*,*
4. $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$*,*
5. $M(x, y,) : (0, \infty) \to [0, 1]$ *is continuous.*

Let $(X, M, *)$ *be a fuzzy metric space. For* > 0*, the open ball* $B(x, r, t)$ *with centre* $x \in X$ *and radius* $0 < r < 1$ *is defined by* $B(x, r, t) = \{y \in X : M(x, y, t) > 1 - r\}$*.*

A subset $A \subset X$ *is called open if for each* $x \in A$ *there exist* $t > 0$ *and* $0 < r < 1$ *such that* $B(x, r, t) \subset A$*. Let* τ *denote the family of all open subsets of X. Then* τ *is called the topology on X induced by the fuzzy metric M. This topology is Hausdorff and first countable. A subset A of X is said to be F-bounded if there exist* $t > 0$ *and* $0 < r < 1$ *such that* $M(x, y, t) > 1 - r$ *for all* $x, y \in A$*.*

***Lemma* 1.1** [9] *Let* $(X, M, *)$ *be a fuzzy metric space. Then* $M(x, y, t)$ *is nondecreasing with respect to t for all* x, y *in X.*

***Definition* 1.3** [17] *Let* $(X, M, *)$ *be a fuzzy metric space. M is said to be continuous on* $X^2 \times (0, \infty)$ *if*

$$(1.1)\quad \lim_{n\to\infty} M(x_n, y_n, t_n) = M(x, y, t),$$

whenever a sequence $\{(x_n, y_n, t_n)\}$ *in* $X^2 \times (0, \infty)$ *converges to a point* $(x, y, t) \in X^2 \times (0, \infty)$*, that is, whenever*

$$\lim_{n\to\infty} M(x_n, x, t) = \lim_{n\to\infty} M(y_n, y, t) = M(x, y, t) = 1,$$

$$(1.2)\quad \lim_{n\to\infty} (x, y, t_n) = M(x, y, t).$$

***Lemma* 1.2** [17] *Let* $(X, M, *)$ *be a fuzzy metric space. Then M is a continuous function on* $X^2 \times (0, \infty)$*. Also let us take the condition:*

$$(1.3)\quad \lim_{t\to\infty} M(x, y, t) = 1,\ \forall\, x, y \in X.$$

***Lemma* 1.3** [13] *Let* $\{y_n\}$ *be a sequence in fuzzy metric space* $(X, M, *)$ *satisfying condition (3). If there exists a positive number* $k < 1$ *such that*

$$(1.4)\quad M(y_n, y_{n+1}, kt) \geq M(y_{n-1}, y_n, t), \quad t > 0, n = 1, 2....$$

***Definition* 1.4** [17] *Let B be a nonempty subset of a fuzzy metric space* $(X, M, *)$*. For* $a \in X$ *and* $t > 0$*, define* $M(a, B, t) = \sup\left\{\frac{a,b,t}{b} \in B\right\}$*.*

In this paper let $K(X)$ *denotes the class of all non empty compact subsets of X.*

***Lemma* 1.4** [17] *Let* $(X, M, *)$ *be a fuzzy metric space. Then for each* $a \in X$*,* $B \in K(X)$ *and* $t > 0$*, there exists* $b \in B$ *such that* $M(a, B, t) = M(a, b, t)$*.*

***Definition* 1.5** [17] *Let* $(X, M, *)$ *be a fuzzy metric space. For each* $A, B \in K(X)$ *and* $t > 0$*, set*

$$(1.5)\quad H_M(A, B, t) = \min\left\{\inf_{x\in A} M(x, B, t),\ \inf_{y\in B} M(A, y, t)\right\}.$$

The 3-tuple $\left(K(X), H_M, *\right)$ *is called a Hausdorff fuzzy metric space.*

***Lemma* 1.5** [10] *Let X be a nonempty set and* $g : X \to X$ *be a mapping. Then there exists a subset* $E \subseteq X$ *such that* $g(E) = g(X)$ *and* $g : X \to X$ *is one one.*

***Definition* 1.6** *Let X be a nonempty set,* $T : X\times X\times X\times X \to 2^X$ *(collection of all nonempty subset of X) and* $f : X \to X$*.*
(i) The point $(s, x, y, z) \in X \times X \times X \times X$ *is called a quadruple fixed point of T if*

$$(1.6)\quad s \in T(s, x, y, z),$$

$$y \in T(y, z, s, x),$$

$$z \in T(z, s, x, y).$$

(ii) The point $(s, x, y, z) \in X \times X \times X \times X$ *is called a quadruple coincident point of T and f if*

$$(1.7)\quad f_s \in T(s, x, y, z),$$

$f_x \in T(x, y, z, s),$

$f_y \in T(y, z, s, x),$

$f_z \in T(z, s, x, y).$

(iii) The point $(s, x, y, z) \in X \times X \times X \times X$ *is called a quadruple common fixed point of* T *and* f *if*

(1.8) $s = f_s \in T(s, x, y, z),$

$x = f_x \in T(x, y, z, s),$

$y = f_y \in T(y, z, s, x),$

$z = f_z \in T(z, s, x, y).$

***Definition* 1.7** *Let* $T : X \times X \times X \times X \to 2^X$ *be a multivalued map and* f *be a self map on* X*. The Hybrid pair* $\{T, f\}$ *is called w-compatible if* $f(T(s, x, y, z)) \subseteq T(fs, fx, fy, fz)$ *whenever* (s, x, y, z) *is quadruple coincident point of* T *and* f.

2 Main Result

Let us prove a slightly different result from Lemma 1.3 which we will use to prove our main result.

***Lemma* 2.1** *Let* $\{s_n\}, \{x_n\}, \{y_n\}$ *and* $\{z_n\}$ *be sequences in fuzzy metric space* $(X, M, *)$ *satisfying condition (1.3). If there exists a positive number* $k < 1$ *such that*

(2.1) $$\min\left\{M(s_n, s_{n+1}, kt)(x_n, x_{n+1}, kt)(y_n, y_{n+1}, kt)(z_n, z_{n+1}, kt)\right\}$$
$$\geq \min\left\{M(s_{n-1}, s_n, t)(x_{n-1}, x_n, t)(y_{n-1}, y_n, t)(z_{n-1}, z_n, t)\right\},$$

for all $t > 0,\ n = 1, 2, ...,$ *then* $\{s_n\}, \{x_n\}, \{y_n\}$ *and* $\{z_n\}$ *are Cauchy sequences in X.*

Proof. We have

(2.2) $$\min\left\{M(s_n, s_{n+1}, kt), M(x_n, x_{n+1}, kt), M(y_n, y_{n+1}, kt), M(z_n, z_{n+1}, kt)\right\}$$
$$\geq \min\left\{M(s_{n-1}, s_n, t), M(x_{n-1}, x_n, t), M(y_{n-1}, y_n, t), M(z_{n-1}, z_n, t)\right\}$$
$$\geq \min\left\{M\left(s_{n-2}, s_{n-1}, \frac{t}{k^2}\right), M\left(x_{n-2}, x_{n-1}, \frac{t}{k^2}\right), M\left(y_{n-2}, y_{n-1}, \frac{t}{k^2}\right),\right.$$
$$\left.M\left(z_{n-2}, z_{n-1}, \frac{t}{k^2}\right)\right\}$$
$$\vdots$$
$$\geq \min\left\{M\left(s_0, s_1, \frac{t}{k^n}\right), M\left(x_0, x_1, \frac{t}{k^n}\right), M\left(y_0, y_1, \frac{t}{k^n}\right), M\left(z_0, z_1, \frac{t}{k^n}\right)\right\}$$

.

Hence,

(2.3) $$M(s_n, s_{n+1}, t) \geq \min\left\{M\left(s_0, s_1, \frac{t}{k^n}\right), M\left(x_0, x_1, \frac{t}{k^n}\right), M\left(y_0, y_1, \frac{t}{k^n}\right), M\left(z_0, z_1, \frac{t}{k^n}\right)\right\}.$$

Now, for any positive integer p,

(2.4) $$M\left(s_n, s_{n+p}, t\right) \geq M\left(s_n, s_{n+1}, \frac{t}{p}\right) * M\left(s_{n+1}, s_{n+2}, \frac{t}{p}\right) * ... * M\left(s_{n+p-1}, s_{n+p}, \frac{t}{p}\right)$$
$$\geq \min\left\{M\left(s_0, s_1, \frac{t}{pk^n}\right), M\left(x_0, x_1, \frac{t}{pk^n}\right), M\left(y_0, y_1, \frac{t}{pk^n}\right), M\left(z_0, z_1, \frac{t}{pk^n}\right)\right\}$$
$$* \min\left\{M\left(s_0, s_1, \frac{t}{pk^{n+1}}\right)\left(x_0, x_1, \frac{t}{pk^{n+1}}\right)\left(y_0, y_1, \frac{t}{pk^{n+1}}\right)\left(z_0, z_1, \frac{t}{pk^{n+1}}\right)\right\}$$
$$*...* \geq \min\left\{M\left(s_0, s_1, \frac{t}{pk^{n+p-1}}\right), M\left(x_0, x_1, \frac{t}{pk^{n+p-1}}\right), M\left(y_0, y_1, tpk^{n+p-1}\right), M(z_0, z_1, \frac{t}{pk^{n+p-1}})\right\}.$$

Letting $n \to \infty$ and using condition (3), we have

$$\lim_{n\to\infty} M\Big(x_n, x_{n+p}, t\Big) \geq 1 * 1 * ... * 1 = 1.$$

Hence,

$$\lim_{n\to\infty} M\Big(x_n, x_{n+p}, t\Big) = 1. \quad (2.5)$$

Thus $\{s_n\}$ is a Cauchy sequence in X. Similarly, we can show that $\{x_n\}, \{y_n\}$ and $\{z_n\}$ are Cauchy sequences in X.

Now, let us prove our first main result.

***Theorem* 2.1** *Let $(X, M, *)$ be a complete fuzzy metric space satisfying condition (1.3) and $F : X \times X \times X \times X \to K(X)$ be a set valued mapping satisfying*

$$H_M\Big(F(s, x, y, z), F(h, u, v, w), kt\Big) \geq \min\Big\{M(s, h, t), M(x, u, t)M(y, v, t), M(z, w, t)\Big\} \quad (2.6)$$

for each $s, x, y, z, h, u, v, w \in X,\ t > 0$, where $0 < k < 1$.

Then F has a quadruple fixed point.

Proof. Let $s_0, x_0, y_0, z_0 \in X$.

Choose $s_1 \in F(s_0, x_0, y_0, z_0), x_1 \in F(x_0, y_0, z_0, s_0), y_1 \in (y_0, z_0, s_0, x_0), z_1 \in F(z_0, s_0, x_0, y_0)$.

Since F is compact valued, by Lemma 1.4, there exists $s_2 \in F(s_1, x_1, y_1, z_1)$ such that

$$\begin{aligned} M(s_1, s_2, kt) &= \sup_{x\in F(s_1,x_1,y_1,z_1)} M(s_1, s, kt) \\ &\geq H_M\Big(F(s_0, x_0, y_0, z_0), F(s_1, x_1, y_1, z_1), kt\Big) \\ &\geq \min\Big\{M(s_0, s_1, t), M(x_0, x_1, t), M(y_0, y_1, t), M(z_0, z_1, t)\Big\}. \end{aligned} \quad (2.7)$$

Since F is compact valued, by Lemma 1.4, there exists $x_2 \in F(x_1, y_1, z_1, s_1)$ such that

$$\begin{aligned} M(x_1, x_2, kt) &= \sup_{x\in F(x_1,y_1,z_1,s_1)} M(x_1, x, kt) \\ &\geq H_M(F(x_0, y_0, z_0, s_0), F(x_1, y_1, z_1, s_1), kt) \\ &\geq \min\{M(x_0, x_1, t), M(y_0, y_1, t), M(z_0, z_1, t), M(s_0, s_1, t)\} \\ &\geq \min\{M(s_0, s_1, t), M(x_0, x_1, t), M(y_0, y_1, t), M(z_0, z_1, t)\}. \end{aligned} \quad (2.8)$$

Since F is compact valued, by Lemma 1.4, there exists $y_2 \in F(y_1, z_1, s_1, x_1)$ such that

$$\begin{aligned} M(y_1, y_2, kt) &= \sup_{x\in F(y_1,z_1,s_1,x_1)} M(y_1, y, kt) \\ &\geq H_M(F(y_0, z_0, s_0, x_0), F(y_1, z_1, s_1, x_1), kt) \\ &\geq \min\{M(y_0, y_1, t), M(z_0, z_1, t), M(s_0, s_1, t), M(x_0, x_1, t)\}. \\ &\geq \min\{M(s_0, s_1, t), M(x_0, x_1, t), M(y_0, y_1, t), M(z_0, z_1, t)\}. \end{aligned} \quad (2.9)$$

Since F is compact valued, by Lemma 1.4, there exists $z_2 \in F(z_1, s_1, x_1, y_1)$ such that

$$\begin{aligned} M(z_1, z_2, kt) &= \sup_{x\in F(z_1,s_1,x_1,y_1)} M(z_1, z, kt) \\ &\geq H_M(F(z_0, s_0, x_0, y_0), F(z_1, s_1, x_1, y_1), kt) \\ &\geq \min\{M(z_0, z_1, t), M(s_0, s_1, t), M(x_0, x_1, t), M(y_0, y_1, t)\} \\ &\geq \min\{M(s_0, s_1, t), M(x_0, x_1, t), M(y_0, y_1, t), M(z_0, z_1, t)\}. \end{aligned} \quad (2.10)$$

Thus,

$$\begin{aligned} &\min\{M(s_1, s_2, kt), M(x_1, x_2, kt), M(y_1, y_2, kt), M(z_1, z_2, kt)\} \\ &\quad \geq \min\{M(s_0, s_1, t), M(x_0, x_1, t), M(y_0, y_1, t), M(z_0, z_1, t)\}. \end{aligned} \quad (2.11)$$

Continuing in this way we can find the sequences $\{s_n\}, \{x_n\}, \{y_n\}$ and $\{z_n\}$ in X such that

$$s_{n+1} \in F(s_n, x_n, y_n, z_n), x_{n+1} \in F(x_n, y_n, z_n, s_n), y_{n+1} \in F(y_n, z_n, s_n, x_n),$$
$$z_{n+1} \in F(z_n, s_n, x_n, y_n)$$

Such that

(2.12) $\min\{M(s_n, s_{n+1}, kt), M(x_n, x_{n+1}, kt), M(y_n, y_{n+1}, kt), M(z_n, z_{n+1}, kt)\}$
$\geq \min\{M(s_{n-1}, s_n, t), M(x_{n-1}, x_n, t), M(y_{n-1}, y_n, t), M(z_{n-1}, z_n, t)\}$.

Hence, by Lemma 2.1, $\{s_n\}, \{x_n\}, \{y_n\}$ and $\{z_n\}$ are Cauchy sequences in X.

Since X is complete, there exists $s, x, y, z \in X$ such that $\lim_{n\to\infty}\{s_n\} = s, \lim_{n\to\infty}\{x_n\} = x, \lim_{n\to\infty}\{y_n\} = y, \lim_{n\to\infty}\{z_n\} = z$.

Consider

(2.13) $H_M(F(s_n, x_n, y_n, z_n), F(s, x, y, z), kt)$
$\geq \min\{M(s_n, s, t), M(x_n, x, t), M(y_n, y, t), M(z_n, z, t)\}$.

Let $n \to \infty$, we get

(2.14) $\lim_{n\to\infty} H_M(F(s_n, x_n, y_n, z_n), F(s, x, y, z), kt) = 1$ so that
$\lim_{n\to\infty} H_M(F(s_n, x_n, y_n, z_n), F(s, x, y, z), t) = 1$.

Similarly we can show that

(2.15) $\lim_{n\to\infty} H_M(F(x_n, y_n, z_n, s_n), F(x, y, z, s), t) = 1$,
$\lim_{n\to\infty} H_M(F(y_n, z_n, s_n, x_n), F(y, z, s, x), t) = 1$,
$\lim_{n\to\infty} H_M(F(z_n, s_n, x_n, y_n), F(z, s, x, y), t) = 1$.

Since

$s_{n+1} \in F(s_n, x_n, y_n, z_n), x_{n+1} \in F(x_n, y_n, z_n, s_n), y_{n+1} \in F(y_n, z_n, s_n, x_n)$,
$z_{n+1} \in F(z_n, s_n, x_n, y_n)$,

From (2.12) and (2.13), we have

(2.16) $\lim_{n\to\infty} \sup_{a\in F(s,x,y,z)} M(s_{n+1}, a, t) = 1$,
$\lim_{n\to\infty} \sup_{b\in F(x,y,z,s)} M(x_{n+1}, b, t) = 1$,
$\lim_{n\to\infty} \sup_{c\in F(y,z,s,x)} M(y_{n+1}, c, t) = 1$,
$\lim_{n\to\infty} \sup_{d\in F(z,s,x,y)} M(z_{n+1}, d, t) = 1$.

Hence there exist sequences $l_n \in F(s, x, y, z), p_n \in F(x, y, z, s), q_n \in F(y, z, s, x)$ and $r_n \in F(z, s, x, y)$ such that

(2.17) $\lim_{n\to\infty} M(s_{n+1}, l_n, t) = 1$,

$\lim_{n\to\infty} M(x_{n+1}, p_n, t) = 1$,
$\lim_{n\to\infty} M(y_{n+1}, q_n, t) = 1$,
$\lim_{n\to\infty} M(z_{n+1}, r_n, t) = 1$.

for each $t > 0$.

Now for each $n \in N$, we have

(2.18) $M(l_n, s, t) \geq M(l_n, s_{n+1}, t/2) * M(s_{n+1}, s, t/2)$.

Letting $n \to \infty$, we obtain

(2.19) $\lim_{n\to\infty} M(l_n, s, t) = 1$ so that $\lim_{n\to\infty} l_n = s$.

Similarly, we can show that

(2.20) $\lim_{n\to\infty} p_n = x$, $\lim_{n\to\infty} q_n = y$, $\lim_{n\to\infty} r_n = z$.

Since $F(s, x, y, z), F(x, y, z, s), F(y, z, s, x)$ and $F(z, s, x, y)$ are compact, we have
$s \in F(s, x, y, z), x \in F(x, y, z, s), y \in F(y, z, s, x)$ and $z \in F(z, s, x, y)$.

Thus (s, x, y, z) is a quadruple fixed point of F.

***Theorem* 2.2** *Let $(X, M, *)$ be a complete fuzzy metric space satisfying condition (1.3) and $F : X \times X \times X \times X \to K(X)$ and $gX \to X$ be a mappings satisfying*

$$(2.21)\quad H_M\Big(F(s,x,y,z), F(h,u,v,w), kt\Big) \geq \min\Big\{M(gs,gh,t), M(gx,gu,t), M(gy,gv,t), M(gz,gw,t)\Big\},$$

for all $s,x,y,z,h,u,v,w \in X$, $t > 0$ and $0 < k < 1$. further assume that $F(X \times X \times X \times X) \subseteq g(x)$, then F and g have a quadrupled coincidence point. Moreover, F and g have a quadrupled common fixed point if one of the following conditions holds.
(a) The pair F, g is called w-compatible and there exists $\mu, \alpha, \beta, \gamma \in X$ such that $\lim_{n\to\infty} g^n s = \mu \lim_{n\to\infty} g^n x = \alpha, \lim_{n\to\infty} g^n y = \beta, \lim_{n\to\infty} g^n z = \gamma$, whenever (s,x,y,z) is a quadrupled coincidence point of F and g and g is continuous at $\mu, \alpha, \beta, \gamma$.
(b) There exist $\mu, \alpha, \beta, \gamma \in X$ such that $\lim_{n\to\infty} g^n\mu = s, \lim_{n\to\infty} g^n\alpha = x, \lim_{n\to\infty} g^n\beta = y, \lim_{n\to\infty} g^n\gamma = z$, whenever (s,x,y,z) is a quadrupled coincidence point of F and g and g is continuous at s,x,y,z.

Proof. By ***Lemma* 1.5**, there exists $E \subseteq X$ such that $g : E \to X$ is one to one and $g(E) = g(X)$.

Now, define $G : g(E) \times g(E) \times g(E) \times g(E) \to K(X)$ by $G(gs,gx,gy,gz) = F(x,y,z)$ for all $gs,gx,gy,gz \in g(E)$. Since g is one-one one E, G is well defined.

Now,

$$(2.22)\quad H_M\Big(G(gs,gx,gy,gz), G(gh,gu,gv,gw), kt\Big) = H_M\Big(F(s,x,y,z), F(h,u,v,w), kt\Big) \geq \min\Big\{M(gs,gh,t), M(gx,gu,t), M(gy,gv,t), M(gz,gw,t)\Big\}.$$

Hence G satisfies (2.6) and all the conditions of ***Theorem* 2.1**.

By ***Theorem* 2.1**, G has a quadruple fixed point $(h,u,v,w) \in g(E) \times g(E) \times g(E) \times g(E)$. Thus,

$$(2.23)\quad \begin{aligned} h &\in (h,u,v,w),\\ u &\in (u,v,w,h),\\ v &\in (v,w,h,u),\\ w &\in (w,h,u,v). \end{aligned}$$

Since $F(X \times X \times X \times X) \subseteq g(x)$, there exist $h,u,v,w \in X \times X \times X \times X$ such that $gh_1 = h, gu_1 = u, gv_1 = v$ and $gw_1 = w$. so from (2.23) we have

$$\begin{aligned} gh_1 &\in G(gh_1,gu_1,gv_1,gw_1) = F(h_1,u_1,v_1,w_1)\\ gu_1 &\in G(gu_1,gv_1,gw_1,gh_1) = F(u_1,v_1,w_1,h_1)\\ gv_1 &\in G(gv_1,gw_1,gh_1,gu_1) = F(v_1,w_1,h_1,u_1)\\ gw_1 &\in G(gv_1,gw_1,gh_1,gu_1) = F(w_1,h_1,u_1,v_1). \end{aligned}$$

This implies that $h_1,u_1,v_1,w_1 \in X \times X \times X \times X$ is a quadruple fixed point of F and g.

Now following as in [15] except from the inequalities satisfied by M we can show that F and g have a quadruple fixed point.

3 Conclusion

Thus, our paper establishes the results regarding the quadruple fixed point in Hausdorff fuzzy metric space.

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