Table **6.6**: Comparison of absolute errors of ***Example*** **6.3** for $2M = 8$.

y	Absolute errors for Haar wavelet	Absolute errors for Hermite wavelet
1/16	$1.9627e-003$	$7.1279e-008$
3/16	$2.0568e-003$	$5.3094e-008$
5/16	$2.2577e-003$	$5.9452e-008$
7/16	$2.5976e-003$	$6.1041e-008$
9/16	$3.1475e-003$	$6.9398e-008$
11/16	$4.0705e-003$	$7.7959e-008$
13/16	$5.7923e-003$	$1.1056e-007$
15/16	$9.7275e-003$	$8.6224e-008$

Table **6.5** represents the comparison of numerical solutions obtained by Haar and Hermite wavelet methods with exact solution of ***Example*** **6.3**. ***Table*** **6.6** represents the comparison of absolute errors obtained by Haar wavelet method and Hermite wavelet method. ***Figure*** **6.5** and ***Figure*** **6.6** show the absolute errors of ***Example*** **6.3** for $2M = 8$.

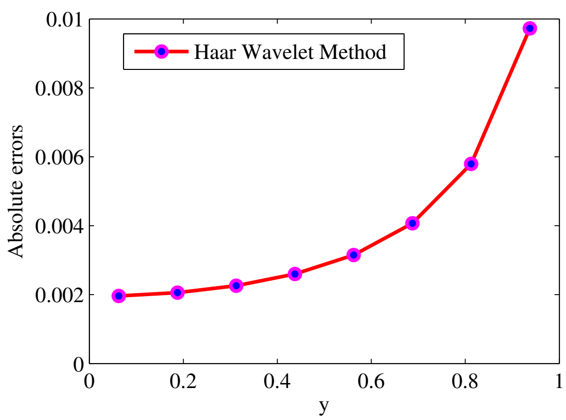

Figure **6.5**: Absolute errors of ***Example*** **6.3** for $2M = 8$.

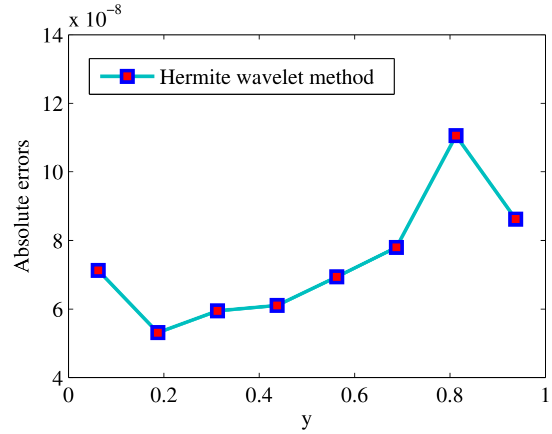

Figure **6.6**: Absolute errors of ***Example*** **6.3** for $2M = 8$.

7 Conclusion

From above discussion, it is concluded that the Hermite wavelet based collocation is much better in comparison to Haar wavelet based collocation method for solving nonlinear differential equations of the Bernoulli's type. For getting the necessary accuracy the number of collocation points may be increased.

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(Dedicated to Honor Dr. R. C. Singh Chandel on His 75^{th} Birth Anniversary Celebrations)

APPLICATION OF INTERVAL VALUED FUZZY SET AND SOFT SET

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Abstract

Molodtsov was a father of soft set approach. We can't easily settle the membership degree in some practical application. So it must be much better to describe interval-valued data instead of explaining membership degree. In this paper, we introduce the latest approach of the interval-valued fuzzy soft set by combining the interval-valued fuzzy set and soft set models. This approach successfully follows distributive, associative and DeMorgans laws as well. In the end, a decision problem is solved by this approach.
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1 Introduction

In 1999 a great researcher Molodtsov was born in Russia who gave an approach which is known as a soft set approach. There are many approach which deal with uncertainty for example, probability, the theory of fuzzy set etc. But this approach is different from all these approach. Molodtsov applied this approach and got positive results.

Many researchers are working on it, because of this approach has many application in several directions. Many operations applied on soft sets by Maji et al. [21]. This approach has also solved many judgment constructing issue by adopting fuzzy mathematics [20]. The characterization devaluation of soft sets analyzed by Chen at al. [10]. The main objective of Kong at al. [16, 17, 18] is to show a devaluation of soft sets and fuzzy soft sets. They also explained algorithm of normal parameter reduction.

By above analysis, we observed that classical soft set [26, 28] approach is the root of all above work. We link this model with any mathematical model. Aktas et al. [1] obtained elementary properties of soft groups by this approach. Feng et al. [11] suggested approach of soft semirings, soft subsemirings, soft ideals, idealistic soft semirings, soft semiring homomorphisms and their characteristics. Approach of soft BCK/BCI-algebras and soft subalgebras are presented by Jun [13, 14, 15]. Maji et al. [20] suggested this approach. It is a amalgam of fuzzy set and soft set models. Yange et al. [29] applied some process on soft sets, these were negation, triangular norm etc. Zou et al. [31] presented soft set, fuzzy soft set into the insufficient situation.

In fuzzy mathematics, The utility of fuzzy set theory in handling uncertainty, arising from deficiencies of information available from a situation in pattern recognition problems have been proposed by many researchers. Controversy has surrounded the concept of fuzziness since its inception. Some maintains that probability theory can handle any kind of uncertainty; other think that fuzziness is probability in disguise or that probability is only sensible way to describe any kind of uncertainty. This theory provide an approximate, effective and more flexible means of classifying the patterns which are too complex or have to ill-defined features to be handled by the classical approach. Both fuzzy logic and probability theory are closely related, the key difference is their meaning. Probability is associated with events and not facts, and those events will either occur or not occur. There is nothing fuzzy about it. Fuzzy Logic is all about degree of truth.

The merging of interval-valued fuzzy set and soft set are shown here. So we get latest soft set model which is Interval-valued Fuzzy Soft Set. To make this work easy, Section 2 having detail about standard soft set and fuzzy soft set. Section 3 having approach about interval-valued fuzzy soft set. We then apply the concept of complement, *AND* and *OR* on interval-valued fuzzy soft sets. Section 4, show that interval-valued fuzzy soft set successfully works on judgment constructing issue. At last we then give summary and direction for new research.We observed these hotels by net and saw their websites. First we saw these attributes and got the rating of all these attributes from their websites.

Triangular norms and conorms are operations which generalize the logical conjunction and logical disjunction to fuzzy logic. They are a natural interpretation of the conjunction and disjunction in the semantics of mathematical fuzzy logics and they are used to combine criteria in multi-criteria decision making.

2 Preliminaries

A fuzzy soft relation is defined as soft set over the fuzzy power set of the Cartesian product of two crisp sets.

By this process Molodtsov [22] explained soft set. Suppose M and N indicate universal set, parameter set respectively. We show soft set by this pair (F, N). In this case F indicates a function of N towards all subsets of set M.

Now make all subsets of M and indicated by $P(M)$, F mapping shown below [10].

$$F : N \rightarrow P(M),$$

$\forall n \in N$, $F(n)$ indicates the family of n-approximate members of soft sets (F, N) [2, 7, 23].

Suppose assemblage of every subsets of M can denote by $P(M)$. So fuzzy soft set for $\mathcal{P}(M)$ as reported by $(\tilde{F}, N)$. Here $\tilde{F}$ transformation represented.

$$\tilde{F} : N \rightarrow \mathcal{P}(M).$$

It indicates the parameterized group of fuzzy subsets of M. It is mapping from parameters to a universe, so this indicates the particular process of soft set.

In general, fuzzy value set for n is indicated by $\tilde{F}(n)$ and it is the subset of M. We can differentiate it from the classic soft set, suppose $(\mu_{\tilde{F}(n)}(HO))$ indicate the membership degree of element HO, which holds the parameter n here. $HO \in M$,$n \in N$. $\tilde{F}$ given below.

$$\tilde{F}(n) = \left\{(HO, \mu_{\tilde{F}(n)}(HO)) : HO \in M\right\}.$$

***Definition* 2.1** *The AND operation on the two interval-valued fuzzy soft sets* $(\tilde{F}, A)$, $(\tilde{G}, B)$ *is defined by*

$$(\tilde{F}, A) \wedge (\tilde{G}, B) = (\tilde{H}, A \times B),$$

where $\tilde{H}(\alpha, \beta) = \tilde{F}(\alpha) \cap \tilde{G}(\beta), \forall(\alpha, \beta) \in (A \times B).$

***Definition* 2.2** *The OR operation on the two interval-valued fuzzy soft sets* $(\tilde{F}, A)$, $(\tilde{G}, B)$ *is defined by*

$$(\tilde{F}, A) \vee (\tilde{G}, B) = (\tilde{H}, A \times B),$$

where $\tilde{H}(\alpha, \beta) = \tilde{F}(\alpha) \cup \tilde{G}(\beta), \forall(\alpha, \beta) \in (A \times B).$

3 Interval-valued fuzzy soft set

3.1 Approach of interval-valued fuzzy soft set

It can obtained with the help of fuzzy set and soft set theory definitely. Note that we can't easily settle the membership degree in some practical application. So it must be much better to describe interval-valued data instead of explaining membership degree.Due to this approach, Zadeh described this approach. Interval-valued fuzzy soft set model can be described by joining interval-valued fuzzy set and soft set. Now, we shortly suggest this theory. Mapping of interval-valued fuzzy set [8, 9, 30] $\hat{S}$ on the universe M.

$$\hat{S} : M \rightarrow Int([0, 1]),$$

Here, collection of every closest subintervals of [0, 1] is indicated by Int([0, 1]). Group of all interval-valued fuzzy sets on M is denoted by $\mathcal{P}(M)$.

Let $\hat{S} \in \tilde{P}(M), \forall m \in M$.

The degree of membership is given by $\mu_{\hat{S}}(HO) = [\mu^-_{\hat{S}}(HO), \mu^+_{\hat{S}}(HO)]$ of element HO to $\hat{S}$.

$\mu^-_{\hat{S}}(HO), \mu^+_{\hat{S}}(HO)$ Indicate the lower and upper membership degree of HO to $\hat{S}$. Here

$$\mu^-_{\hat{S}}(HO), \mu^+_{\hat{S}}(HO).$$

We can describe complement, intersection and other operation of the interval-valued fuzzy set here.

Suppose $\hat{S}$, $\hat{T} \in \tilde{\mathcal{P}}$(M)

1. $\hat{S}^C$ indicates the complement of $\hat{S}$
   $\mu_{\hat{S}^C}(HO) = 1 - \mu_{\hat{S}}(HO) = [1 - \mu^+_{\hat{S}}(HO), 1 - \mu^-_{\hat{S}}(HO)]$;
2. $\hat{S} \cap \hat{T}$ Indicates the intersection of $\hat{S}$ and $\hat{T}$
   $\mu_{\hat{S} \cap \hat{T}}(HO) = inf[\mu_{\hat{S}}(HO), \mu_{\hat{T}}(HO)]$
   $= [inf(\mu^-_{\hat{S}}(HO), \mu^-_{\hat{T}}(HO)), inf(\mu^+_{\hat{S}}(HO), \mu^+_{\hat{T}}(HO))]$;

Table **3.1:** $(I\tilde{v}f_1, \Omega_1)$ serve as interval valued fuzzy soft set.

M	n_1	n_2	n_3	n_4
HO_1	[0.6 – 0.8]	[0.4 – 0.6]	[0.6 – 0.8]	[0.7 – 0.9]
HO_2	[0.6 – 0.8]	[0.5 – 0.7]	[0.6 – 0.8]	[0.4 – 0.6]
HO_3	[0.8 – 1.0]	[0.7 – 0.9]	[0.8 – 1.0]	[0.8 – 1.0]
HO_4	[0.8 – 1.0]	[0.6 – 0.8]	[0.8 – 1.0]	[0.8 – 1.0]
HO_5	[0.6 – 0.8]	[0.5 – 0.7]	[0.7 – 0.9]	[0.7 – 0.9]
HO_6	[0.6 – 0.8]	[0.5 – 0.7]	[0.6 – 0.8]	[0.7 – 0.9]

Table **3.2:** $(I\tilde{v}f_2, \Omega_2)$ serve as interval valued fuzzy soft set.

M	σ_1	σ_2	σ_3
HO_1	[0.6 – 0.8]	[0.4 – 0.6]	[0.6 – 0.8]
HO_2	[0.6 – 0.8]	[0.4 – 0.6]	[0.6 – 0.8]
HO_3	[0.8 – 1.0]	[0.7 – 0.9]	[0.8 – 1.0]
HO_4	[0.8 – 1.0]	[0.7 – 0.9]	[0.8 – 1.0]
HO_5	[0.7 – 0.9]	[0.6 – 0.8]	[0.7 – 0.9]
HO_6	[0.6 – 0.8]	[0.4 – 0.6]	[0.6 – 0.8]

3. $\hat{S} \cup \hat{T}$ indicates the union of $\hat{S}$ and $\hat{T}$.
   $\mu_{\hat{S}\cup\hat{T}}(HO) = sup[\mu_{\hat{S}}(HO), \mu_{\hat{T}}(HO)] = [sup(\mu^-_{\hat{S}}(HO), \mu^-_{\hat{T}}(HO)), sup(\mu^+_{\hat{S}}(HO), \mu^+_{\hat{T}}(HO))]$.

Suppose M and N indicate the universal set, parameters set respectively, an ordered pair $(I\tilde{v}f_1, N)$ indicates interval-valued fuzzy soft set for $\tilde{\mathcal{P}}(M)$, here $I\tilde{v}f_1$ defined below [3, 4, 5].

$$I\tilde{v}f_1 : N \to \tilde{\mathcal{P}}(M).$$

By parameterized family of subsets of M we prepare interval-valued fuzzy soft set. Hence, its universe is the set of all interval-valued fuzzy sets of M, i.e. $\tilde{\mathcal{P}}(M)$.
$\forall n \in N$, the interval fuzzy value set of guideline represented by $I\tilde{v}f_1(n)$, which indicates interval-valued fuzzy set of M where $HO \in M$ and $n \in N$, we write in the form:

$$I\tilde{v}f_1(n) = \{\langle HO, \mu_{I\tilde{v}f_1(n)}(HO)\rangle : HO \in M\},$$

here, $I\tilde{v}f_1(n)$ is the interval-valued fuzzy membership degree of object HO which holds on guideline n $\forall$ $n \in N$, $\forall$ $HO \in M$, $\mu^-_{I\tilde{v}f_1(n)}(HO) = \mu^+_{I\tilde{v}f_1(n)}(HO)$, then $I\tilde{v}f_1(n)$ will degenerated to be a standard fuzzy set and then $(I\tilde{v}f_1, N)$ will be degenerated to be a traditional fuzzy soft set. Let us consider,

- M is the collection of hotel according to choice and M=$\{HO_1, HO_2, HO_3, HO_4, HO_5, HO_6\}$
- Ω_1 is the collection of attributes of hotel,
  Ω_1=$\{n_1, n_2, n_3, n_4\}$={Staff, Value for Money, Facilities, Location}

An interval valued fuzzy soft set $(I\tilde{v}f_1, \Omega_1)$ has given in ***Table* 3.1**.

3.2 Source of Numerical Data

The lower and upper limits of an evaluation is given. For example, we cannot present the precise degree of how beautiful hotel HO_1 is, yet, hotel HO_1 is at least beautiful on the degree of 0.6 and it is at most beautiful on the degree of 0.8. According to these attributes. By questionnaire to 150 visiters who visites these hotels, we collected membership value of these attributes. For example: By net we obtained the staff rating of hotel Le Meridien is 8.7. So 90 percent visiters assign this [0.6-0.8] and 10 percent [0.7-0.9] then we give [0.6-0.8] vale to the staff attribute. likewise, we make interval valued fuzzy soft sets for all attributes of all hotels.

Let $(I\tilde{v}f_1, N)$ indicates interval-valued fuzzy soft set for $\tilde{\mathcal{P}}$(M), $I\tilde{v}f_1$(n) indicates interval fuzzy value set for n, then all interval fuzzy value sets in interval-valued fuzzy soft set $(I\tilde{v}f_1, N)$ are specify to as the interval fuzzy value class of $(I\tilde{v}f_1, N)$, $G_{(I\tilde{v}f_1,N)}$ symbolic representation of it and presented as follows [19, 24, 25].

$$G_{(I\tilde{v}f_1,N)} = \{I\tilde{v}f_1(n) : n \in N\}.$$

***Table* 3.1**, we got $G_{(I\tilde{v}f_1,\Omega_1)} = \{I\tilde{v}f_1(n_1), I\tilde{v}f_1(n_2), I\tilde{v}f_1(n_3), I\tilde{v}f_1(n_4)\}$here:
$I\tilde{v}f_1(n_1) = \{(HO_1, [0.6-0.8]), (HO_2, [0.6-0.8]), (HO_3, [0.8-1.0]), (HO_4, [0.8-1.0]), (HO_5, [0.6-0.8]), (HO_6, [0.6-0.8])\}$
$I\tilde{v}f_1(n_2) = \{(HO_1, [0.4-0.6]), (HO_2, [0.5-0.7]), (HO_3, [0.7-0.9]), (HO_4, [0.6-0.8]), (HO_5, [0.5-0.7]), (HO_6, [0.5-0.7])\}$
$I\tilde{v}f_1(n_3) = \{(HO_1, [0.6-0.8]), (HO_2, [0.6-0.8]), (HO_3, [0.8-1.0]), (HO_4, [0.8-1.0]), (HO_5, [0.7-0.9]), (HO_6, [0.6-0.8])\}$
$I\tilde{v}f_1(n_4) = \{(HO_1, [0.7-0.9]), (HO_2, [0.9-0.6]), (HO_3, [0.8-1.0]), (HO_4, [0.8-1.0]), (HO_5, [0.7-0.9]), (HO_6, [0.7-0.9])\}$

Suppose M and N indicates universal set and parameter set respectively, let $\Omega_1, \Omega_2 \subset N$, $(I\tilde{v}f_1, \Omega_1)$ and $(I\tilde{v}f_2, \Omega_2)$ indicates interval-valued fuzzy soft sets, $(I\tilde{v}f_1, \Omega_1)$ is an interval-valued fuzzy soft subset of $(I\tilde{v}f_2, \Omega_2)$ if

1. $\Omega_1 \subset \Omega_2$
2. $\forall\ n \in \Omega_1, I\tilde{v}f_1(n)$

shows interval-valued fuzzy subset of $I\tilde{v}f_2(n)$; shown as follows $(I\tilde{v}f_1, \Omega_1)\tilde{\subset}(I\tilde{v}f_2, \Omega_2)$.
$(I\tilde{v}f_1, \Omega_1)$ and $(I\tilde{v}f_2, \Omega_2)$ indicates two interval-valued fuzzy soft sets, $M = \{HO_1, HO_2, HO_3, HO_4, HO_5, HO_6\}$. M indicates the collection of hotels, $\Omega_1 = \{n_1, n_2\} = \{Staff, ValueforMoney\}$, Ω_2=$\{n_1, n_2, n_3\}$=$\{Staff, ValueforMoney, Facilities\}$, and
$I\tilde{v}f_1(n_1) = \{(HO_1, [0.6-0.8]), (HO_2, [0.6-0.8]), (HO_3, [0.8-1.0]), (HO_4, [0.8-1.0]), (HO_5, [0.6-0.8]), (HO_6, [0.6-0.8])\}$
$I\tilde{v}f_1(n_2) = \{(HO_1, [0.4-0.6]), (HO_2, [0.5-0.7]), (HO_3, [0.7-0.9]), (HO_4, [0.6-0.8]), (HO_5, [0.5-0.7]), (HO_6, [0.5-0.7])\}$
$I\tilde{v}f_2(n_1) = \{(HO_1, [0.8-1.0]), (HO_2, [0.6-0.8]), (HO_3, [0.9-1.0]), (HO_4, [0.8-1.0]), (HO_5, [0.8-1.0]), (HO_6, [0.8-1.0])\}$
$I\tilde{v}f_2(n_2) = \{(HO_1, [0.6-0.7]), (HO_2, [0.9-1.0]), (HO_3, [0.8-0.9]), (HO_4, [0.7-0.8]), (HO_5, [0.6-0.8]), (HO_6, [0.9-1.0])\}$
$I\tilde{v}f_2(n_3) = \{(HO_1, [0.6-0.8]), (HO_2, [0.6-0.8]), (HO_3, [0.8-1.0]), (HO_4, [0.8-1.0]), (HO_5, [0.6-0.8]), (HO_6, [0.6-0.8])\}$

Obviously, we can see $(I\tilde{v}f_1, \Omega_1)\ \tilde{\subset}\ (I\tilde{v}f_2, \Omega_2)$. Suppose $(I\tilde{v}f_1, \Omega_1)$ and $(I\tilde{v}f_2, \Omega_2)$ indicates two interval-valued fuzzy soft sets, These two sets will equal if

1. $(I\tilde{v}f_1, \Omega_1)$ is a subset of $(I\tilde{v}f_2, \Omega_2)$,
2. $(I\tilde{v}f_2, \Omega_2)$ is a subset of $(I\tilde{v}f_1, \Omega_1)$.

It is represented as $(I\tilde{v}f_1, \Omega_1) = (I\tilde{v}f_2, \Omega_2)$.

Operation on interval-valued fuzzy soft sets $(I\tilde{v}f_1, \Omega_1)^C$ is the complement of $(I\tilde{v}f_1, \Omega_1)$ and it is explained as $(I\tilde{v}f_1, \Omega_1)^C$=$(I\tilde{v}f_1{}^C, \neg\Omega_1)$,
Here $\forall\ \theta_1 \in \Omega_1$, $\neg\ \theta_1$=not θ_1, not belongs to the parameter θ_1, it means it is opposite of θ_1;

$$I\tilde{v}f_1{}^C : \neg\Omega_1 \rightarrow \tilde{\mathcal{P}}(M),$$

is the function given by $I\tilde{v}f_1{}^C(\theta_2) = (I\tilde{v}f_1(\neg\theta_2))^C, \forall\theta_2 \in \neg\Omega_1$.

Another interval-valued fuzzy soft set taken into consideration $(I\tilde{v}f_2, \Omega_2)$ and given in ***Table* 3.2**. Universal set M is same in both table, i.e $M = \{HO_1, HO_2, HO_3, HO_4, HO_5, HO_6\}$ it is the collection of hotels; Ω_2= $\{\sigma_1, \sigma_2, \sigma_3\}$ = {Cleanliness, Free WiFi, Comfort} it is the collection of parameters.

So, with the help **1** of ***Section* 3.1**, we get
$I\tilde{v}f_2{}^C(\neg\sigma_1) = \{(HO_1, [0.2, 0.4]), (HO_2, [0.2, 0.4]), (HO_3, [0.0, 0.2]), (HO_4, [0.0, 0.2]), (HO_5, [0.1, 0.3]), (HO_6, [0.2, 0.4])\}$
$I\tilde{v}f_2{}^C(\neg\sigma_2) = \{(HO_1, [0.4, 0.6]), (HO_2, [0.4, 0.6]), (HO_3, [0.1, 0.3]), (HO_4, [0.1, 0.3]), (HO_5, [0.2, 0.4]), (HO_6, [0.4, 0.6])\}$
$I\tilde{v}f_2{}^C(\neg\sigma_3) = \{(HO_1, [0.2, 0.4]), (HO_2, [0.2, 0.4]), (HO_3, [0.0, 0.2]), (HO_4, [0.0, 0.2]), (HO_5, [0.1, 0.3]), (HO_6, [0.2, 0.4])\}$

The "*AND*" procedure on $(I\tilde{v}f_1, \Omega_1)$ $(I\tilde{v}f_2, \Omega_2)$ is shown below

$$(I\tilde{v}f_1, \Omega_1) \wedge (I\tilde{v}f_2, \Omega_2) = (I\tilde{v}f_3, \Omega_1 \times \Omega_2).$$

Here

$$I\tilde{v}f_3(\theta_1, \theta_2) = I\tilde{v}f_1(\theta_1) \cap I\tilde{v}f_2(\theta_2), \forall(\theta_1, \theta_2) \in \Omega_1 \times \Omega_2.$$

We can apply "AND" procedure on $(I\tilde{v}f_1, \Omega_1)$ and $(I\tilde{v}f_2, \Omega_2)$ and can show in tabular form below.

Table **3.3:** "AND" procedure on *Tables* **3.1** and **3.2**.

M	$n_1\sigma_1$	$n_1\sigma_2$	$n_1\sigma_3$	$n_2\sigma_1$	$n_2\sigma_2$	$n_2\sigma_3$
HO_1	[0.6, 0.8]	[0.4, 0.6]	[0.6, 0.8]	[0.4, 0.6]	[0.4, 0.6]	[0.4, 0.6]
HO_2	[0.6, 0.8]	[0.4, 0.6]	[0.6, 0.8]	[0.5, 0.7]	[0.4, 0.6]	[0.5, 0.7]
HO_3	[0.8, 1.0]	[0.7, 0.9]	[0.8, 1.0]	[0.7, 0.9]	[0.7, 0.9]	[0.7, 0.9]
HO_4	[0.8, 1.0]	[0.7, 0.9]	[0.8, 1.0]	[0.6, 0.8]	[0.6, 0.8]	[0.6, 0.8]
HO_5	[0.6, 0.8]	[0.6, 0.8]	[0.6, 0.8]	[0.5, 0.7]	[0.5, 0.7]	[0.5, 0.7]
HO_6	[0.6, 0.8]	[0.4, 0.6]	[0.6, 0.8]	[0.5, 0.7]	[0.4, 0.6]	[0.5, 0.7]
M	$n_3\sigma_1$	$n_3\sigma_2$	$n_3\sigma_3$	$n_4\sigma_1$	$n_4\sigma_2$	$n_4\sigma_3$
HO_1	[0.6, 0.8]	[0.4, 0.6]	[0.6, 0.8]	[0.6, 0.8]	[0.4, 0.6]	[0.6, 0.8]
HO_2	[0.6, 0.8]	[0.4, 0.6]	[0.6, 0.8]	[0.4, 0.6]	[0.4, 0.6]	[0.4, 0.6]
HO_3	[0.8, 1.0]	[0.7, 0.9]	[0.8, 1.0]	[0.8, 1.0]	[0.7, 0.9]	[0.8, 1.0]
HO_4	[0.8, 1.0]	[0.7, 0.9]	[0.8, 1.0]	[0.8, 1.0]	[0.7, 0.9]	[0.8, 1.0]
HO_5	[0.7, 0.9]	[0.6, 0.8]	[0.7, 0.9]	[0.7, 0.9]	[0.6, 0.8]	[0.7, 0.9]
HO_6	[0.6, 0.8]	[0.4, 0.6]	[0.6, 0.8]	[0.6, 0.8]	[0.4, 0.6]	[0.6, 0.8]

Table **3.4:** "OR" procedure on *Tables* **3.1** and **3.2**.

M	$n_1\sigma_1$	$n_1\sigma_2$	$n_1\sigma_3$	$n_2\sigma_1$	$n_2\sigma_2$	$n_2\sigma_3$
HO_1	[0.6, 0.8]	[0.6, 0.8]	[0.6, 0.8]	[0.6, 0.8]	[0.4, 0.6]	[0.6, 0.8]
HO_2	[0.6, 0.8]	[0.6, 0.8]	[0.6, 0.8]	[0.6, 0.8]	[0.5, 0.7]	[0.6, 0.8]
HO_3	[0.8, 1.0]	[0.8, 1.0]	[0.8, 1.0]	[0.8, 1.0]	[0.7, 0.9]	[0.8, 1.0]
HO_4	[0.8, 1.0]	[0.8, 1.0]	[0.8, 1.0]	[0.8, 1.0]	[0.7, 0.9]	[0.8, 1.0]
HO_5	[0.7, 0.9]	[0.6, 0.8]	[0.7, 0.9]	[0.7, 0.9]	[0.6, 0.8]	[0.7, 0.9]
HO_6	[0.6, 0.8]	[0.6, 0.8]	[0.6, 0.8]	[0.6, 0.8]	[0.5, 0.7]	[0.6, 0.8]
M	$n_3\sigma_1$	$n_3\sigma_2$	$n_3\sigma_3$	$n_4\sigma_1$	$n_4\sigma_2$	$n_4\sigma_3$
HO_1	[0.6, 0.8]	[0.6, 0.8]	[0.6, 0.8]	[0.7, 0.9]	[0.7, 0.9]	[0.7, 0.9]
HO_2	[0.6, 0.8]	[0.6, 0.8]	[0.6, 0.8]	[0.6, 0.8]	[0.4, 0.6]	[0.6, 0.8]
HO_3	[0.8, 1.0]	[0.8, 1.0]	[0.8, 1.0]	[0.8, 1.0]	[0.8, 1.0]	[0.8, 1.0]
HO_4	[0.8, 1.0]	[0.8, 1.0]	[0.8, 1.0]	[0.8, 1.0]	[0.8, 1.0]	[0.8, 1.0]
HO_5	[0.7, 0.9]	[0.7, 0.9]	[0.7, 0.9]	[0.7, 0.9]	[0.7, 0.9]	[0.7, 0.9]
HO_6	[0.6, 0.8]	[0.6, 0.8]	[0.6, 0.8]	[0.7, 0.9]	[0.7, 0.9]	[0.7, 0.9]

By ***Definition*** **2.1**, we obtain
$I\tilde{v}f_3(n_1, \sigma_2) = I\tilde{v}f_1(n_1) \cap I\tilde{v}f_2(\sigma_2)$
$=\{(HO_1, [0.4, 0.6]), (HO_2, [0.4, 0.6]), (HO_3, [0.7, 0.9]), (HO_4, [0.7, 0.9]), (HO_5, [0.6, 0.8]), (HO_6, [0.4, 0.6])\}$
By ***Table*** **3.3** we obtain the value of $(I\tilde{v}f_1, \Omega_1) \wedge (I\tilde{v}f_2, \Omega_2)$

The "OR" procedure on $(I\tilde{v}f_1, \Omega_1)$ $(I\tilde{v}f_2, \Omega_2)$ is shown below [6, 12].

$$(I\tilde{v}f_1, \Omega_1) \vee (I\tilde{v}f_2, \Omega_2) = (I\tilde{v}f_3, \Omega_1 \times \Omega_2),$$

Here $I\tilde{v}f_3(\theta_1,\theta_2)=I\tilde{v}f_1(\theta_1) \cup I\tilde{v}f_2(\theta_2)$, $\forall\ (\theta_1,\theta_2) \in \Omega_1 \times \Omega_2$.

The output of the "OR" procedure on $(I\tilde{v}f_1, \Omega_1)$ and $(I\tilde{v}f_2, \Omega_2)$ in ***Tables*** **3.1** and **3.2** is given in ***Table*** **3.4**.

Let us, we apply DeMorgan's Laws on two interval-valued fuzzy soft sets $(I\tilde{v}f_1, \Omega_1)$, $(I\tilde{v}f_2, \Omega_2)$ as shown below.
$((I\tilde{v}f_1, \Omega_1) \wedge (I\tilde{v}f_2, \Omega_2))^C = (I\tilde{v}f_1, \Omega_1)^C \vee (I\tilde{v}f_2, \Omega_2)^C$
$((I\tilde{v}f_1, \Omega_1) \vee (I\tilde{v}f_2, \Omega_2))^C = (I\tilde{v}f_1, \Omega_1)^C \wedge (I\tilde{v}f_2, \Omega_2)^C$
Proof. $(I\tilde{v}f_1, \Omega_1)^C \vee (I\tilde{v}f_2, \Omega_2)^C = (I\tilde{v}f_1{}^C, \neg\Omega_1) \vee (I\tilde{v}f_2{}^C, \neg\Omega_2) = (I\tilde{v}f_4, \neg\Omega_1 \times \neg\Omega_2)$
$where, I\tilde{v}f_4(\neg\theta_1, \neg\theta_2) = I\tilde{v}f_1{}^C(\neg\theta_1) \cup I\tilde{v}f_2{}^C(\neg\theta_2)$
$= (I\tilde{v}f_4, \neg(\Omega_1 \times \Omega_2))$.
Assume $(I\tilde{v}f_1, \Omega_1) \wedge (I\tilde{v}f_2, \Omega_2) = (I\tilde{v}f_3, \Omega_1 \times \Omega_2)$, then we have $((I\tilde{v}f_1, \Omega_1) \wedge (I\tilde{v}f_2, \Omega_2))^C = (I\tilde{v}f_3, \Omega_1 \times \Omega_2)^C = (I\tilde{v}f_3{}^C, \neg(\Omega_1 \times \Omega_2))$.

***Table* 3.5:** Consequence interval-valued fuzzy soft set $(I\tilde{v}f_3, \Omega_3)$

M	$n_1\sigma_2$	$n_2\sigma_1$	$n_3\sigma_2$	$n_3\sigma_3$	$n_4\sigma_1$
HO_1	[0.4 – 0.6]	[0.4 – 0.6]	[0.4 – 0.6]	[0.6 – 0.8]	[0.6 – 0.8]
HO_2	[0.4 – 0.6]	[0.5 – 0.7]	[0.4 – 0.6]	[0.6 – 0.8]	[0.4 – 0.6]
HO_3	[0.7 – 0.9]	[0.7 – 0.9]	[0.7 – 0.9]	[0.8 – 1.0]	[0.8 – 1.0]
HO_4	[0.7 – 0.9]	[0.6 – 0.8]	[0.7 – 0.9]	[0.8 – 1.0]	[0.8 – 1.0]
HO_5	[0.6 – 0.8]	[0.5 – 0.7]	[0.6 – 0.8]	[0.7 – 0.9]	[0.7 – 0.9]
HO_6	[0.4 – 0.6]	[0.5 – 0.7]	[0.4 – 0.6]	[0.6 – 0.8]	[0.6 – 0.8]

$\forall\, (\theta_1, \theta_2) \in \Omega_1 \times \Omega_2$, we know
$I\tilde{v}f_3^{\,C}(\neg\theta_1, \neg\theta_2) = (I\tilde{v}f_3(\theta_1, \theta_2))^C = (I\tilde{v}f_1(\theta_1) \cap I\tilde{v}f_2(\theta_2))^C = (I\tilde{v}f_1(\theta_1))^C \cup (I\tilde{v}f_2(\theta_2))^C = (I\tilde{v}f_1^{\,C}(\neg\theta_1)) \cup (I\tilde{v}f_2^{\,C}(\neg\theta_2))$

By above analysis,we got $((I\tilde{v}f_1, \Omega_1) \wedge (I\tilde{v}f_2, \Omega_2))^C = (I\tilde{v}f_1, \Omega_1)^C \vee (I\tilde{v}f_2, \Omega_2)^C$.

In the same way, we can prove that $((I\tilde{v}f_1, \Omega_1) \vee (I\tilde{v}f_2, \Omega_2))^C = (I\tilde{v}f_1, \Omega_1)^C \wedge (I\tilde{v}f_2, \Omega_2)^C$.

Suppose we have interval-valued fuzzy soft sets, $(I\tilde{v}f_1, \Omega_1), (I\tilde{v}f_2, \Omega_2) and (I\tilde{v}f_3, \Omega_3)$.

[**Associative law**]
$(I\tilde{v}f_1, \Omega_1) \wedge ((I\tilde{v}f_2, \Omega_2) \wedge (I\tilde{v}f_3, \Omega_3)) = ((I\tilde{v}f_1, \Omega_1) \wedge (I\tilde{v}f_2, \Omega_2)) \wedge (I\tilde{v}f_3, \Omega_3)$.
$(I\tilde{v}f_1, \Omega_1) \vee ((I\tilde{v}f_2, \Omega_2) \vee (I\tilde{v}f_3, \Omega_3)) = ((I\tilde{v}f_1, \Omega_1) \vee (I\tilde{v}f_2, \Omega_2)) \vee (I\tilde{v}f_3, \Omega_3)$.

[**Distributive law**]
$(I\tilde{v}f_1, \Omega_1) \wedge ((I\tilde{v}f_2, \Omega_2) \vee (I\tilde{v}f_3, \Omega_3)) = ((I\tilde{v}f_1, \Omega_1) \wedge (I\tilde{v}f_2, \Omega_2)) \vee ((I\tilde{v}f_1, \Omega_1) \wedge (I\tilde{v}f_3, \Omega_3))$.
$(I\tilde{v}f_1, \Omega_1) \vee ((I\tilde{v}f_2, \Omega_2) \wedge (I\tilde{v}f_3, \Omega_3)) = ((I\tilde{v}f_1, \Omega_1) \vee (I\tilde{v}f_2, \Omega_2)) \wedge ((I\tilde{v}f_1, \Omega_1) \vee (I\tilde{v}f_3, \Omega_3))$.

Proof. $\forall\, \theta_1 \in \Omega_1, \theta_2 \in \Omega_2$ and $\forall \theta_3 \in \Omega_3$, Now
$I\tilde{v}f_1(\theta_1) \cap (I\tilde{v}f_2(\theta_2) \cap I\tilde{v}f_3(\theta_3)) = (I\tilde{v}f_1(\theta_1) \cap I\tilde{v}f_2(\theta_2)) \cap I\tilde{v}f_3(\theta_3)$
from it, we can say $(I\tilde{v}f_1, \Omega_1) \wedge ((I\tilde{v}f_2, \Omega_2) \wedge (I\tilde{v}f_3, \Omega_3)) = ((I\tilde{v}f_1, \Omega_1) \wedge (I\tilde{v}f_2, \Omega_2)) \wedge (I\tilde{v}f_3, \Omega_3)$.

In the same way, we can say $(I\tilde{v}f_1, \Omega_1) \vee ((I\tilde{v}f_2, \Omega_2) \vee (I\tilde{v}f_3, \Omega_3)) = ((I\tilde{v}f_1, \Omega_1) \vee (I\tilde{v}f_2, \Omega_2)) \vee (I\tilde{v}f_3, \Omega_3)$.
$\forall\, \theta_1 \in \Omega_1, \theta_2 \in \Omega_2$ and $\forall \theta_3 \in \Omega_3$, we have $I\tilde{v}f_1(\theta_1) \cap (I\tilde{v}f_2(\theta_2) \cup I\tilde{v}f_3(\theta_3)) = (I\tilde{v}f_1(\theta_1) \cap I\tilde{v}f_2(\theta_2)) \cup (I\tilde{v}f_1(\theta_1) \cap I\tilde{v}f_3(\theta_3))$

In the same way, we can say $(I\tilde{v}f_1, \Omega_1) \wedge ((I\tilde{v}f_2, \Omega_2) \vee (I\tilde{v}f_3, \Omega_3)) = ((I\tilde{v}f_1, \Omega_1) \wedge (I\tilde{v}f_2, \Omega_2)) \vee ((I\tilde{v}f_1, \Omega_1) \wedge (I\tilde{v}f_3, \Omega_3))$.

Similarity, we also have $(I\tilde{v}f_1, \Omega_1) \vee ((I\tilde{v}f_2, \Omega_2) \wedge (I\tilde{v}f_3, \Omega_3)) = ((I\tilde{v}f_1, \Omega_1) \vee (I\tilde{v}f_2, \Omega_2)) \wedge ((I\tilde{v}f_1, \Omega_1) \vee (I\tilde{v}f_3, \Omega_3))$.

4 Application of interval-valued fuzzy soft set

Object can be identified by using algorithm, given by Roy et al. [27], different objects are compared in it. Kong et al. [18] said that Roy's innovation has error, Kong gave correct innovation, in it they compare choice values of different objects.

***Rule* 4.1**:Take the set of interval-valued fuzzy soft sets.
$(I\tilde{v}f_1, \Omega_1)$ and $(I\tilde{v}f_2, \Omega_2)$ demonstrate in ***Tables* 3.1** and **3.2** are under deliberation.
Here, we obtain the value of sets $(I\tilde{v}f_1, \Omega_1)$ and $(I\tilde{v}f_2, \Omega_2)$ from ***Tables* 3.1** and **3.2**.

***Rule* 4.2**:Input the guideline set X as detected by the viewer.
We take these two sets $(I\tilde{v}f_1, \Omega_1)$ and $(I\tilde{v}f_2, \Omega_2)$,***Table* 3.3** obtained by applying "*AND*" procedure on these sets. Suppose we are taking these values of parameters X such that X={{$n_1\sigma_2$},{$n_2\sigma_1$},
{$n_3\sigma_2$},{$n_3\sigma_3$},{$n_4\sigma_1$}}.

***Rule* 4.3**:We obtain resultant $(I\tilde{v}f_3, \Omega_3)$ set from $(I\tilde{v}f_1, \Omega_1)$ and $(I\tilde{v}f_2, \Omega_2)$.
***Table* 3.5** the resultant interval-valued fuzzy soft set is obtained from ***Table* 3.3** according to the set of parameters.

***Rule* 4.4**: $\forall HO_i \in M$, In this way, we calculate the choice value d for every hotel HO.

$$d_i = [d_i^-, d_i^+] = \Big[\sum_{r\in R} \mu^-_{I\tilde{v}f_3(r)}(HO_i), \sum_{r\in R} \mu^+_{I\tilde{v}f_3(r)}(HO_i)\Big].$$

The result is shown in ***Table* 4.1**.

Table **4.1:** Choice value.

M	$n_1\sigma_2$	$n_2\sigma_1$	$n_3\sigma_2$	$n_3\sigma_3$	$n_4\sigma_1$	d_i
HO_1	$[0.4, 0.6]$	$[0.4, 0.6]$	$[0.4, 0.6]$	$[0.6, 0.8]$	$[0.6, 0.8]$	$d_1 = [2.4, 3.4]$
HO_2	$[0.4, 0.6]$	$[0.5, 0.7]$	$[0.4, 0.6]$	$[0.6, 0.8]$	$[0.4, 0.6]$	$d_2 = [2.3, 3.3]$
HO_3	$[0.7, 0.9]$	$[0.7, 0.9]$	$[0.7, 0.9]$	$[0.8, 1.0]$	$[0.8, 1.0]$	$d_3 = [3.7, 4.7]$
HO_4	$[0.7, 0.9]$	$[0.6, 0.8]$	$[0.7, 0.9]$	$[0.8, 1.0]$	$[0.8, 1.0]$	$d_4 = [3.6, 4.6]$
HO_5	$[0.6, 0.8]$	$[0.5, 0.7]$	$[0.6, 0.8]$	$[0.7, 0.9]$	$[0.7, 0.9]$	$d_5 = [3.1, 4.1]$
HO_6	$[0.4, 0.6]$	$[0.5, 0.7]$	$[0.4, 0.6]$	$[0.6, 0.8]$	$[0.6, 0.8]$	$d_6 = [2.5, 3.5]$

Rule **4.5**:$\forall HO_i \in M$,calculate the value of t_i for each hotel HO_i in this way.

$$t_i = \sum_{HO_i \in M} ((d_i^- - d_j^-) + (d_i^+ - d_j^+)).$$

By calculation, we obtained $t_1 = -6.4, t_2 = -7.4, t_3 = 9.2, t_4 = 8.0, t_5 = 2.0, t_6 = -5.2$.
Rule **4.6**:The solution is any one of the elements in Z here $Z = max_{HO_i \in M}(t_i)$
In this problem, hotel HO_3 is the best option because $max_{HO_i \in M}(t_i) = (HO_3)$. This result is reasonable because we can see that $d_3 > d_i$ here i=1,2,3,4,5,6 i.e. HO_3 has the highest choice value.

5 Conclusion

Soft set theory solves problems which contain uncertainty, fuzziness or vagueness. At last, an example shows that interval-valued fuzzy soft set works properly in judgment constructing issue. In previous work Yang at al. [29] took house problem which can be solved by Fuzzy Soft Maximum Minimum Decision Making Method. In the present work we took very important daily life problem which can only solved by this method. There are parameterizations reduction of interval-valued fuzzy soft set which is new topic for further research. In the end, this work also having the utilization in the field of automobile for buying car, buying laptop according to our desire parameter.

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UPPER BOUND ON FOURTH HANKEL DETERMINANT FOR CERTAIN SUBCLASS OF MULTIVALENT FUNCTIONS

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Abstract

The present investigation is concerned with the estimation of the upper bound to the $H_4(p)$ Hankel determinant for a subclass of p-valent functions in the open unit disc $E = \{z :| z |< 1\}$. This work will motivate the researchers to work in the direction of investigation of fourth Hankel determinant for several other subclasses of univalent and multivalent functions.
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1 Introduction

Let P denote the class of analytic functions $p(z)$ of the form

$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n,$

whose real parts are positive in E.

By A_p, we denote the class of functions of the form

(1.1) $f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k, (p \in N = \{1, 2, 3, ...\}),$

which are analytic in the unit disc $E = \{z :| z |< 1\}$ and normalized by $f(0) = f'(0) - 1 = 0$.

Let S be the class $A_1 \equiv A$ consisting of functions of the form (1.1) and which are univalent in E.

Let R represent the class of functions $f \in A$, which satisfy the condition

$Re(f'(z)) > 0.$

The class R was introduced by MacGregor [12] and functions in this class are called bounded turning functions.

By R_1, we denote the class of functions $f \in A$, with the condition that

$$Re\left(\frac{f(z)}{z}\right) > 0.$$

R_1 is a subclass of close-to-star functions and was studied by MacGregor [13].

Further, Murugusundramurthi and Magesh [15] introduced the following class:

$$R(\alpha) = \left\{f : f \in A, Re\left\{(1-\alpha)\frac{f(z)}{z} + \alpha f'(z)\right\} > 0, 0 \le \alpha \le 1, z \in E\right\}.$$

In particular, $R(1) \equiv R$ and $R(0) \equiv R_1$.

Later on, Vamshee Krishna et al. [8] introduced a subclass of p-valent functions as follows:

$$RT_p = \left\{f : f \in A_p, Re\left(\frac{f'(z)}{pz^{p-1}}\right) > 0, z \in E\right\}.$$

For $p = 1$, $RT_1 \equiv R$.

Motivated by the above defined classes, Amourah et al. [2] defined the following subclass of p-valent functions:

$$R_p(\alpha) = \left\{f : f \in A_p, Re\left\{(1-\alpha)\frac{f(z)}{z^p} + \alpha\frac{f'(z)}{pz^{p-1}}\right\} > 0, 0 \le \alpha \le 1, z \in E\right\}.$$

The following observations are obvious:

(i) $R_1(\alpha) \equiv R(\alpha)$,
(ii) $R_p(1) \equiv RT_p$,
(iii) $R_1(1) \equiv R$,
(iv) $R_1(0) \equiv R_1$.

In 1976, Noonan and Thomas [16] stated the qth Hankel determinant for $q \geq 1$ and $n \geq 1$ as

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q+1} \\ a_{n+1} & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a_{n+q+1} & \cdots & \cdots & a_{n+2q-2} \end{vmatrix}.$$

In the particular cases, $q = 2, n = p, a_1 = 1$ and $q = 2, n = p + 1$, the Hankel determinant simplifies respectively to $H_2(p) = |a_{p+2} - a_{p+1}^2|$ and $H_2(p+1) = |a_{p+1}a_{p+3} - a_{p+2}^2|$.

This paper is concerned with the Hankel determinant in the case $q = 3$ and $n = p$ as

$$H_3(p) = \begin{vmatrix} a_p & a_{p+1} & a_{p+2} \\ a_{p+1} & a_{p+2} & a_{p+3} \\ a_{p+2} & a_{p+3} & a_{p+4} \end{vmatrix},$$

which is known as Hankel determinant of order 3.

For $f \in A_p$ and $a_p = 1$, we have

$$H_3(p) = a_{p+2}(a_{p+1}a_{p+3} - a_{p+2}^2) - a_{p+3}(a_{p+3} - a_{p+1}a_{p+2}) + a_{p+4}(a_{p+2} - a_{p+1}^2),$$

and using the triangle inequality, it yields

$$(1.2)\quad |H_3(p)| \leq |a_{p+2}||a_{p+1}a_{p+3} - a_{p+2}^2| + |a_{p+3}||a_{p+3} - a_{p+1}a_{p+2}| + |a_{p+4}||a_{p+2} - a_{p+1}^2|.$$

For any $f \in A_p$ of the form (1.1), we can represent the fourth Hankel determinant as

$$(1.3)\quad H_{4,p}(f) = a_{p+6}H_3(p) - a_{p+5}D_1 + a_{p+4}D_2 - a_{p+3}D_3,$$

where D_1, D_2 and D_3 are determinants of order 3 given by

$$(1.4)\quad D_1 = (a_{p+2}a_{p+5} - a_{p+3}a_{p+4}) - a_{p+1}(a_{p+1}a_{p+5} - a_{p+2}a_{p+4}) + a_{p+3}(a_{p+1}a_{p+3} - a_{p+2}^2),$$

$$(1.5)\quad D_2 = (a_{p+3}a_{p+5} - a_{p+4}^2) - a_{p+1}(a_{p+2}a_{p+5} - a_{p+3}a_{p+4}) + a_{p+2}(a_{p+2}a_{p+4} - a_{p+3}^2),$$

$$(1.6)\quad D_3 = a_{p+1}(a_{p+3}a_{p+5} - a_{p+4}^2) - a_{p+2}(a_{p+2}a_{p+5} - a_{p+3}a_{p+4}) + a_{p+3}(a_{p+2}a_{p+4} - a_{p+3}^2).$$

Hankel determinant has been considered by several authors. For example, Noor [17] determined the rate of growth of $H_q(n)$ as $n \to \infty$ for the functions given by Eq.(1.1) with bounded boundary. Ehrenborg [5] studied the Hankel determinant of exponential polynomials and in [10], the Hankel transform of an integer sequence is defined and some of its properties have been discussed by Layman.

Second Hankel determinant for various classes has been extensively studied by various authors including Mehrok and Singh [14], Janteng et al.[7] and many others. Third Hankel determinants for various classes were studied by some of the researchers including Babalola [3], Shanmugam et al.[18], Altinkaya and Yalcin [1] and Singh and Singh [19]. Also the Hankel determinant for various subclasses of p-valent functions were studied by various authors including Krishna and Ramreddy [8] and Hayami and Owa [6].

In this paper, we seek upper bound for the functional $H_{4,p}(f)$ for the functions belonging to the class $R_p(\alpha)$. This paper will motivate the future researchers to investigate the fourth Hankel determinant for some other subclasses of univalent and multivalent functions.

2 Preliminary results

***Lemma* 2.1**[4,11] *If* $p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n \in P$, *then for* $n, k \in N = \{1, 2, 3, ...\}$, *we have the following inequalities*:

$$|c_{n+k} - \lambda c_n c_k| \leq 2, 0 \leq \lambda \leq 1,$$

and

$$|c_n| \leq 2.$$

***Lemma* 2.2** *If* $p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n \in P$, *then for* $n, k \in N = \{1, 2, 3, ...\}$, *we have*:

$$|c_{n+k} - \lambda c_n c_k| \leq 4\lambda - 2, \lambda \geq 1.$$

Proof. For $\lambda \geq 1$, we have

$$|c_{n+k} - \lambda c_n c_k| \leq |c_n c_k - c_{n+k}| + (\lambda - 1)|c_n c_k|.$$

Using ***Lemma* 2.1**, the above inequality yields

$$|c_{n+k} - \lambda c_n c_k| \le 4\lambda - 2.$$

***Lemma* 2.3**[2] *If $f \in R_p(\alpha)$, then*

$$|a_{p+j}| \le \frac{2p}{p + j\alpha}.$$

***Lemma* 2.4**[2] *If $f \in R_p(\alpha)$, then*

$$|a_{p+2} - a_{p+1}^2| \le \frac{2p}{p + 2\alpha}.$$

***Lemma* 2.5**[9] *If $f \in R_p(\alpha)$, then*

$$|a_{p+1}a_{p+3} - a_{p+2}^2| \le \frac{4p^2}{(p + 2\alpha)^2}.$$

***Lemma* 2.6**[2] *If $f \in R_p(\alpha)$, then*

$$|a_{p+1}a_{p+2} - a_{p+3}| \le \begin{cases} 2 & \text{if } \alpha = 0, \\ \dfrac{2p(6\alpha^2 + 3p\alpha + p^2)^{\frac{3}{2}}}{3\sqrt{6}\alpha(p+\alpha)(p+2\alpha)(p+3\alpha)} & \text{if } 0 < \alpha \le 1. \end{cases}$$

***Lemma* 2.7** *If $f \in R_p(\alpha)$, then*

$$|H_3(p)| \le \begin{cases} 16 & \text{for } \alpha = 0, \\ \dfrac{4p^2}{p+2\alpha}\left[\dfrac{2p}{(p+2\alpha)^2} + \dfrac{1}{p+4\alpha} + \dfrac{(6\alpha^2 + 3p\alpha + p^2)^{\frac{3}{2}}}{3\sqrt{6}\alpha(p+\alpha)(p+3\alpha)^2}\right] & \text{for } 0 < \alpha \le 1. \end{cases}$$

Proof. From ***Lemma* 2.3**, we have

(2.1) $|a_{p+2}| \le \dfrac{2p}{p+2\alpha}$,

(2.2) $|a_{p+3}| \le \dfrac{2p}{p+3\alpha}$,

and

(2.3) $|a_{p+4}| \le \dfrac{2p}{p+4\alpha}$.

Using equations (2.1),(2.2) and (2.3), ***Lemma* 2.4**, ***Lemma* 2.5** and ***Lemma* 2.6** in (1.2), the result is obvious.
For $p = 1$, ***Lemma* 2.7** yields the following result:
***Corollary* 2.1** *If $f \in R(\alpha)$, then*

$$|H_3(1)| \le \begin{cases} 16 & \text{for } \alpha = 0, \\ \dfrac{4}{1+2\alpha}\left[\dfrac{2}{(1+2\alpha)^2} + \dfrac{1}{1+4\alpha} + \dfrac{(6\alpha^2 + 3\alpha + 1)^{3/2}}{3\sqrt{6}\alpha(1+\alpha)(1+3\alpha)^2}\right] & \text{for } 0 < \alpha \le 1. \end{cases}$$

For $p = 1, \alpha = 1$, ***Lemma* 2.7** gives the following result proved by Babalola [3]:
***Corollary* 2.2** *If $f \in R$, then*

$$|H_3(1)| \le 0.7423.$$

3 Fourth Hankel determinant for the class $R_p(\alpha)$

***Theorem* 3.1** *If $f \in R_p(\alpha)$, then*

$$(3.1)\quad |H_4(p)| \le \begin{cases} 152.0866 & \text{for } \alpha = 0, \\ \dfrac{8p^3}{(p+2\alpha)(p+6\alpha)}\left[\dfrac{2p}{(p+2\alpha)^2} + \dfrac{1}{p+4\alpha} + \dfrac{(6\alpha^2 + 3p\alpha + p^2)^{3/2}}{3\sqrt{6}\alpha(p+\alpha)(p+3\alpha)^2}\right] & \\ +\dfrac{2p}{(p+5\alpha)}u(p,\alpha) + \dfrac{2p}{(p+4\alpha)}v(p,\alpha) + \dfrac{2p}{(p+3\alpha)}w(p,\alpha) & \text{for } 0 < \alpha \le 1, \end{cases}$$

where

$$(3.2)\quad u(p,\alpha)=2p^2(4p-2)\left[\frac{1}{(p+\alpha)^2(p+5\alpha)}+\frac{1}{(p+3\alpha)(p+2\alpha)^2}+\frac{1}{(p+\alpha)(p+3\alpha)^2}\right]+\frac{174p^2(4p-2)+4p^2}{48(p+\alpha)(p+2\alpha)(p+4\alpha)},$$

$$(3.3)\quad v(p,\alpha)=\left[\frac{63p^2(4p-2)}{25(p+\alpha)(p+2\alpha)(p+5\alpha)}+\frac{18p^2(4p-2)}{5(p+4\alpha)(p+2\alpha)^2}+\frac{150p^2(4p-2)+4p^2}{75(p+2\alpha)(p+3\alpha)^2}\right]$$

and

$$(3.4)\quad w(p,\alpha)=2p^2(4p-2)\times\left[\frac{1}{(p+2\alpha)^2(p+5\alpha)}+\frac{1}{(p+\alpha)(p+3\alpha)(p+5\alpha)}+\frac{2}{(p+3\alpha)^3}+\frac{1}{(p+\alpha)(p+4\alpha)^2}\right]+\frac{34p^2(4p-2)}{16(p+2\alpha)(p+3\alpha)(p+4\alpha)}+\frac{p^2}{(p+\alpha)(p+2\alpha)^2(p+3\alpha)(p+4\alpha)^2(p+5\alpha)}.$$

Proof. Using ***Lemma* 2.3** in (1.4), (1.5) and (1.6), it gives

$$(3.5)\quad D_1=\frac{p^2c_2c_5}{(p+2\alpha)(p+5\alpha)}-\frac{p^2c_3c_4}{(p+3\alpha)(p+4\alpha)}-\frac{p^3c_1^2c_5}{(p+\alpha)^2(p+5\alpha)}+\frac{p^3c_1c_2c_4}{(p+\alpha)(p+2\alpha)(p+4\alpha)}+\frac{p^3c_1c_3^2}{(p+\alpha)(p+3\alpha)^2}-\frac{p^3c_3c_2^2}{(p+3\alpha)(p+2\alpha)^2},$$

$$(3.6)\quad D_2=\frac{p^2c_3c_5}{(p+3\alpha)(p+5\alpha)}-\frac{p^2c_4^2}{(p+4\alpha)^2}-\frac{p^3c_1c_2c_5}{(p+\alpha)(p+2\alpha)(p+5\alpha)}+\frac{p^3c_1c_3c_4}{(p+\alpha)(p+3\alpha)(p+4\alpha)}+\frac{p^3c_4c_2^2}{(p+2\alpha)^2(p+4\alpha)}-\frac{p^3c_2c_3^2}{(p+2\alpha)(p+3\alpha)^2}$$

and

$$(3.7)\quad D_3=\frac{p^3c_1c_3c_5}{(p+\alpha)(p+3\alpha)(p+5\alpha)}-\frac{p^3c_1c_4^2}{(p+\alpha)(p+4\alpha)^2}-\frac{p^3c_2^2c_5}{(p+2\alpha)^2(p+5\alpha)}+\frac{2p^3c_2c_3c_4}{(p+2\alpha)(p+3\alpha)(p+4\alpha)}-\frac{p^3c_3^3}{(p+3\alpha)^3}.$$

On rearranging the terms in (3.5), (3.6) and (3.7), it yields

$$(3.8)\quad D_1=\frac{p^2c_5(c_2-pc_1^2)}{(p+\alpha)^2(p+5\alpha)}+\frac{p^2c_3(c_4-pc_2^2)}{(p+3\alpha)(p+2\alpha)^2}-\frac{p^2c_3(c_4-pc_1c_3)}{(p+\alpha)(p+3\alpha)^2}-\frac{67p^2c_4(c_3-pc_1c_2)}{48(p+\alpha)(p+2\alpha)(p+4\alpha)}+\frac{19p^2c_2(c_5-pc_1c_4)}{48(p+\alpha)(p+2\alpha)(p+4\alpha)}+\frac{p^2c_2c_5}{48(p+\alpha)(p+2\alpha)(p+4\alpha)},$$

$$(3.9)\quad D_2=\frac{p^2c_5(c_3-pc_1c_2)}{(p+\alpha)(p+2\alpha)(p+5\alpha)}-\frac{p^2c_4(c_4-pc_2^2)}{(p+4\alpha)(p+2\alpha)^2}-\frac{p^2c_3(c_5-pc_2c_3)}{(p+2\alpha)(p+3\alpha)^2}-\frac{4p^2c_4(c_4-pc_1c_3)}{5(p+4\alpha)(p+2\alpha)^2}-\frac{13p^2c_3(c_5-pc_1c_4)}{50(p+\alpha)(p+2\alpha)(p+5\alpha)}+\frac{p^2c_3c_5}{75(p+2\alpha)(p+3\alpha)^2}$$

and

$$(3.10)\quad D_3=\frac{p^2c_5(c_4-pc_2^2)}{(p+2\alpha)^2(p+5\alpha)}-\frac{p^2c_5(c_4-pc_1c_3)}{(p+\alpha)(p+3\alpha)(p+5\alpha)}+\frac{p^2c_3(c_6-pc_3^2)}{(p+3\alpha)^3}-\frac{p^2c_3(c_6-pc_2c_4)}{(p+3\alpha)^3}+\frac{p^2c_4(c_5-pc_1c_4)}{(p+\alpha)(p+4\alpha)^2}-\frac{17p^2c_4(c_5-pc_2c_3)}{16(p+2\alpha)(p+3\alpha)(p+4\alpha)}+\frac{p^2c_4c_5}{4(p+\alpha)(p+2\alpha)^2(p+3\alpha)(p+4\alpha)^2(p+5\alpha)}.$$

Using ***Lemma* 2.2** and applying triangle inequality in (3.8), (3.9) and (3.10), we obtain

$$(3.11)\quad |D_1|\le u(p,\alpha),$$

$$(3.12)\quad |D_2|\le v(p,\alpha)$$

and

(3.13) $|D_3| \leq w(p,\alpha)$,

where $u(p,\alpha), v(p,\alpha)$ and $w(p,\alpha)$ are defined in (3.2), (3.3) and (3.4) respectively.
Hence using ***Lemma* 2.3**, ***Lemma* 2.7** and equations (3.11), (3.12), (3.13) in equation (1.3) and applying triangle inequality, the result (3.1) is obvious.
On putting $p = 1$ in ***Theorem* 3.1**, we obtain the following result:
***Corollary* 3.1** *If $f \in R(\alpha)$, then*

$$|H_4(1)| \leq \begin{cases} 152.0866 & \text{for } \alpha = 0, \\ \dfrac{8}{(1+2\alpha)(1+6\alpha)}\left[\dfrac{2}{(1+2\alpha)^2} + \dfrac{1}{1+4\alpha} + \dfrac{(6\alpha^2+3\alpha+1)^{\frac{3}{2}}}{3\sqrt{6}\alpha(1+\alpha)(1+3\alpha)^2}\right] & \\ +\dfrac{2}{(1+5\alpha)}p(\alpha) + \dfrac{2}{(1+4\alpha)}q(\alpha) + \dfrac{2}{(1+3\alpha)}r(\alpha) & \text{for } 0 < \alpha \leq 1, \end{cases}$$

where

$$p(\alpha) = 4\left[\frac{1}{(1+\alpha)^2(1+5\alpha)} + \frac{1}{(1+3\alpha)(1+2\alpha)^2} + \frac{1}{(1+\alpha)(1+3\alpha)^2}\right] + \frac{29}{4(1+\alpha)(1+2\alpha)(1+4\alpha)},$$

$$q(\alpha) = 4\left[\frac{63}{50(1+\alpha)(1+2\alpha)(1+5\alpha)} + \frac{9}{5(1+4\alpha)(1+2\alpha)^2} + \frac{76}{75(1+2\alpha)(1+3\alpha)^2}\right]$$

and

$$r(\alpha) = 4\left[\frac{1}{(1+2\alpha)^2(1+5\alpha)} + \frac{1}{(1+\alpha)(1+3\alpha)(1+5\alpha)} + \frac{2}{(1+3\alpha)^3} + \frac{1}{(1+\alpha)(1+4\alpha)^2}\right] + \frac{68}{16(1+2\alpha)(1+3\alpha)(1+4\alpha)} + \frac{1}{(1+\alpha)(1+2\alpha)^2(1+3\alpha)(1+4\alpha)^2(1+5\alpha)}.$$

On putting $p = 1, \alpha = 1$ in ***Theorem* 3.1**, the following result is obvious:
***Corollary* 3.2** *If $f \in R$, then*

$$|H_{4,1}(f)| \leq 0.7973.$$

4 Conclusion.

In the present work, we estimated the bounds for the fourth Hankel determinant for a subclass of multivalent bounded turning functions. The estimation of fourth Hankel determinant for the various subclasses of analytic functions is a new concept in the field of geometric function theory. Till now much work has been done on the study of second and third Hankel determinants for various subclasses of univalent functions, so this paper will work as a milestone to the future researchers in this field.

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